Generalized Dewitt-Schwinger point-splitting expansions for the charged scalar field in curved space
by Rhett Byron Herman

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics
Montana State University

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The theory of DeWitt-Schwinger point-splitting is developed for a massive, charged quantized scalar field coupled to an electromagnetic field possessing U(1) symmetry in a general, four-dimensional spacetime. The infinite regularization counterterms required to renormalize the vacuum polarization (φ2), the current (j μ) associated with the scalar field, and the stress-energy tensor (Tμ ν) associated with the scalar field are presented. The DeWitt-Schwinger point-splitting expansions in this thesis are carried out until they reach terms of order m-2 for (Tμν), where m is the mass of the quantized field. Presented here for the first time is an analytic approximation for (T μν) for a massive, quantized charged scalar field in a general spacetime with a general electromagnetic field possessing U(1) symmetry.

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STATEMENT OF PERMISSION TO USE

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CONVENTIONS

The sign conventions used in this thesis are those of Misner, Thorne, and Wheeler [1]. Natural units, where $G = c = \hbar = 1$ are used throughout except where explicitly indicated. Derivatives are indicated as follows:

$\dfrac{\partial}{\partial x^\mu}$ or ∂_μ or $_{,\mu}$	partial derivative
∇_μ or $_{;\mu}$	covariant derivative
$\nabla_\mu - ieA_\mu$ or $_{\vert\mu}$	gauge covariant derivative.

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The theory of DeWitt-Schwinger point-splitting is developed for a massive, charged quantized scalar field coupled to an electromagnetic field possessing $U(1)$ symmetry in a general, four-dimensional spacetime. The infinite regularization counterterms required to renormalize the vacuum polarization $\langle\phi^2\rangle$, the current $\langle j^\mu\rangle$ associated with the scalar field, and the stress-energy tensor $\langle T^{\mu\nu}\rangle$ associated with the scalar field are presented. The DeWitt-Schwinger point-splitting expansions in this thesis are carried out until they reach terms of order m^{-2} for $\langle T^{\mu\nu}\rangle$, where m is the mass of the quantized field. Presented here for the first time is an analytic approximation for $\langle T^{\mu\nu}\rangle$ for a massive, quantized charged scalar field in a general spacetime with a general electromagnetic field possessing $U(1)$ symmetry.

CHAPTER 1

Introduction

The study of quantum fields in curved space follows the typical path in physics of moving from models that are initially simple and easier to understand to models that are more complicated but more physically realistic. Successive models build upon the knowledge gained in the study of previous models. The goal of this thesis is to build the theoretical foundation for the study of quantized charged scalar fields in a general four-dimensional spacetime which has an arbitrary electromagnetic field of $U(1)$ symmetry. The DeWitt-Schwinger point-splitting procedure [2] is developed for the case of the charged scalar field coupled to the electromagnetic field of the curved spacetime. The first set of the two major results derived here are the regularization counterterms for the vacuum expectation values (VEVs) for the vacuum polarization $\langle\phi^2\rangle$, the current $\langle j^\mu\rangle$, and the stress-energy tensor $\langle T^{\mu\nu}\rangle$ associated with the charged scalar field.

The DeWitt-Schwinger point-splitting procedure is known to be capable of yielding analytic expressions that are approximations to both $\langle\phi^2\rangle$ and $\langle T^{\mu\nu}\rangle$. The DeWitt-Schwinger point-splitting expansions in this thesis are carried out until they reach terms of order m^{-2} for $\langle T^{\mu\nu}\rangle$, where m is the mass of the quantized field. Presented here for the first time is an analytic approximation for $\langle T^{\mu\nu}\rangle$ for a massive, quantized charged scalar field in a general spacetime with a general electromagnetic field possessing $U(1)$ symmetry. After subtracting the infinites of the stress-energy tensor, which are contained in the terms in the expansion proportional to non-negative powers of m, the remaining expression, proportional to m^{-2}, serves as the "DeWitt-Schwinger approximation" to the actual renormalized value for $\langle T^{\mu\nu}\rangle$. This expression, along with an analytic expression for $\langle\phi^2\rangle$ of order m^{-4}, constitute the second of the two major results derived here.

The theory of DeWitt-Schwinger geodesic point-splitting began with the classic work of Schwinger [3]. Schwinger calculated the Green function $G(x,x')$ associated with a fermion current produced by and external electromagnetic field,

$$\langle j^\mu(x)\rangle = \lim_{x'\to x} ie\ \mathrm{tr}[\gamma^\mu G(x,x')] \tag{1}$$

where e is the charge of the fermion field and the γ^μ are the Dirac matrices. He calculated the Green function by introducing a fictitious, non-quantum-mechanical Hilbert space in which calculations were performed [4]. This Hilbert space was constructed in 4 + 1-dimensions, with the 4 familiar spacetime dimensions being supplemented by a fifth dimension identified as the proper time parameter s in this fictitious space. Working within this fictitious Hilbert space, Schwinger was able to isolate the divergences that appeared in the quantum field integrals involved with the fermion field current, and use those divergences to renormalize the charge of the fermion current and the strength of the external field.

While Schwinger's original work was performed in flat spacetime, DeWitt recognized that Schwinger's method could provide a way to isolate the divergences that appeared in quantum field theory calculations in curved spacetime. Consider the semiclassical Einstein-Maxwell field equations,

$$G_{\mu\nu} = 8\pi\langle T_{\mu\nu}\rangle, \tag{2}$$

and

$$F^{\mu\nu}{}_{;\nu} = 4\pi\langle j^\mu\rangle. \tag{3}$$

Eqs.(2) and (3) treat the gravitational and electromagnetic fields classically, while treating the sources for these fields quantum mechanically. It is well known that, when the transition is made from classical to quantized fields, infinities appear in the expectation values on the right hand sides of Eqs.(2)–(3) in both flat [5, 6, 7] and curved [8] spacetimes. These infinities can not be physical in the context of curved spacetime physics. For example, in non-gravitational physics, any infinities that appear in energy density calculations are considered to be "zero-point energies" and are summarily discarded. This rescaling of the zero point of energy in flat space does not change the physics and is allowed. However, in gravitational physics, all energy is a source of the gravitational field and thus a source of curvature. Zero-point energy, infinite or finite, may not be naïvely thrown away since doing so throws away a source of curvature and thus change the physics. Yet there must be a way to correctly remove any unphysical infinities from the theory.

The geodesic point-splitting regularization scheme developed by DeWitt [2, 4] is a fully covariant method whereby these unphysical infinities may be isolated, a process known as *regularization*. Then, in the process known as *renormalization*, the infinities are discarded by subtracting them from the unrenormalized field equations, leaving behind finite quantities that represent the physical universe. All of the quantities to be renormalized are VEVs constructed from products of the quantized field $\phi(x)$ and its derivatives. Products such as $\langle\phi(x)\phi(x')\rangle$ have their two constituent quantum fields evaluated at two spacetime points, x and the nearby x'. This product is finite so long as the two points are separated in spacetime. The infinities will be shown to arise when these two spacetime points are brought together.

DeWitt based his scheme on the earlier proper-time method method used by Schwinger to calculate the Feynman Green function associated with a quantized fermion field. While there are other regularization methods, point-splitting has proven to be the most robust and trustworthy method of the lot [8]. This is because the point-splitting procedure is well-developed for a general spacetime of arbitrary curvature.

Although algebraically quite complicated, point-splitting works in every case, fully isolating all infinite quantities.

Other regularization methods do exist [9]. Pauli-Villars regularization [10] requires the introduction of fictitious fields whose own divergences are chosen to exactly cancel the divergences of the physical field. The number of these fields introduced is chosen according to the number of divergences the physical fields possess, and these fields are allowed to either commute or anti-commute depending on whether they are required to add to or subtract from the divergences in the stress-energy tensor of the physical field.

Dimensional regularization involves the continuation of quantum field calculations to non-integral spacetime dimensions. Physical parameters requiring renormalization are shown to have bare values proportional to $(n-4)^{-1}$, where n is the dimension of the spacetime. This fact requires the introduction into the original physical Lagrangian terms proportional to bare coefficient of adiabatic order 4 that will serve to absorb the infinities. However, the renormalized quantities that result must have restrictions on their size in order to be consistent with observations [11, 12].

Adiabatic regularization has been used extensively in calculations of conformally flat spacetimes [13, 14]. In this method, the subtraction of infinite quantities is based on the adiabatic expansion of the modes of the quantized field. However, the subtractions necessary to renormalize the physical parameters are often too difficult to evaluate. This means that, while the infinities are isolated, they can not serve to renormalize the VEVs of interest such as $\langle T^{\mu\nu}\rangle$.

