Embeddings of inverse limit spaces
by James Lee Kassebaum

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences
Montana State University

Abstract:
Barge and Martin have shown that the inverse limit of any interval map may be topologically realized as an attractor of a planar homeomorphism H in such a way that H restricted to the attractor is conjugate to the induced homeomorphism on the inverse limit. They extend the interval map to a near homeomorphism of a disk containing the interval and use a theorem of Morton Brown to obtain their result. We present similar results for degree -1, 0, or 1 circle maps and give necessary conditions for a map to be extended to a near homeomorphism. Then, it is shown that for a given finite, connected, planar, graph, G, containing a branch point, the set of surjective, continuous maps of G which cannot be extended to a near homeomorphism on any neighborhood of G are open and dense in the set of all such maps, with the C^0-topology. However, an example of such a map for which the inverse limit can be embedded as an attractor in the plane is given. Next, we prove the inverse limit of any n-od ($n \geq 3$) can be embedded as an attractor in 3-space. We then give necessary and sufficient conditions for the inverse limit of a finite, connected, planar, graph with a surjective bonding map to be chainable. Since any chainable continuum may be embedded in the plane, such an inverse limit is planar. Finally, we give an example of a chainable continuum for which there exists a homeomorphism which is not essentially extendable.

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James Lee Kassebaum

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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Date 10/31/95

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ABSTRACT

Barge and Martin have shown that the inverse limit of any interval map may be topologically realized as an attractor of a planar homeomorphism H in such a way that H restricted to the attractor is conjugate to the induced homeomorphism on the inverse limit. They extend the interval map to a near homeomorphism of a disk containing the interval and use a theorem of Morton Brown to obtain their result. We present similar results for degree -1, 0, or 1 circle maps and give necessary conditions for a map to be extended to a near homeomorphism. Then, it is shown that for a given finite, connected, planar, graph, G, containing a branch point, the set of surjective, continuous maps of G which cannot be extended to a near homeomorphism on any neighborhood of G are open and dense in the set of all such maps, with the C^0-topology. However, an example of such a map for which the inverse limit can be embedded as an attractor in the plane is given. Next, we prove the inverse limit of any n-od ($n \geq 3$) can be embedded as an attractor in 3-space. We then give necessary and sufficient conditions for the inverse limit of a finite, connected, planar, graph with a surjective bonding map to be chainable. Since any chainable continuum may be embedded in the plane, such an inverse limit is planar. Finally, we give an example of a chainable continuum for which there exists a homeomorphism which is not essentially extendable.

CHAPTER 1

INTRODUCTION

We begin our discussion with the following preliminary definitions:

Definition 1.1 *Let Z be a compact metric space and $g : Z \to Z$ be continuous. Define* $\varprojlim g = \{(z_0, z_1, \ldots) | \, z_i \in Z \text{and } g(z_{i+1}) = z_i\}$*. Then* $\varprojlim g$ *is a subset of the product space* $\{(z_0, z_1, \ldots) | z_i \in Z\}$ *with the product topology.*

Definition 1.2 *Let $h : X \to X$ be a map and $U \subset X$ be relatively compact with $h(cl(U)) \subset U$. Then $\Lambda = \cap_{n \geq 0} h^n(U)$ is called an attractor for h.*

Inverse limit spaces provide interesting examples of continua. The pseudo-arc, a homogeneous indecomposable plane continuum (Bing, [5]), is homeomorphic to the inverse limit of an interval map ([11], [18]). Much attention has been paid to the embeddability of inverse limit spaces in Euclidean space. Ralph Bennett ([4]), R. H. Bing ([6], [7]), J. R. Isbell ([12]), James Keesling ([13], [14]), Michael C. McCord ([17]), and David C. Wilson ([14]) have all published on this topic. In addition, inverse limit spaces have been used to model attractors of homeomorphisms. For example, the solenoid (Smale, [23]) is homeomorphic to the inverse limit space of a circle map.

Definition 1.3 *Define the* **induced homeomorphism** $\hat{g} : \varprojlim g \to \varprojlim g$ *by* $\hat{g}((z_0, z_1, \ldots)) = (g(z_0), z_0, z_1, \ldots)$.