The technique of ζ-function regularization allows the effective Lagrangian to be written as a derivative of a ζ function *resembling* Riemann's ζ function on the curved space [15]. This formal technique for regularizing the effective action uses a generalized ζ function, $\zeta(\nu)$, whose argument ν must be analytically continued from regions where $\zeta(\nu)$ converges into regions where $\zeta(\nu)$ does not converge. This is necessary since $\nu = 0$, a value for which the generalized ζ function does not converge, is the value of interest in quantum field calculations.

Point-splitting was chosen for the purposes of this thesis for two main reasons. First, the major advantage that point-splitting possesses over these and any other known regularization schemes is that it is the most efficient method to use when computing actual values for the quantized fields instead of working with purely formal manipulations of the effective action of the theory. Second, by using point-splitting for the present calculations of a complex scalar field in curved space, a connection may be made with the previous work of Christensen in using the point-splitting procedure for regularization of the real scalar field in curved space [31, 30]. The results of this thesis will explicitly show that, if the charge of the complex scalar field is allowed to vanish, then the regularization counterterms presented here reduce exactly to those of Christensen.

After the point-splitting calculations have been performed and the infinities of the field equations are isolated then, in principle, they may be subtracted from the unrenormalized equations, leaving a finite remainder which contains the relevant physical information. In practice, the renormalized quantities are often too algebraically complicated to evaluate analytically in the case of general spacetimes. In the case of conformally invariant fields in conformally invariant spacetimes, the renormalized quantities been evaluated analytically [16, 17, 13, 18]. These calculations have been performed by first renormalizing the Hadamard elementary function, $G^{(1)}(x, x')$, from which the quantities $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$ are constructed. The VEV $\langle\phi^2\rangle$ is directly proportional to $G^{(1)}(x, x')$ and is renormalized when $G^{(1)}(x, x')$ is renormalized. The stress-energy tensor $\langle T_{\mu\nu}\rangle$ is constructed from $G^{(1)}(x, x')$ and its covariant derivatives. The presence of these derivatives requires a slight modification of the renormalization of $G^{(1)}(x, x')$, yet these modifications in essence amount to carrying out certain power series expansions involving the two spacetime points x and the nearby x' to a higher order than when renormalizing $G^{(1)}(x, x')$.

Point-splitting renormalization calculations have been performed for specific spacetimes for both $\langle \phi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$ for the real scalar field. Candelas [19] computed a renormalized value for $\langle \phi^2 \rangle$, or $\langle \phi^2 \rangle_{ren}$, for the massless scalar field in Boulware, Hartle-Hawking, and Unruh vacuum states in the region exterior to the horizon of a Schwarzschild black hole. Some of the components of $\langle T_{\mu\nu} \rangle_{ren}$ were also renormalized. Frolov [20, 21] generalized the work of Candelas such that $\langle \phi^2 \rangle_{ren}$ for the massless field could be calculated on the event horizon of a Reissner-Nordström black hole and near the pole of the event horizon of a charged Kerr black hole. Candelas and Howard [22] calculated $\langle \phi^2 \rangle_{ren}$ for the Hartle-Hawking vacuum in the Schwarzschild spacetime in the region exterior to the horizon. Candelas and Jensen [23] analytically continued the calculation of $\langle \phi^2 \rangle_{ren}$ across the event horizon of a Schwarzschild black hole, giving an expression which is valid for a range Schwarzschild radial coordinate r. They numerically evaluated $\langle \phi^2 \rangle_{ren}$ for the range of r given by $0.5M < r < 2M$.

Following the work of the investigators mentioned above, Anderson [24] has decribed a method whereby $\langle \phi^2 \rangle_{ren}$ may be numerically computed for free scalar fields in a general static spherically symmetric spacetime. This method assumes the spacetime is asymptotically flat for the purpose of defining initial conditions for the modes sums to be computed for the fields. The fields may be massive or massless, at zero or non-zero temperature, and the spacetime may have arbitrary curvature coupling ξ. This scheme is fully renormalized, and the computations may be carried to arbitrary numerical precision.

Renormalization calculations for $\langle T_{\mu\nu} \rangle$ in specific spacetimes have also been performed. These calculations include the work by Howard and Candelas [25] wherein

$\langle T_\mu{}^\nu \rangle_{ren}$ was calculated for a massless, conformally invariant scalar field in a Hartle-Hawking vacuum state in a Schwarzschild spacetime. Their work allowed numerical computations of $\langle T_\mu{}^\nu \rangle_{ren}$ in the region exterior to the event horizon and described methods for increasing the efficiency of the computations. Frolov and Zel'nikov [20, 26] calculated an approximate expression for $\langle T_\mu{}^\nu \rangle_{ren}$ for massive scalar, spinor, and vector fields in Ricci-flat spacetimes. This was done by calculating an approximation for the effective action using the generalization of the DeWitt-Schwinger technique developed by Barvinsky and Vilkovisky [27]. Brown, Ottewill, and Page derived an analytic approximation for $\langle T^{\mu\nu} \rangle$ in conformal spacetimes using the one-loop effective action for massless, conformally invariant scalar, spinor, and vector fields on static Einstein spaces [28].

Building on the studies of $\langle T_{\mu\nu} \rangle$ above, Anderson, Hiscock, and Samuel [29] have recently decribed a method whereby $\langle T_{\mu\nu} \rangle_{ren}$ may be calculated to arbitrary numerical precision for a general static spherically symmetric spacetime. The scalar field can be massless or massive, in a zero temperature or non-zero temperature state, and the coupling ξ to the scalar curvature can be arbitrary. While calculationally quite intensive, this method is very powerful since it provides a way of using a fully renormalized stress-energy tensor in quantum field theory calculations in these static spherically symmetric spacetimes.

These renormalization calculations rely on the regularization counterterms previously derived by Christensen [30, 31]. They use these counterterms to subtract infinities from the unrenormalized expressions for $\langle \phi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$, leaving behind finite remainders which correctly describe the physics. These subtractions have been performed for the specific spacetimes mentioned above and not for completely general spacetimes. Also, all of these studies have been performed using uncharged fields. The first of the two major results of this thesis are the point-splitting regularization counterterms that will be required for the study of quantized charged scalar fields interacting with the electromagnetic fields that may be present in curved spacetimes.

The quantities $\langle\phi^2\rangle$, $\langle j^\mu\rangle$, and $\langle T_{\mu\nu}\rangle$ are constructed from the Hadamard elementary function, $G^{(1)}(x,x')$, and its derivatives. The new feature present in this thesis is that these derivatives are now *gauge covariant* derivatives. As explained further at the end of this chapter, the coupling of the charged scalar field to the electromagnetic field of the curved spacetime will introduce new physics into the structure of $G^{(1)}$ and its derivatives. This new physics will not only modify the results of Christensen for a real scalar field, but will also be responsible for the generation of the current, $\langle j^\mu\rangle$, associated with this charged scalar field.

The Hadamard elementary function is constructed from $\langle\phi(x)\phi(x')\rangle$ and is finite so long as the points x and x' in the argument of $G^{(1)}(x,x')$ are not coincident. The point-splitting scheme is actually an asymptotic expansion of the biscalar $G^{(1)}(x,x')$ in powers of the mass m of the quantized field, an expansion known as a "DeWitt-Schwinger (DS) expansion." The expansion of $G^{(1)}(x,x')$ goes as,

$$G^{(1)}(x,x') = A_{+2}(x,x')m^2 + A_0(x,x')m^0 + A_{-2}(x,x')m^{-2} + \ldots, \tag{4}$$

where the A_n are coefficients constructed from curvature and electromagnetic tensors, and m is the mass of the quantized field. Eq.(4) has been shown to contain infinities that arise when the points x and x' are brought together [30, 31, 8, 4]. These infinities have been shown to be contained in the first two terms of the right hand side of Eq.(4), or, in the terms proportional to nonnegative powers of m. These infinities may then be, in principle, subtracted from the unrenormalized expressions involving the charged quantized field, with the finite result containing real physical information. Unfortunately, these subtractions are difficult at best to evaluate analytically, as mentioned previously.

Fortunately, point-splitting provides a way out of this dilemma of difficult subtractions, at least in the case of massive quantized fields. The magnitudes of the A_n depend directly on the strength of the curvature and electromagnetic fields of the spacetime since they are constructed from curvature and electromagnetic invariants. As the inverse power of m becomes large, the A_n contain higher and higher derivatives of both the spacetime metric and electromagnetic field functions. Yet, since Eq.(4) is an asymptotic series for $G^{(1)}(x, x')$, for some large value of n, the magnitude of the A_n becomes large when compared with the magnitude of the m^{-n}. Beyond this value for n, the asymptotic expansion breaks down, and the series represention of $G^{(1)}(x, x')$ must be abandoned. Subtracting the infinite terms from the unrenormalized expression for $G^{(1)}(x, x')$ leaves a series finite in length containing finite terms. The first few, or even the first one, of these terms may be kept as a "DeWitt-Schwinger approximation" for the actual value of $G^{(1)}(x, x')$. The accuracy of this approximation will depend on how large the mass m of the field is chosen when compared to the magnitude of the spacetime curvature invariants. Hereafter, the phrases "DS expansion" or "DS approximation" will refer to the large mass expansion of any quantity.

The point-splitting procedure is capable of yielding a DS approximation for both the vacuum polarization $\langle \phi^2 \rangle$ (which is directly proportional to $G^{(1)}$) and the vacuum expectation value of the stress energy tensor $\langle T_{\mu\nu} \rangle$ (which is assembled from $G^{(1)}$ and its derivatives). Unfortunately, one major limitation of the point-splitting procedure is exposed whenever real particle production occurs. As will be shown in greater detail later, point-splitting is incapable of yielding either the imaginary part of the expansion of $G^{(1)}$ [2], or odd powers of m in Eq.(4). These two limitations mean that point-splitting is incapable of yielding a DS approximation for the current of the quantized charged scalar field $\langle j^\mu \rangle$.

This thesis calculates the regularization counterterms for the three VEVs $\langle \phi^2 \rangle$, $\langle j^\mu \rangle$, and $\langle T_{\mu\nu} \rangle$. All of these will be derived for the case of a complex scalar field interacting with a classical background electromagnetic in a spacetime of arbitrary curvature. A scalar field is chosen in the present work in order to make a connection with the regularization work previously done by Christensen [30, 31] and for simplicity. The added complication of spin need not be considered since essentially no new physics is expected to arise when spin effects are considered.

Chapter 2 contains a discussion of the degree of divergence in QED will be presented. The relationship between the momentum space representations of VEVs and the geodesic separation of spacetime points x and x' will be discussed. This discussion will show $G^{(1)}(x, x')$ (and equivalently $\langle\phi^2\rangle$) to have an expected quadratic divergence in the spacetime separation distance $|x - x'|$ as the two points x and x' are brought together. The quantities $\langle j^\mu\rangle$ and $\langle T_{\mu\nu}\rangle$ will be shown to potentially contain cubic and quartic divergences, respectively, in the separation distance $|x - x'|$.

Chapter 3 begins with a discussion of the local structure of spacetime upon which point-splitting regularization depends crucially. A Hamilton-Jacobi analysis of geodesic motion in the presence of gravitational and electromagnetic fields will lead to the biscalar of the geodetic interval, $\sigma(x, x')$. The biscalar $\sigma(x, x')$ will be shown to be equal to one-half the square of the geodesic distance between the points x and x', and will be shown to carry information about the structure of spacetime along the geodesic between the two points. The Van Vleck-Morette determinant, symbolized D, arises in a discussion of caustic surfaces in curved spacetimes, and will be shown to place constraints on how the points x and x' are separated. Differential equations that define $\sigma(x, x')$ and D for the point-splitting procedure will be derived. The bivector of parallel transport, $g^\mu{}_{\nu'}$ will be shown as the means for conveying information from the nearby point x' back to the stationary point x. A differential equation involving $g^\mu{}_{\nu'}$ that is required for the point-splitting procedure will be derived.

Chapter 4 presents a derivation of the geodesic point-splitting procedure using the quantities derived in Chapter 3. This chapter will start with Schwinger's original proper time method of calculating the Feynman Green function, and end with the presentation of the DS expansion of $G^{(1)}$. This will illustrate the need for $G^{(1)}$ and its derivatives to be rewritten in terms of local geometric quantities. The point-splitting "recursion relations" derived by Christensen [30] are a set of differential equations that are used to construct $G^{(1)}$ and its derivatives. In this chapter, a derivation of the recursion relations of the familiar form presented by Christensen [30] will follow. The crucial new feature in this thesis is the presence of the electromagnetic gauge field. The original recursion relations will be cast into the gauge-invariant form necessary for calculating effects due to *both* the gauge and gravitational fields. The gravitational field emerged in Christensen's work while solving the differential equations, or recursion relations, through the commutator of the covariant derivative

$$[\nabla_\alpha, \nabla_\beta]V^\mu = R^\mu{}_{\nu\alpha\beta}V^\nu, \tag{5}$$

where ∇_α is the covariant derivative, V^μ is a purely geometric vector, and $R^\mu{}_{\nu\alpha\beta}$ is the Riemann tensor. In this thesis, the addition of the electromagnetic field requires the use of the gauge covariant derivative whose commutator is given by

$$[\nabla_\alpha - ieA_\alpha, \nabla_\beta - ieA_\beta]W = ieF_{\beta\alpha}W, \tag{6}$$

where W is a scalar that explicitly depends on the gauge field, A_α is the vector potential for the gauge field, and $F_{\beta\alpha}$ is the Maxwell tensor. The effects of this gauge field in quantum field theory calculations in curved spacetime is the heart of the present work.

Chapter 5 is devoted to solving the recursion relations in order to construct the expansion of Eq.(4). The recursion relations can not be solved analytically and so they must be iteratively solved order by order. This iterative process is detailed and a listing of the geometric quantities that form the A_n of Eq.(4) will be given.

In Chapter 6, the DS point-splitting expansions for $G^{(1)}$, $\langle j^\mu \rangle$, and $\langle T_{\mu\nu} \rangle$ will be given, with the divergent and finite parts of each clearly identified. These point-splitting expansions will be asymptotic series as in Eq.(4). The expansions for all three VEVs will explicitly show that the divergent parts are proportional to non-negative powers of m. The expansion for $G^{(1)}$ will include terms of order m^{-2} and m^{-4}, and these terms may be used as a DS approximation to the true value for $G^{(1)}$ as discussed above. This chapter will discuss why the coefficients B_n in the expansion for $\langle j^\mu \rangle$,

$$\langle j^\mu \rangle = B_{+2}(x,x')m^2 + B_0(x,x')m^0 + B_{-2}(x,x')m^{-2} + \ldots, \tag{7}$$

vanish for all negative n. The DS approximations for $\langle \phi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$ will be presented in their most general form as combinations of the electromagnetic and curvature tensors. Previous workers derived expressions for $\langle \phi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$ for uncharged fields that contained only geometric terms. Here, the contributions of the electromagnetic field to these expressions will be shown.

Chapter 7 will discuss the possible future use of this work in studying the evolution of charged black hole interiors. The regularization counterterms presented here for $\langle j^\mu \rangle$ and $\langle T_{\mu\nu} \rangle$ are a first step towards studies of both gravitational and electromagnetic backreation effects. Also, the use of the general DeWitt-Schwinger approximation presented here for $\langle T_{\mu\nu} \rangle$ will be discussed.

The appendices following the Bibliography contain expressions which are too large to list in the body of this thesis. The equations in these appendices are quite long. So long, in fact, that there was no practical way to apply every simplification rule in order to write them in their most compact form. Rewriting or cancelling a few terms will make no difference in the physics contained in the longer expressions. These appendices are intended as an archive for the expressions derived in the DeWitt-Schwinger point-splitting procedure. Appendices A–E contain the coincidence limits of derivatives of the biscalars $\sigma(x,x')$, $\Delta^{1/2}(x,x')$, $a_0(x,x')$, $a_1(x,x')$, and $a_2(x,x')$. Appendix F contains the purely geometric regularization counterterms first derived by Christensen [30] that contribute to $\langle T_{\mu\nu}\rangle_{finite}$. In the interest of space, Chapter 6 contains only those terms in the DS expansion of $\langle T_{\mu\nu}\rangle$ which depend on the electromagnetic field. Appendix G contains the purely geometric terms of order m^{-2} term of the DS approximation of $\langle T_{\mu\nu}\rangle$.

CHAPTER 2

Degree of Divergence in Scalar QED

As mentioned in Chapter 1, the biscalar $\langle\phi(x)\phi(x')\rangle$ has its two constituent quantum fields evaluated at two spacetime points, x and the nearby x'. This is finite so long as the two points are separated in spacetime, but diverges as the two points are brought together. The so-called *degree of divergence* indicates whether the quantities $\langle\phi^2\rangle$, $\langle j^\mu\rangle$, and $\langle T_{\mu\nu}\rangle$ diverge logarithmically, linearly, etc., when the points are brought together. Quantities that have a logarithmic, linear, etc., degree of divergence contain terms proportional to $\ln(|x-x'|)$, $|x-x'|^{-1}$, etc. The DeWitt-Schwinger point-splitting procedure isolates the infinities which appear in $\langle\phi(x)\phi(x')\rangle$ as $x' \to x$ and in the VEVs of $\langle\phi^2\rangle$, $\langle j^\mu\rangle$, and $\langle T_{\mu\nu}\rangle$. This chapter discusses the origin and structure of these infinities using the familiar tool of the Fourier expansion of quantum field operators.