Definition 1.4 *Let G be a finite, connected, planar graph and $f : G \to G$ be a continuous map. If there exists a homeomorphism $H : X \to X$ such that K is an attractor of H, K is homeomorphic to $\varprojlim f$, and $H|_K$ is conjugate to the shift homeomorphism, $\hat{f}$, on $\varprojlim f$, then we will say that $\varprojlim f$ can be embedded as an attractor in X.*

Barge and Martin ([1]) have shown that for <u>any</u> interval map f, $\varprojlim f$ can be embedded as an attractor in $\mathbb{R}^2$. They have thus insured a plethora of interesting planar attractors. In the proof of their result, they use the following definitions and theorem of Morton Brown ([10]).

Definition 1.5 *(Brown [10]) Let X be a metric space. A map $f : X \to X$ is a* **near homeomorphism** *if for any $\epsilon > 0$ there is a homeomorphism H_ϵ of X onto itself such that $||H_\epsilon - f|| < \epsilon$.*

Definition 1.6 *(Brown [10]) Let X_i be a sequence of compact metric spaces, and for $i \geq 2$ let f_i map X_i into X_{i-1}. Let $f_{ij} = f_{i+1}f_{i+2}\dots f_j$ and $f_{ii} = 1$. If z is a point of X_i then z_i will always denote the i^{th} coordinate of z. Hence $z = (z_i)$. Then the subspace $S = \{z \in \prod_1^\infty X_i | f_{ij}(z_j) = z_i\}$ of $\prod_1^\infty X_i$ is the limit space of the inverse system (X_i, f_i).*

Theorem 1.1 *(Brown [10]) Let $S = Lim(X_i, f_i)$ where the X_i are all homeomorphic to a compact metric space X, and for all i, f_i is a near homeomorphism. Then S is homeomorphic to X.*

The method of Barge and Martin is to extend the interval map f to F–a near homeomorphism of a disk containing the interval in its interior–and use the theorem of Morton Brown.

In chapter 2, we obtain similar results for degree -1,0, or 1 circle maps. Also, two results on extendibility to a near homeomorphism are given. In chapter 3, we show that if f is a continuous, onto map of a triod which can be approximated by planar embeddings, $\varprojlim f$ is at least planar. Now let G be a finite, connected planar graph. If G contains a branch point, the set of continuous maps, f, of G onto itself such that f cannot be approximated by planar embeddings is open and dense in the set of all maps of G onto itself. Since such a map f cannot be extended to a near homeomorphism on any planar neighborhood of G, the Barge-Martin construction cannot be used to embed $\varprojlim f$ as an attractor in $\mathbb{R}^2$. However, an example of a map $f : G \to G$ (onto, with G a triod) which cannot be approximated by planar embeddings, but for which $\varprojlim f$ can be embedded as an attractor in $\mathbb{R}^2$ is supplied. Sanford and Walker have shown that "a positive entropy map of the product of a Cantor Set and an arc (which covers a homeomorphism) cannot be "embedded" into a near homeomorphism of the 2-disk. Thus, a theorem of M. Brown cannot be used to embed the induced shift map on the corresponding inverse limit space into a 2-disk homeomorphism" ([22]).

In chapter 4, the inverse limit of any n-od ($n \geq 3$) is embedded as an attractor in $\mathbb{R}^3$ and in chapter 5 we return to the question of which maps $f : G \to G$ have the property that $\varprojlim f$ can be embedded as an attractor in the plane. As a first step toward that goal, we give necessary and sufficient conditions that $\varprojlim f$, $f : G \to G(onto)$, be snake-like or chainable–a condition insuring that $\varprojlim f$ is planar. Finally, after a few examples, propositions, and a corollary, we turn to the question of extending a homeomorphism of a given planar continuum to a homeomorphism of all of $\mathbb{R}^2$. For, once we embed $\varprojlim f$ in the plane, we need to be able to extend $\hat{f}$ to a homeomorphism of all of $\mathbb{R}^2$ in order to embed $\varprojlim f$ as an attractor in the plane. Brechner ([9]) has shown that there exists a chainable continuum $M \subseteq S^2$ and a homeomorphism $h : M \to M(onto)$ such that h is not "essentially extendable". We give a similar example of a chainable continuum K and a homeomorphism $f : K \to K(onto)$ such that f is not essentially extendable to all of $\mathbb{R}^2$. Barge and Walker also have such an example ([2]). Brechner ([8]) and Wayne Lewis ([15]) both have results that imply that if K is any chainable continuum and $f : K \to K(onto)$ is any homeomorphism, f is essentially extendable to all of $\mathbb{R}^3$. We then close with an example that shows that even if $\varprojlim f$ is planar the shift homeomorphism, $\hat{f}$, may not be essentially extendable to $\mathbb{R}^2$.

CHAPTER 2

EMBEDDING CIRCLE-LIKE CONTINUA AS ATTRACTORS

In the following, it is shown that for a circle map f, $\varprojlim f$ can be embedded as an attractor in the plane for $\deg(f) = 0$ and the plane less the origin for $\deg(f) \in \{-1, 1\}$. Dr. James Keesling has pointed out that Michael C. McCord has shown the following related result ([17], pg. 323):

Proposition 2.1 *(McCord [17]) A circle-like continuum X can be embedded in the plane if and only if $H^1(X) = 0$ or $H^1(X) \approx \mathbb{Z}$.*

($H^1(X)$ denotes the 1-dimensional Čech cohomology $\mathbb{Z}$-module of X.)
Finally, two theorems on extending a map to a near homeomorphism are given. The importance of these results lies in the fact that in order to embed $\varprojlim f$ as an attractor in a space Y, we first need to extend $f : S^1 \to S^1$ to a near homeomorphism of Y.

Definition 2.7 *Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and $A = \{z \in \mathbb{C} \mid \frac{1}{2} \leq |z| \leq \frac{3}{2}\}$ and define the* **covering maps:**
$\pi : \mathbb{R} \to S^1$ *by* $\pi(x) = e^{2\pi i x}$ *and* $P : \mathbb{R} \times [\frac{1}{2}, \frac{3}{2}] \to A$ *by* $P(x, y) = ye^{2\pi i x}$.

Definition 2.8 *Let $f : S^1 \to S^1$ be continuous. A* **lift** *of f is a continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $\pi \circ \tilde{f} = f \circ \pi$.*

Definition 2.9 *$deg(f) = \tilde{f}(1) - \tilde{f}(0)$, where $\tilde{f}$ is a lift of f.*

Lemma 2.1 *Let $f : S^1 \to S^1$ be continuous with $deg(f) = 0$. Then there is an interval I and a map $g : I \to I$ so that $\varprojlim f$ is homeomorphic to $\varprojlim g$.*

Proof: Let $\tilde{f}$ be a lift of f, $a = \min_{t\in\mathbb{R}}\{\tilde{f}(t)\}$, $b = max\{a+1, \max_{t\in\mathbb{R}}\{\tilde{f}(t)\}\}$, and $g = \tilde{f}|_{[a,b]}$. Note that since $deg(f) = 0$, $\tilde{f}(x+k) = \tilde{f}(x)$ for each real number x and integer k. Define $\Pi : \varprojlim g \to \varprojlim f$ via $\Pi(t_0, t_1, \ldots) = (\pi(t_0), \pi(t_1), \ldots)$, where π is the covering map $\pi(x) = e^{2\pi ix}$. $(t_0, t_1, \ldots) \in \varprojlim g \Rightarrow \tilde{f}(t_{n+1}) = t_n$
$\Rightarrow \pi \circ \tilde{f}(t_{n+1}) = \pi(t_n) \Rightarrow f(\pi(t_{n+1})) = \pi(t_n)$, so $(\pi(t_0), \pi(t_1), \ldots) \in \varprojlim f$.

Π **is one-to-one:** Suppose that $\Pi(t_0, t_1, \ldots) = \Pi(t'_0, t'_1, \ldots)$. For each positive integer n, $\pi(t_{n+1}) = \pi(t'_{n+1}) \Rightarrow t_{n+1} = t'_{n+1} + k$, k an integer.

So, $\tilde{f}(t_{n+1}) = t_n = \tilde{f}(t'_{n+1} + k) = \tilde{f}(t'_{n+1}) = t'_n$. Therefore, $t_n = t'_n$, for each positive integer n.