Relativistic quantum field theory calculations of VEVs of operators may be performed by Fourier transforming to wavenumber space. It is known that infinities arise as the integration limits over the wavenumbers involved are extended to infinity. As an example of divergences appearing due to divergent integration over wavenumbers, consider the Wightman function,

$$G^+(x,x') \equiv \langle 0|\underline{\phi}(x)\underline{\phi}(x')|0\rangle, \tag{8}$$

in flat space. The Fourier transform of the operator $\underline{\phi}(x)$ is given by,

$$\underline{\phi}(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[\underline{a}(k)e^{-ik_\alpha x^\alpha} + \underline{b}^\dagger(k)e^{ik_\alpha x^\alpha}\right], \tag{9}$$

where k^α is the wavenumber of the basis functions $e^{\pm ik_\alpha x^\alpha}$, $\omega_{\vec{k}} \equiv k_0 = \sqrt{\vec{k}\cdot\vec{k}+m^2}$, and the limits of integration extend from zero to infinity in each wavenumber integral. The action of the particle annihilation and creation operators on the vacuum state $|0\rangle$ is given by

$$\underline{a}(\vec{k})|0\rangle = 0 \qquad , \qquad \underline{a}^\dagger(\vec{k})|0\rangle = |1(\vec{k})\rangle_{part}, \tag{10}$$

where $|1(\vec{k})\rangle_{part}$ is a state with one particle of wavenumber $\vec{k}$. The action of the antiparticle annihilation and creation operators on the vacuum state $|0\rangle$ is given by

$$\underline{b}(\vec{k})|0\rangle = 0 \qquad , \qquad \underline{b}^\dagger(\vec{k})|0\rangle = |1(\vec{k})\rangle_{anti}\,. \tag{11}$$

where $|1(\vec{k})\rangle_{anti}$ is a state with one antiparticle of wavenumber $\vec{k}$. Substituting Eq.(9) into Eq.(8), and using Eqs.(10)–(11) along with the commutation relations,

$$[\underline{a}(\vec{k}), \underline{a}^\dagger(\vec{k}')] = [\underline{b}(\vec{k}), \underline{b}^\dagger(\vec{k}')] = \delta^3(\vec{k}-\vec{k}'), \tag{12}$$

gives

$$\begin{aligned} G^+(x,x') &= \int \frac{d^3k\ d^3k'}{(2\pi)^3\sqrt{2\omega_{\vec{k}}2\omega_{\vec{k}'}}} e^{-ik_\alpha x^\alpha + ik'_\alpha x'^\alpha} \langle 0|\underline{a}(\vec{k})\underline{a}^\dagger(\vec{k}')|0\rangle \\ &= \int \frac{d^3k\ d^3k'}{(2\pi)^3\sqrt{2\omega_{\vec{k}}2\omega_{\vec{k}'}}} e^{-ik_\alpha x^\alpha + ik'_\alpha x'^\alpha} \left[\langle 0|\underline{a}^\dagger(\vec{k}')\underline{a}(\vec{k})|0\rangle + \langle 0|\delta^3(\vec{k}-\vec{k}')|0\rangle\right] \\ &= \int \frac{d^3k\ d^3k'}{(2\pi)^3\sqrt{2\omega_{\vec{k}}2\omega_{\vec{k}'}}} e^{-ik_\alpha x^\alpha + ik'_\alpha x'^\alpha} \delta^3(k-k')\langle 0|0\rangle \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} e^{-ik_\alpha(x^\alpha - x^{\alpha\prime})}. \end{aligned} \tag{13}$$

The last line is finite so long as the points x and x' are not coincident. As the points x and x' are brought together, $G^+(x,x')$ diverges quadratically since $G^+(x,x') \sim \int k\ dk \sim k^2$ as $k \to \infty$. This is where a direct correlation exists between point-splitting and wavenumber space integrations. States with high wavenumbers, or high energy values, have short wavelengths. These short wavelength states are probing the spacetime at ever-decreasing length scales. As the wavenumbers diverge, the spacetime is probed to ever-decreasing length scales, and the vacuum self-energy diverges. Instead of investigating the behavior VEVs using divergent wavenumber integrations, point-splitting studies the behavior of VEVs explicitly in terms of small length scales.

It is straightforward to show that one and two derivatives of $G^+(x,x')$ give the cubic and quartic divergences

$$\begin{aligned} \frac{\partial}{\partial x} G^+(x,x') &\sim \int k^2\, dk \sim k^3, \quad k \to \infty, \\ \frac{\partial^2}{\partial x^2} G^+(x,x') &\sim \int k^3\, dk \sim k^4, \quad k \to \infty\,, \end{aligned} \tag{14}$$

where spacetime indices have been suppressed for simplicity. It should be noted that $G^+(x,x')$ is similar to the Hadamard elementary function $G^{(1)}(x,x')$ in that they are both constructed from the VEV of products of the fields $\phi(x)$ and $\phi(x')$.

While the above discussion of the divergences of the Wightman function is in flat space, it gives an intuitive picture of the physical origin of the divergences in quantum field theory. With $\langle\phi^2\rangle$, $\langle j^\mu\rangle$, and $\langle T_{\mu\nu}\rangle$ being constructed from $G^{(1)}(x,x')$ and its derivatives, it is possible to consider an analogy between the divergences of $G^+(x,x')$ and its derivatives and $G^{(1)}(x,x')$ and its derivatives. For example, the stress-energy tensor $\langle T_{\mu\nu}\rangle$ will be constructed from up to two derivatives of $G^{(1)}(x,x')$. With the result of Eq.(14), the stress-energy tensor would potentially contain up to quartic divergences. This will be shown to be true. The current $\langle j^\mu\rangle$ will be constructed from up to one derivatives of $G^{(1)}(x,x')$ and would potentially contain up to cubic divergences. However, $\langle j^\mu\rangle$ will be shown to actually contain only a linear divergence.

While the above discussion provides a general framework within which to view the origin of the divergences in scalar QED, there exists a method for determining the specific degree of divergence for $\langle\phi^2\rangle$, $\langle j^\mu\rangle$, and $\langle T_{\mu\nu}\rangle$ individually. To calculate these degrees of divergence, it is necessary to consider the action functional for a complex scalar field coupled to the electromagnetic field in an arbitrary curved background [1];

$$S[\phi, A_\mu, g_{\mu\nu}] = -\frac{1}{2}\int \mathcal{L}\ dV \tag{15}$$
$$= -\frac{1}{2}\int \mathcal{L}\ (-g)^{1/2}\ d^4x \tag{16}$$
$$= -\frac{1}{2}\int (-g)^{1/2}\left[(D_\mu\phi)(D^\mu\phi)^* + (m^2+\xi R)\phi\phi^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] d^4x, \tag{17}$$

where $\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$ is the complex scalar field, g is the determinant of the metric $g_{\mu\nu}$, $D_\mu \equiv (\nabla_\mu - ieA_\mu)$ is the gauge covariant derivative, A^μ is the classical electromagnetic vector potential, e is the coupling between the complex scalar and the electromagnetic fields, m is the mass of the complex scalar field, ξ is the scalar curvature coupling, and R is the scalar curvature. The Lagrangian density $\mathcal{L}$ may be rewritten in terms of interaction Lagrangians

$$\begin{aligned}\mathcal{L} &= \left[(D_\mu\phi)(D^\mu\phi)^* + (m^2+\xi R)\phi\phi^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]\\ &= \left[(\nabla_\mu - ieA_\mu)\phi(\nabla^\mu + ieA^\mu)\phi^* + (m^2+\xi R)\phi\phi^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]\\ &= \frac{1}{2}\{\partial_\mu\phi, \partial^\mu\phi^*\} + \frac{1}{2}(m^2+\xi R)\{\phi,\phi^*\} +\\ &\quad \frac{ie}{2}\left[\{D^\mu\phi,\phi^*\} - \{D^\mu\phi,\phi^*\}^*\right]A_\mu - \frac{e^2}{2}A_\mu A^\mu\{\phi,\phi^*\} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\\ &= \frac{1}{2}\{\partial_\mu\phi, \partial^\mu\phi^*\} + \frac{1}{2}(m^2+\xi R)\{\phi,\phi^*\} + j^\mu A_\mu - \frac{e^2}{2}A_\mu A^\mu\{\phi,\phi^*\} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\end{aligned} \tag{18}$$

where $j^\mu \equiv \frac{ie}{2}\left[\{D^\mu\phi,\phi^*\} - \{D^\mu\phi,\phi^*\}^*\right]$ is the scalar field current, and the anti-commutators have arisen by symmetrizing with respect to the fields. Eq.(18) may be rewritten in terms of three interaction Lagrangian densities and the classical electromagnetic Lagrangian density;

$$\mathcal{L} \equiv \mathcal{L}_{I,1} + \mathcal{L}_{I,2} + \mathcal{L}_{I,3} + \mathcal{L}_{EM}, \tag{19}$$

where

$$\mathcal{L}_{I,1} \equiv \frac{1}{2}\{\partial_\mu\phi, \partial^\mu\phi^*\} + \frac{1}{2}(m^2+\xi R)\{\phi,\phi^*\} \tag{20}$$

is the interaction Lagrangian density for the scalar field in curved space studied by Christensen [30],

$$\mathcal{L}_{I,2} \equiv j^{\mu} A_{\mu} = \frac{ie}{2} \left[\{ D^{\mu}\phi, \phi^* \} - \{ D^{\mu}\phi, \phi^* \}^* \right] A_{\mu} \tag{21}$$

is the interaction Lagrangian density for the scalar field current j^{μ} and the classical background electromagnetic field A_{μ},

$$\mathcal{L}_{I,3} \equiv -\frac{e^2}{2} A_{\mu} A^{\mu} \{ \phi, \phi^* \} \tag{22}$$

is the interaction Lagrangian density for the scalar field and classical background electromagnetic field, and

$$\mathcal{L}_{EM} \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{23}$$

is the Lagrangian density for the classical background electromagnetic field. Note that Eq.(21) is an interaction first order in the coupling constant e, while Eq.(22) is second order in e.