Π **is onto:** Let $(s_0, s_1, \ldots) \in \varprojlim f$ and $T_n = \{\pi^{-1}(s_n)\} \cap [a,b]$, for each positive integer n. Then for $n \geq 1$, $\tilde{f}(T_n) \in T_{n-1}$, since $f(s_n) = s_{n-1}$, $\pi \circ \tilde{f} = f \circ \pi$, and $\{\pi^{-1}(s_n)\} = \{t_n + k | t_n \in [0,1], k \text{ an integer}\}$. Construct an element of $\varprojlim g$ by letting $t_0 = \tilde{f}(T_1), t_1 = \tilde{f}(T_2), t_2 = \tilde{f}(T_3), \ldots$. Then $(t_0, t_1, \ldots) \in \varprojlim g$ and $\Pi((t_0, t_1, \ldots)) = (s_0, s_1, \ldots)$.

Π **is continuous:** Since the metric topology on $\varprojlim f$ is equivalent to the product topology and for each $\tilde{t} \in \varprojlim f$, $\Pi(\tilde{t}) = (\pi(\pi_j(\tilde{t})))_{j\in N}$ (π_j is projection in the j^{th} coordinate), Π is continuous.

$\varprojlim g$ is compact and $\varprojlim f$ is Hausdorff, so Π is a homeomorphism.

PTL

Lemma 2.2 *(* $\varprojlim g, \tilde{g}$*) is conjugate to (* $\varprojlim f, \tilde{f}$*).*

Proof: $\Pi : \varprojlim g \to \varprojlim f$ is a homeomorphism and $(\tilde{f} \circ \Pi)((t_0, t_1, \ldots)) = \tilde{f}(e^{2\pi i t_0}, e^{2\pi i t_1}, \ldots)$ $= (e^{2\pi i \tilde{f}(t_0)}, e^{2\pi i t_0}, e^{2\pi i t_1}, \ldots) = \Pi(\tilde{f}(t_0), t_0, t_1, \ldots) = (\Pi \circ \tilde{g})((t_0, t_1, \ldots))$

PTL

Theorem 2.2 *If $deg(f) = 0$, then $\varprojlim f$ can be embedded as an attractor in the plane.*

Proof: By a result of Barge and Martin [1], there is an embedding h from $\varprojlim g$ into the plane, $\mathbb{R}^2$, a homeomorphism $k = h \circ \tilde{g} \circ h^{-1} : h(\varprojlim g) \to h(\varprojlim g)$, and a homeomorphism, K of the plane, so that $K|_{h(\varprojlim g)} = k$, and $h(\varprojlim g)$ is a global attractor for K. So, we have the following commutative diagram:

$$
\begin{array}{ccccccc}
\varprojlim f & \xrightarrow{\Pi^{-1}} & \varprojlim g & \xrightarrow{h} & h(\varprojlim g) & \xrightarrow{i} & \mathbb{R}^2 \\
\downarrow \tilde{f} & & \downarrow \tilde{g} & & \downarrow k & & \downarrow K \\
\varprojlim f & \xrightarrow{\Pi^{-1}} & \varprojlim g & \xrightarrow{h} & h(\varprojlim g) & \xrightarrow{i} & \mathbb{R}^2
\end{array}
$$

where i is the inclusion map. $(h \circ \Pi^{-1})(\varprojlim f) = h(\varprojlim g)$ embeds $\varprojlim f$ in $\mathbb{R}^2$ and $h \circ \Pi^{-1} \circ \tilde{f} \circ \Pi \circ h^{-1} = h \circ \tilde{g} \circ h^{-1} = k$, so that the shift map on this embedding is just k and the result follows.

PTL

Theorem 2.3 *Let $f : S^1 \to S^1$ be continuous with $deg(f) \in \{-1, 1\}$. Then $\varprojlim f$ can be embedded as an attractor in the plane less the origin.*

Proof: Let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be a lift of f. Since $\tilde{f}(x+1) = \tilde{f}(x) + 1$ for every $x \in \mathbb{R}$, there is a $b > 0$ so that $graph(\tilde{f}) \subset S = \{(x,y) \in \mathbb{R}^2 | y \in \mathbb{R}, y - b < x < y + b\}$. For every $y \in \mathbb{R}$, define a homeomorphism $h_y : [y-b, y+b] \times \{y\} \to [y-b, y+b] \times \{y\}$ by

$$h_y(x,y) = \begin{cases} (\tilde{f}(y) + \frac{\tilde{f}(y)-y+b}{b}(x-y), y), x \in [y-b, y] \\ (\tilde{f}(y) + \frac{y+b-\tilde{f}(y)}{b}(x-y), y), x \in [y, y+b] \end{cases}$$