The three VEVs $\langle \phi^2 \rangle$, $\langle j^{\mu} \rangle$, and $\langle T_{\mu\nu} \rangle$ arise from functional variation of Eq.(17) for the total action with respect to the the fields ϕ^*, A^{μ}, and $g_{\mu\nu}$, respectively. The degree of divergence of each VEV may be predicted from analysis of the Feynman diagrams for the interaction Lagrangian density pertaining to each. For example, the total interaction Lagrangian density for the scalar field and the electromagnetic field is

$$\mathcal{L}_{\phi, A^{\mu}} \equiv \mathcal{L}_{I,2} + \mathcal{L}_{I,3}. \tag{24}$$

The first term gives rise to the three-point Feynman graph of Figure 1, while the second term is the four-point graph.

In the 3-point diagram, there are two external scalar (boson) lines, one external photon line, and one vertex. For scalar QED, the degree of divergence D in four dimensions is given by [5]

$$D = 4 - P_e - Q_e, \tag{25}$$

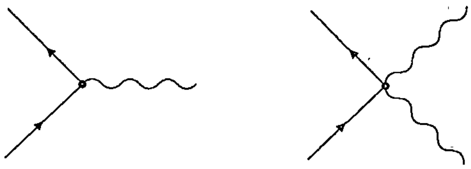

Figure 1: Feynman diagrams for scalar field-photon interaction

where P_e and Q_e are the number of external scalar and photon lines, respectively, in the Feynman diagrams for the interaction. The values $D = 0, 1, 2, \ldots$ imply logarithmic, linear, quadratic, etc., divergences, while $D < 0$ implies the interaction is not divergent. DeWitt has pointed out for the generalized Yang-Mills field in the presence of the gravitational field that the simplest possible diagram will be the most divergent [2]. With the total interaction given by Eq.(24), the three-point graph will be the most divergent. Thus, the expected degree of divergence is $D = 1$, a linear divergence. The direct correspondence between the degree of divergence in the wavenumber Fourier transforms of the operators $\underline{\phi}(x)$ and the dependence upon the splitting of points in spacetime indicates that, when all calculations of the VEV of the current $\langle j^\mu \rangle$ are complete, the infinities should be proportional to $|x - x'|^{-1}$.

The vacuum polarization $\langle \phi^2 \rangle$ for a real scalar field in curved space has well-known quadratic and logarithmic divergences as the split points x and x' are brought together [30]. The degree of this divergence should not change in the case of a complex scalar field since the complex field is constructed from the complex sum of two real scalar fields. Thus, $\langle \phi\phi^* \rangle \equiv \langle \phi^2 \rangle$ should also have a quadratic divergence. The interaction Lagrangian density of Eq.(20) gives rise to a two-point graph with a degree of divergence $D = 2$. This gives an expected quadratic divergence in full agreement with previous point-splitting results. With no gauge field contribution to this interaction, all of the divergences should be exactly the same as when no gauge field is present in the spacetime.

The VEV of the stress-energy tensor in curved space has well-known quartic, quadratic, linear, and logarithmic divergences in the case of a real scalar field with no gauge field present [30]. According to DeWitt [2], the addition of the gauge field should contribute additional divergences within the existing logarithmic ones. These may be considered as arising from the Lagrangian density of Eq.(22) which has the logarithmic degree of divergence $D = 0$.

Point-splitting will be shown capable of isolating all of the divergences of $\langle\phi^2\rangle$, $\langle j^\mu\rangle$, and $\langle T_{\mu\nu}\rangle$ predicted above.

CHAPTER 3

The Local Structure of Spacetime

The DeWitt-Schwinger geodesic point-splitting procedure depends crucially upon geometric quantities describing the local structure of spacetime. At the heart of point-splitting lies the idea of N-tensors [32, 33, 2, 31]. An N-tensor is simply a tensor that may be evaluated at more than one point in spacetime. For example, $\langle\phi^2\rangle \equiv \langle\phi(x)\phi(x')\rangle$ is a biscalar constructed from the product of two scalars, with each individual scalar being evaluated at its own spacetime point. Another example would be the bivector $\langle\phi(x)\phi(x')\rangle^{;\mu'} \equiv \langle\phi(x)\phi^{;\mu'}(x')\rangle$, where the derivative with respect to x' does not affect the scalar $\phi(x)$, a scalar which is associated with the nearby point x. (Note how the prime on the derivative index indicates the spacetime point on which the derivative acts.) With derivatives being taken with respect to a particular spacetime point, all derivatives with respect to primed indices commute with all derivatives with respect to umprimed indices, and vice versa. In the present work, we will only be concerned with bitensors. More general quantities such as $\langle\phi(x)\phi(x')\phi(x'')\rangle$ will not arise.

A concept of importance associated with N-tensors is that of tensor weight [1]. Consider, for example, the familiar transformation involving volume elements in flat space:

$$dx^0dx^1dx^2dx^3 \equiv d^4x = \left|\frac{\partial x}{\partial x'}\right| d^4x' \tag{26}$$

$$= (-g)^{1/2} d^4x', \tag{27}$$

where $|\partial x/\partial x'|$ is the Jacobian of the transformation from the coordinates $\{x^{\alpha'}\}$ to the coordinates $\{x^\alpha\}$. The number of factors of $|\partial x/\partial x'|$ determines the weight of the density. Here, d^4x' with a tensor density of weight +1 transforms like an ordinary tensor such that $(-g)^{1/2}d^4x'$ is an invariant volume element. Another example of tensor weight involves the basis kets $|x\rangle$ such as those of Eqs.(10)–(11). These transform as densities of weight $\frac{1}{2}$ under coordinate transformations in flt space [4]:

$$|x'\rangle = (-g)^{1/4}|x\rangle. \tag{28}$$

The most important biscalar for the point-splitting procedure is $\sigma(x, x')$, the biscalar of the geodesic interval [2], which will be defined as one-half the square of the geodesic distance between the points x and x'. The biscalar $\sigma(x, x')$ contains information about the local structure of spacetime and arises from a Hamilton-Jacobi analysis of motion along geodesics.

The Lagrangian describing the motion of a charged particle in combined external gravitational and electromagnetic fields is given by [34],

$$L = \frac{1}{2}g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + eA_\mu\frac{dx^\mu}{d\lambda}, \tag{29}$$

where A^μ is the electromagnetic gauge field and e is the charge of the particle. The invariant interval is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -d\tau^2. \tag{30}$$

Following Carter [34], we are free to choose the normalization constant, or mass m of the particle moving along the geodesic, such that

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = -\left(\frac{d\tau}{d\lambda}\right)^2 = -m^2 = \left(\frac{ds}{d\lambda}\right)^2 = constant. \tag{31}$$

Given this normalization condition, the action for the motion of the particle becomes, for the parameter λ along any curve $x^\mu(\lambda)$,

$$S = \int_{x'(\lambda')}^{x(\lambda)} L\ d\lambda \tag{32}$$

$$= -\frac{1}{2}m^2 \int_{x'(\lambda')}^{x(\lambda)} d\lambda + e \int_{x'(\lambda')}^{x(\lambda)} A_\mu \frac{dx^\mu}{d\lambda} d\lambda \tag{33}$$

$$= \frac{1}{2}\left(\frac{ds}{d\lambda}\right)^2 (\lambda - \lambda') + e \int_{x'(\lambda')}^{x(\lambda)} A_\mu \frac{dx^\mu}{d\lambda} d\lambda \tag{34}$$

$$\equiv \frac{\sigma(x,x')}{(\lambda - \lambda')} + e \int_{x'}^{x} A_\mu dx^\mu. \tag{35}$$

The transition from Eq.(34) to Eq.(35) was made by requiring

$$\sigma(x,x') \equiv \frac{1}{2}\left(\frac{ds}{d\lambda}\right)^2 (\lambda - \lambda')^2 \tag{36}$$

showing $\sigma(x,x')$ must be defined as one-half of the square of the geodesic distance between x and x' in the limit that $(\lambda - \lambda')$ is an infinitesimal quantity.

It is not generally possible to write the right hand side of Eq.(36) in terms of metric functions and their derivatives. Doing so would require directly evaluating

$$ds = \int_{x'(\lambda')}^{x(\lambda)} \left(\frac{ds}{d\lambda}\right) d\lambda = \int_{\lambda'}^{\lambda} \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda = constant(\lambda - \lambda') \tag{37}$$

and expressing the results in terms of metric functions and their derivatives. Then, one would have to solve the geodesic equations in order to also express $(\lambda - \lambda')$ in terms of metric functions and their derivatives. Point-splitting will instead rely upon a differential equation for $\sigma(x,x')$ that may be solved iteratively. This differential equation may be obtained by moving from the Lagrangian of Eq.(29) to a Hamiltonian study of geodesic motion. In the Hamiltonian formalism, the conjugate momenta and Hamiltonian are given by,

$$p_\mu = \frac{\partial L}{\partial(\frac{dx^\mu}{d\lambda})} = g_{\mu\nu}\frac{dx^\nu}{d\lambda} + eA_\mu \equiv \pi_\mu + eA_\mu, \tag{38}$$

and,

$$H = p_\mu \frac{dx^\mu}{d\lambda} - L = \frac{1}{2}g_{\mu\nu}(p^\mu - eA^\mu)(p^\nu - eA^\nu). \tag{39}$$

The Hamilton-Jacobi equation is given by [35, 34],

$$\frac{\partial S}{\partial \lambda} = -H(x^\alpha, \frac{\partial S}{\partial x^\alpha}) = \frac{1}{2}g^{\mu\nu}(\frac{\partial S}{\partial x^\mu} - eA_\mu)(\frac{\partial S}{\partial x^\nu} - eA_\nu). \tag{40}$$

Using

$$\frac{\partial S}{\partial \lambda} = -\frac{\sigma(x,x')}{(\lambda-\lambda')^2}, \tag{41}$$

and

$$\frac{\partial S}{\partial x^\alpha} = -\frac{\sigma_{,\alpha}(x,x')}{(\lambda-\lambda')} + eA_\alpha, \tag{42}$$

the Hamilton-Jacobi equation becomes

$$\begin{aligned}\frac{\sigma(x,x')}{(\lambda-\lambda')^2} &= \frac{1}{2}\left[\frac{\sigma_{,\mu}(x,x')}{(\lambda-\lambda')} + eA_\mu - eA_\mu\right]\left[\frac{\sigma^{,\mu}(x,x')}{(\lambda-\lambda')} + eA^\mu - eA^\mu\right] \\ &= \frac{1}{2}\frac{\sigma_{,\mu}(x,x')}{(\lambda-\lambda')}\frac{\sigma^{,\mu}(x,x')}{(\lambda-\lambda')},\end{aligned} \tag{43}$$

where $+eA^\mu - eA^\mu$ has been included to emphasize the explicit cancellation of the gauge field terms. Multiplying both sides by $(\lambda-\lambda')^2$, and defining $\sigma \equiv \sigma(x,x')$ gives,

$$\sigma = \frac{1}{2}\sigma^{,\mu}\sigma_{,\mu} = \frac{1}{2}\sigma^{;\mu}\sigma_{;\mu} \equiv \frac{1}{2}\sigma^\mu\sigma_\mu, \tag{44}$$

where $\sigma^\mu \equiv \sigma^{;\mu}$. This differential equation will be used extensively in the point-splitting procedure.

The result of Eq.(44) is very important. The insensitivity of the differential equation (44) to the presence of the gauge field demonstrates that the biscalar $\sigma(x,x')$ is a purely geometric quantity. The first covariant derivative of σ is a simple partial derivative, requiring no gauge, A^μ, or Christoffel, $\Gamma^\rho{}_{\mu\nu}$, connections. Second and higher covariant derivatives will require Christoffel connections but no gauge connections. Like all other purely geometric quantities such as curvature tensors, etc., the only connections required for the covariant derivatives of $\sigma(x,x')$ are the Christoffel connections. Point-splitting regularization in the presence of a gauge field requires careful identification of those quantities whose covariant derivatives will require not only the Christoffel connections but also the gauge connection A^μ.

The components on the right hand side of Eq.(44) may be given a physical meaning. Consider the geodesic connecting the points x and x' of Figure 2.

The vector σ^μ is a tangent to the curve at x, being the derivative of a scalar with

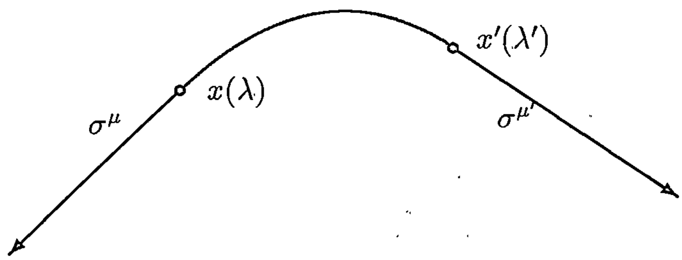

Figure 2: Geodesic connecting points x and x' with tangent vector at each endpoint

respect to x^μ. It has length equal to the arc length of the geodesic between x and x', and points in the direction $x' \to x$. An equation similar to Eq.(44) may be derived for $\sigma^{\mu'}$, the vector tangent at the other end of the geodesic and pointing in the direction $x \to x'$:

$$\sigma(x, x') = \frac{1}{2}\sigma^{\mu'}\sigma_{\mu'} = \sigma(x', x). \tag{45}$$

$\sigma^{\mu'}$ is a vector tangent to the curve at x', has length equal to the arc length of the geodesic between x and x', and points in the direction $x \to x'$.

The biscalar of the geodesic interval, $\sigma(x, x')$, describes the spacetime geometry along a geodesic between the separated points x and x'. Point-splitting requires these points to be separated by a *unique* geodesic so that $\sigma(x, x')$ is a single-valued function of the points x and x'. In a general Riemannian manifold, the geodesics emanating from one spacetime point begin to overlap and cross one another at some distance from that point [2]. Consider the points x, x', and x'' and the two geodesics of Figure 3.

While there are an infinite number of geodesics emanating from the stationary point x, this diagram is only concerned with the two shown. These two are shown because the point x'' is the nearest point where the geodesics emanating from x start to cross themselves. This diagram shows the points x and x'' are not sufficiently

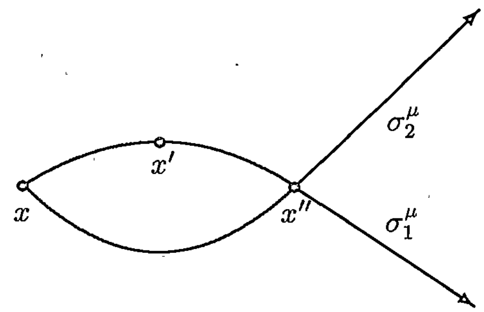

Figure 3: Two of the infinite number of geodesics emanating from x that cross at x''

close in this spacetime for $\sigma(x, x'')$ to be a single-valued function. The point x' is closer to x, close enough such that no other geodesic coming from x connects to x', and thus $\sigma(x, x')$ is a single-valued function as shown in Figure 3. The surface at which geodesics start to overlap is known as a *caustic surface.* It should be noted here that there are no caustic surfaces in flat spacetime. (An example of caustic surfaces occurring in two-dimensional space is the two-sphere. The infinite number of geodesics (great circles) emanating from the "north" pole cross themselves at the "south" pole of the two-sphere.) If the points x and x' are "sufficiently close," caustic surfaces may be avoided. A way of quantifying the criterion of "sufficiently close" is provided by calculating the Van Vleck-Morette determinant [36, 37].

The Van Vleck-Morette determinant arises when considering how variation of the tangent vectors at x' relates to the single-valuedness of $\sigma(x, x")$. A geodesic may be characterized either by its two endpoints $\{x^\mu, x^{\nu'}\}$, or by one of its endpoints along with the tangent vector at that endpoint, $\{x^{\mu'}, \sigma_{\nu'}\}$. This last characterization may not seem adequate to specify the geodesic at first glance, but it actually does. Consider the geodesic of Figure 2. There are an infinite number of geodesics emanating from the point x'. However, only that geodesic which has $\sigma^{\mu'}(x, x')$ as its tangent vector is the one that connects the point x' to the point x. Any other of the infinite number of tangent vectors at x' would specify a geodesic connecting x' to a point other than x.