Now, define $H : cl(S) \to cl(S)$ by $H(x,y) = h_y(x,y)$. Note that $H(x,x) = (\tilde{f}(x), x)$, for every $x \in \mathbb{R}$. H is one-to-one and onto, since each h_y is. Let $\tilde{S} = \{(x,y) \in \mathbb{R}^2 | y \in [0,1], y - b \le x \le y + b\}$. Then, let $\tilde{H} = H|_{\tilde{S}}$.

$\tilde{H}$ **is continuous:** Fix $(x,y) \in \tilde{S}$. Since any two norms on a finite dimensional vector space are equivalent, we'll use the one-norm, i.e., $\|(x,y)\| = |x| + |y|$.

Case I: $x < y$. If $\|(x,y) - (x_1,y_1)\|$ is small, $x_1 < y_1$ and $\|\tilde{H}(x,y) - \tilde{H}(x_1,y_1)\|$
$= \|(\tilde{f}(y) + \frac{\tilde{f}(y)-y+b}{b}(x-y), y) - (\tilde{f}(y_1) + \frac{\tilde{f}(y_1)-y_1+b}{b}(x_1-y_1), y_1)\| \le |\tilde{f}(y) - \tilde{f}(y_1)|$
$+ |\frac{y(x-y)-b(x-y)-\tilde{f}(y)(x-y)-y_1(x_1-y_1)+b(x_1-y_1)+\tilde{f}(y_1)(x_1-y_1)}{b}| + |y - y_1| \le |\tilde{f}(y) - \tilde{f}(y_1)|$
$+ \frac{1}{b}|y(x-y) - y_1(x_1-y_1)| + |(x-y) - (x_1-y_1)| + \frac{1}{b}|\tilde{f}(y_1)(x_1-y_1) - \tilde{f}(y)(x-y)| + |y - y_1|$
$\le |\tilde{f}(y) - \tilde{f}(y_1)| + \frac{1}{b}(|y||(x-y) - (x_1-y_1)| + |x_1-y_1||y-y_1|) + |(x-y) - (x_1-y_1)|$
$+ \frac{1}{b}(|\tilde{f}(y_1)||(x_1-y_1) - (x-y)| + |x-y||\tilde{f}(y_1) - \tilde{f}(y)|) + |y-y_1|$ which can be made small by the continuity of $\tilde{f}$ and boundedness of the region $cl(S)$.

Case II: $x = y$ and without loss of generality $x_1 > y_1$. $\|\tilde{H}(x,y) - \tilde{H}(x_1,y_1)\|$
$= \|(\tilde{f}(y), y) - (\tilde{f}(y_1) + \frac{y_1+b-\tilde{f}(y_1)}{b}(x_1-y_1), y_1)\| \le |\tilde{f}(y) - \tilde{f}(y_1)| + |\frac{y_1+b-\tilde{f}(y_1)}{b}||x_1 - y_1|$
$+ |y - y_1|$ is small if $\|(x,y) - (x_1,y_1)\|$ is small.

Case III: $x > y$.

Similar to case I.

Therefore, $\tilde{H} : \tilde{S} \to \tilde{S}$ is continuous. Since $\tilde{H}$ is bijective continuous and $\tilde{S}$ is closed and compact, $\tilde{H}$ is a homeomorphism. Since $cl(S)$ can be partitioned into closed sets $\{\tilde{S}_i\}$ so that $H|_{\tilde{S}_i}$ is a homeomorphism for every i, $H : cl(S) \to cl(S)$ is a homeomorphism by the pasting lemma ([19], pg. 108).

$\underline{deg(f) = 1:}$ Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation on $\mathbb{R}^2$ induced by the matrix $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Also, let $\alpha : \mathbb{R} \times [\frac{1}{2}, \frac{3}{2}] \to \mathbb{R} \times [-b, b]$ be the homeomorphism defined by $\alpha(x, y) = (x, 2b(y-1))$. Define $K : A \to A$ by $K(ye^{2\pi ix})$
$= (P \circ \alpha^{-1} \circ T^{-1} \circ H \circ T \circ \alpha \circ P^{-1})(ye^{2\pi ix})$.

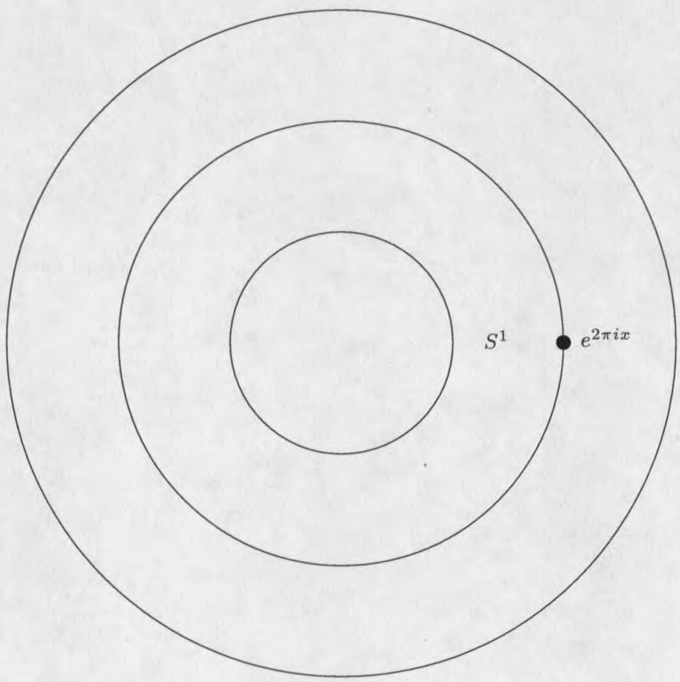

Figure 1: A

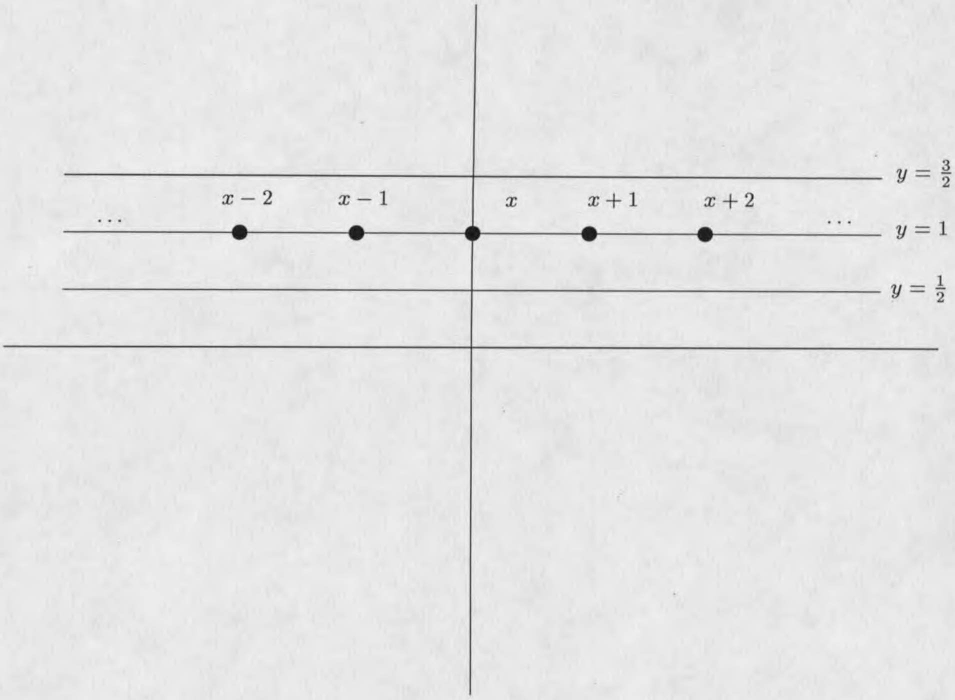

Figure 2: $P^{-1}(A)$

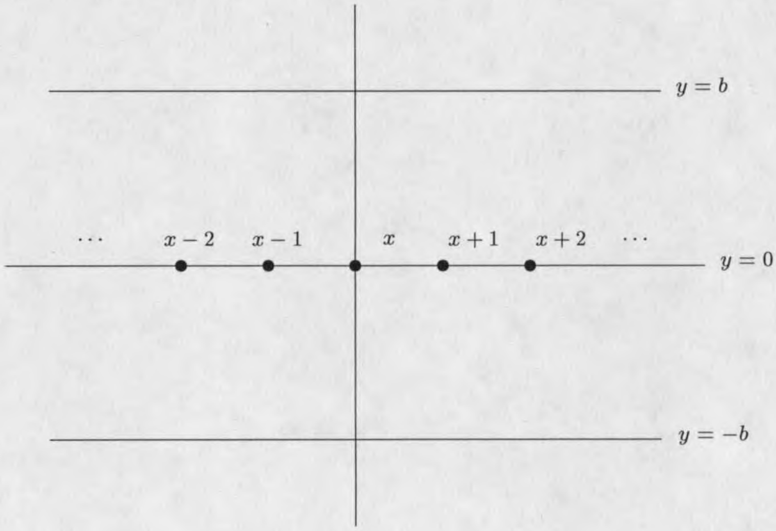

Figure 3: $\alpha \circ P^{-1}(A)$