The transformation from characterizing a geodesic by $\{x^\mu, x^{\nu'}\}$ to $\{x^\mu, \sigma_{\nu'}\}$ is given by the Jacobian,

$$\frac{\partial(\sigma_{\nu'}, x^{\rho'})}{\partial(x_\mu, x^{\tau'})} = \begin{vmatrix} \frac{\partial \sigma_{\nu'}}{\partial x_\mu} & \frac{\partial \sigma_{\nu'}}{\partial x^{\tau'}} \\ \frac{\partial x^{\rho'}}{\partial x_\mu} & \frac{\partial x^{\rho'}}{\partial x^{\tau'}} \end{vmatrix} = \frac{\partial(\sigma_{\nu'})}{\partial(x_\mu)} = \det(\sigma_{,\mu\nu'}), \tag{46}$$

where the fact that partials with respect to different coordinates commute has been used. Variation of the tangent vector $\sigma_{\nu'}$ produces a corresponding change in the other endpoint at x;

$$\delta\sigma_{\nu'} = \delta(\sigma_{,\nu'}) = \delta\left(\frac{\partial\sigma}{\partial x^{\nu'}}\right) = \frac{\partial}{\partial x^\mu}\left(\frac{\partial\sigma}{\partial x^{\nu'}}\right)\delta x^\mu = \sigma_{,\mu\nu'}\delta x^\mu. \tag{47}$$

Inverting this gives the variation of the endpoint at x in terms of the variation of the tangent vector $\delta\sigma_{\nu'}$,

$$\delta x^\mu = -(-\sigma_{\mu\nu'})^{-1}\delta\sigma_{\nu'} \equiv -(D^{-1\mu\nu'})\delta\sigma_{\nu'}, \tag{48}$$

where $D_{\mu\nu'} \equiv -\sigma_{\mu\nu'}$ for a given x and x', and $(D^{-1\mu\nu'})$ is the inverse of $(-\sigma_{\mu\nu'})$. Eq.(48) means that, when $(D^{-1\mu\nu'})$ is zero for a given x and x', variations of the tangent vector $\sigma_{\nu'}$ do not produce a corresponding change in the original endpoint of the geodesic located at x. Figure 3 illustrates an example wherein $(D^{-1\mu\nu'})$ is zero. Variation of the tangent vector from the vector σ_1^μ to the vector σ_2^μ at point x'' changes from the upper geodesic between x and x'' to the lower geodesic between these points. Yet both of these geodesics still have x as their other endpoint. Thus $\sigma(x, x'')$ is not single-valued, x'' is not "sufficiently close" to x, and the point x'' is not appropriate to choose for point-splitting.

With more than one geodesic emanating from x crossing at the point x'', a caustic surface has been found at x''. This occurs if $(D^{\mu\nu'})^{-1} = 0$, or, equivalently, $det(D^{-1\mu\nu'}) \equiv D^{-1} = 0$. If $D^{-1} = 0$, then $det(D_{\mu\nu'})$ blows up. The Van Vleck-Morette determinant D is defined by

$$D(x, x') \equiv \det(D_{\mu\nu'}). \tag{49}$$

Evidently, when the Van Vleck-Morette determinant diverges, a caustic surface has been found, and the points of chosen for geodesic point-splitting are not "sufficiently close."

An important identity involving the Van Vleck-Morette determinant may be derived from Eq.(44). Taking one covariant derivative with respect to the stationary point x^α gives

$$\sigma_{;\alpha} = \sigma_{;\mu\alpha}\sigma^{;\mu} = \sigma_{;\alpha\mu}\sigma^{;\mu}. \tag{50}$$

Taking one covariant derivative with respect to the nearby point $x^{\beta'}$ gives

$$\sigma_{;\alpha\beta'} = \sigma_{;\alpha\mu\beta'}\sigma^{;\mu} + \sigma_{;\alpha\mu}\sigma^{;\mu}{}_{\beta'} = \sigma_{;\alpha\beta'\mu}\sigma^{;\mu} + \sigma_{;\alpha}{}^{\mu}\sigma_{;\mu\beta'} \tag{51}$$

where the fact that covariant derivatives with respect to different spacetime points commute has been used. This may be rewritten as

$$D_{\alpha\beta'} = D_{\alpha\beta';\mu}\sigma^{;\mu} + D_{\mu\beta'}\sigma_{;\alpha}{}^{\mu}. \tag{52}$$

Multiplying by $(D^{\alpha\beta'})^{-1}$, and using $tr\ ln(A) = ln\ det(A)$ for any matrix A, gives

$$(D^{\alpha\beta'})^{-1}D_{\alpha\beta'} = (D^{\alpha\beta'})^{-1}D_{\alpha\beta';\mu}\sigma^{;\mu} + (D^{\alpha\beta'})^{-1}D_{\mu\beta'}\sigma_{;\alpha}{}^{\mu} \tag{53}$$

$$4 = [\mathrm{tr}\ln(D_{\alpha\beta'})]_{;\mu}\sigma^{;\mu} + \delta^{\alpha}{}_{\mu}\sigma_{;\alpha}{}^{\mu} \tag{54}$$

$$4 = [\ln\det(D_{\alpha\beta'})]_{;\mu}\sigma^{;\mu} + \sigma_{;\mu}{}^{\mu} \tag{55}$$

$$4 = D^{-1}D_{;\mu}\sigma^{;\mu} + D^{-1}D\sigma_{;\mu}{}^{\mu} \tag{56}$$

$$4 = D^{-1}(D\sigma^{;\mu})_{;\mu}. \tag{57}$$

This has the form of a divergence equation for the vector $\sigma^{;\mu}$. Eq.(57) may be rewritten in a more convenient form. First, note the action of the operator, $\sigma^{\mu}\frac{\partial}{\partial x^\mu}$ on the scalar f:

$$\sigma^{\mu}f_{,\mu} = (\lambda - \lambda')\frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} = (\lambda - \lambda')\frac{df}{d\lambda}. \tag{58}$$

This operator gives the change of a function along the geodesic parameterized by λ, or, in essence, the change in the function as it is parallel transported along that geodesic. Also,

$$\sigma^{\mu'} f_{,\mu'} = (\lambda' - \lambda)\frac{dx^{\mu'}}{d\lambda'}\frac{\partial f}{\partial x^{\mu'}} = (\lambda' - \lambda)\frac{df}{d\lambda} \tag{59}$$

gives the change of the f when it is parallel transported in the opposite direction along the geodesic. Defining,

$$\Delta(x, x') \equiv g^{-1/2}(x) D(x, x') g^{-1/2}(x'), \tag{60}$$

and using Eq.(58), Eq.(57) may now be rewritten in terms of the divergence of the vector σ^{μ}:

$$\begin{aligned}
\sigma^{\mu}{}_{;\mu} &= 4 - \Delta^{-1}\sigma^{;\mu}\Delta_{;\mu} && (61)\\
&= 4 - \Delta^{-1}(\lambda - \lambda')\frac{d\Delta}{d\lambda} && (62)\\
&= 4 - \frac{d\Delta}{\Delta}\frac{(\lambda - \lambda')}{d\lambda} && (63)\\
&= 4 - \frac{d(ln\Delta)}{d(ln\lambda)}, && (64)
\end{aligned}$$

where the transition to the last line was made by setting the parameter $\lambda' = 0$. In flat spacetime, there are no caustic surfaces, the Van Vleck-Morette determinant is zero, and this divergence is 4. If D, and thus Δ, blows up, then there are an infinite number of geodesics emanating from the point x crossing at x', and there is a caustic surface at x'.

Finally, there is the matter how to interpret N-tensors having weight at a spacetime point that is not the stationary point x. For example, consider the vector $A^{\nu'}$ which is evaluated at the nearby point x'. A way is needed for the information contained in this vector to be conveyed to the stationary point x. This is a concept familiar in field theory in that Green functions accomplish such transport of information. Consider the flat space Klein-Gordon equation

$$(\partial_\alpha \partial^\alpha - m^2)\phi(x) = j(x), \tag{65}$$

where $\partial_\alpha \partial^\alpha$ is the flat space D'Alembertian and $j(x)$ is a source for the field $\phi(x)$. Solutions to Eq.(65) will be generated by [41]

$$\phi(x) = \phi^{(0)}(x) + \int G(x, x') j(x') \, d^4x', \tag{66}$$

where $\phi^{(0)}$ is a solution to the homogeneous form of Eq.(65). The Green function $G(x, x')$ takes information at x' and conveys it to the point x. In a general spacetime, the entity analogous to $G(x, x')$ is $g^{\mu}{}_{\nu'}$, the *bivector of parallel transport*, defined by

$$A^\mu = g^{\mu}{}_{\nu'} A^{\nu'}, \tag{67}$$

where $g^{\mu}{}_{\nu'}$ takes the information at the nearby point x' and conveys it to the stationary point x.

An important property of $g^{\mu}{}_{\nu'}$ may be demonstrated by using the operator of Eq.(58) to parallel transport A^μ along a geodesic parameterized by λ. Parallel transport of A^μ along the geodesic connecting the points x and x' requires

$$\sigma^\rho A^{\mu}{}_{;\rho} = 0, \tag{68}$$

and thus

$$\sigma^\rho A^{\mu}{}_{;\rho} = g^{\mu}{}_{\nu';\rho} \sigma^{;\rho} A^{\nu'} = 0. \tag{69}$$

For this to be true for arbitrary $\sigma^{;\rho}$ and $A^{\nu'}$ requires

$$\sigma^{;\rho} g^{\mu}{}_{\nu';\rho} = 0. \tag{70}$$

This equation will be needed for the point-splitting procedure.