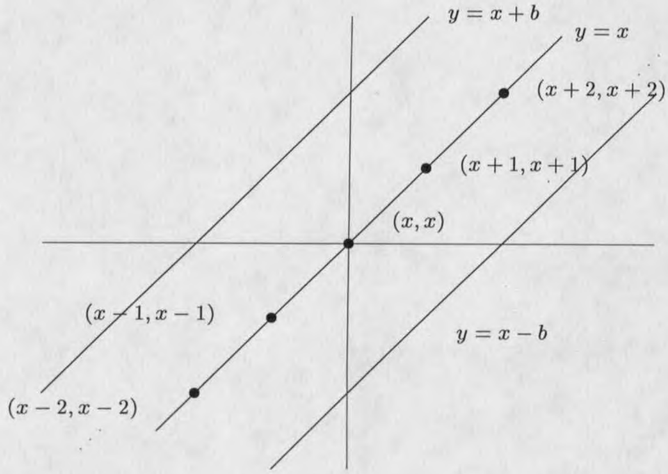

Figure 4: $T \circ \alpha \circ P^{-1}(A)$

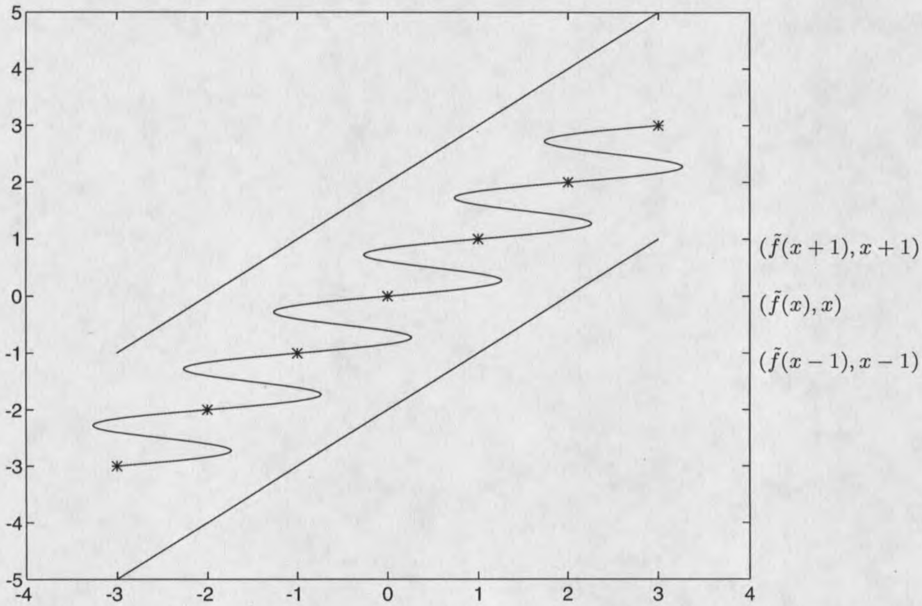

Figure 5: $H \circ T \circ \alpha \circ P^{-1}(A)$