CHAPTER 4

DeWitt-Schwinger Geodesic Point-Splitting

The geodesic point-splitting scheme of DeWitt [2] was developed from Schwinger's [3] original method of calculating the Feynman Green function $G_F(x, x')$ associated with a quantum field. At the heart of Schwinger's method lies the idea that the Feynman Green function is determined by the differential operator of the gauge invariant wave equation. The wave equation for the charged complex scalar field is obtained by varying the action in Eq.(17) with respect to ϕ^* and is given by

$$g^{1/2}[D_\mu D^\mu - (m^2 + \xi R)]\phi(x) \equiv \phi_{|\mu}{}^{\mu} - (m^2 + \xi R)\phi = 0. \tag{71}$$

It will be shown that, once the Feynman Green function is determined, the Hadamard function $G^{(1)}(x, x')$ will be determined. Then the DS point-splitting expansions for $\langle\phi^2\rangle$, $\langle j^\mu\rangle$, and $\langle T_{\mu\nu}\rangle$ may be constructed from $G^{(1)}$ and its derivatives. This chapter will detail the steps for obtaining a DeWitt-Schwinger expansion for $G^{(1)}(x, x')$ from Schwinger's original proper time method.

The classical current for a charged scalar field is obtained by varying the action in Eq.(17) with respect to A^μ and is given by

$$j^\mu = \frac{1}{4\pi} F^{\mu\nu}{}_{;\nu} = \frac{1}{2} ie \left[\{D^\mu\phi, \phi^*\} - \{D^\mu\phi, \phi^*\}^*\right], \tag{72}$$

where the braces $\{\}$ denote the anticommutator. The components of the classical stress-energy tensor are given by,

$$\begin{aligned} T^{\mu\nu} = &\frac{1}{2}\Big[\frac{1}{2}(1-2\xi)\{\phi^{|\mu}, \phi^{*|\nu}\} + \frac{1}{2}(2\xi - \frac{1}{2})g^{\mu\nu}\{\phi_{|\sigma}, \phi^{*|\sigma}\} \\ &-\xi\{\phi^{|\mu\nu}, \phi^*\} + \xi g^{\mu\nu}\{\phi_{|\sigma}{}^{\sigma}, \phi^*\} \\ &+\frac{1}{2}\xi(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)\{\phi, \phi^*\} - \frac{1}{4}m^2 g^{\mu\nu}\{\phi, \phi^*\} + c.c.\Big], \end{aligned} \tag{73}$$

where *c.c.* denotes the complex conjugate of all of the previous terms. The anti-commutators above arise from symmetrizing with respect to the fields ϕ and ϕ^*.

Consider now the transition from the classical form of $\{\phi(x), \phi(x)\}$ to its quantized form in the point-splitting procedure. The functions $\phi(x)$ become the field operators $\underline{\phi}(x)$. The first field operator is moved to the nearby point x' and evaluated between the vacuum states $\langle 0|$ and $|0\rangle$. A similar expression is then obtained by taking the second field operator to the point x'. The two results are then averaged, yielding the biscalar Hadamard function $G^{(1)}(x, x')$, given by

$$\{\phi(x), \phi^*(x)\} \rightarrow G^{(1)}(x, x') \equiv \langle 0|\{\underline{\phi}(x), \underline{\phi}(x')\}|0\rangle. \tag{74}$$

With this definition, the vacuum polarization is given by

$$\langle \phi^2 \rangle = \lim_{x' \to x} \frac{1}{2} G^{(1)}(x, x'). \tag{75}$$

Constructing the VEV of the current and stress-energy tensor will require not only $G^{(1)}(x, x')$, but also gauge covariant derivatives of $G^{(1)}(x, x')$. Remembering that derivatives with respect to a spacetime point only act on functions of that point, it is straightforward to show the first few gauge covariant derivatives of $G^{(1)}(x, x')$ are

$$G^{(1)|\mu} \equiv \langle 0|\{\underline{\phi}^{|\mu}(x), \underline{\phi}^*(x')\}|0\rangle, \tag{76}$$

$$G^{(1)|\mu'} \equiv \langle 0|\{\underline{\phi}(x), \underline{\phi}^{*|\mu'}(x')\}|0\rangle, \tag{77}$$

$$G^{(1)|\mu\nu} \equiv \langle 0|\{\underline{\phi}^{|\mu\nu}(x), \underline{\phi}^*(x')\}|0\rangle, \tag{78}$$

$$G^{(1)|\mu\nu'} \equiv \langle 0|\{\underline{\phi}^{|\mu}(x), \underline{\phi}^{*|\nu'}(x')\}|0\rangle, \tag{79}$$

and

$$G^{(1)|\mu'\nu'} \equiv \langle 0|\{\underline{\phi}(x), \underline{\phi}^{*|\mu'\nu'}(x')\}|0\rangle. \tag{80}$$

In all of these, the unprimed and primed indices refer to differentiation with respect to the spacetime points x and x', respectively. Since $G^{(1)}(x, x')$ is understood to be a biscalar, the (x, x') argument of $G^{(1)}(x, x')$ will hereafter often be dropped for brevity in the long expressions to follow.

The expectation value of the current associated with the charged scalar field may be written using Eq.(72) and the definitions of Eqs.(76-(80) and is given by

$$\langle \underline{j}^{\mu}(x)\rangle = \lim_{x'\to x} \frac{ie}{4}\left[\left(G^{(1)|\mu} + g^{\mu}{}_{\tau'}G^{(1)|\tau'}\right) - \left(G^{(1)|\mu} + g^{\mu}{}_{\tau'}G^{(1)|\tau'}\right)^{*}\right]. \qquad (81)$$

The bivector of parallel transport $g^{\mu}{}_{\tau'}$ has been used to transport the information from the spacetime point x' back to the stationary point x. Note that the form of Eq.(81) guarantees that the current will be real. The expectation value of the stress-energy tensor may similarly be written in terms of $G^{(1)}$ and its derivatives as

$$\begin{aligned}\langle \underline{T}^{\mu\nu}(x)\rangle = \lim_{x'\to x} Re \Big[&\frac{1}{2}(\frac{1}{2}-\xi)(g^{\mu}{}_{\tau'}G^{(1)|\tau'\nu} + g^{\nu}{}_{\rho'}G^{(1)|\mu\rho'}) + (\xi - \frac{1}{4})g^{\mu\nu}g^{\alpha\rho'}G^{(1)}{}_{|\alpha\rho'}\\ &-\frac{1}{2}\xi(G^{(1)|\mu\nu} + g^{\mu}{}_{\tau'}g^{\nu}{}_{\rho'}G^{(1)|\tau'\rho'}) + \frac{1}{4}\xi g^{\mu\nu}(G^{(1)}{}_{|\alpha}{}^{\alpha} + g^{\alpha}{}_{\tau'}g_{\alpha\rho'}G^{(1)|\tau'\rho'})\\ &+\frac{1}{2}\xi(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)G^{(1)} - \frac{1}{4}m^2 g^{\mu\nu}G^{(1)}\Big]. \qquad (82)\end{aligned}$$

Eqs.(75)-(82) are divergent for the same reasons that the Wightman function $G^{+}(x, x')$ of Eq.(13) is divergent. The Hadamard elementary function, $G^{(1)}$, is constructed from the VEV of the product of two fields in a manner similar to $G^{+}(x, x')$ as shown by Eqs.(74) and (13). Just as $G^{+}(x, x')$ is quadratically divergent so, too, is $G^{(1)}(x, x')$ quadratically divergent. By direct analogy with Eq.(14), one derivative of $G^{(1)}$ will potentially have a cubic divergence, while two derivatives will potentially have a quartic divergence. Point-splitting will isolate these divergences in preparation for renormalizing Eqs.(75), (81), and (82).

Evaluating $G^{(1)}$ and its derivatives requires relating $G^{(1)}$ to the Feynman Green function $G_F(x, x') \equiv G_F$. The Feynman Green function for the complex scalar field is given by the VEV of the time-ordered product of the field and its conjugate,

$$iG_F(x, x') = \langle 0|T[\phi(x)\phi^{*}(x')]|0\rangle \qquad (83)$$

$$= \langle 0|\Theta(t - t')\underline{\phi}(x)\underline{\phi}^{*}(x') + \Theta(t' - t)\underline{\phi}^{*}(x')\underline{\phi}(x)|0\rangle, \qquad (84)$$

where

$$\Theta(t-t') = \begin{cases} 1, & t > t' \\ 0, & t' > t \,. \end{cases} \tag{85}$$

The Pauli-Jordan, or Schwinger function, $G(x,x')$, is given by the VEV of the commutator of the field operators $\underline{\phi}(x)$,

$$iG(x,x') \equiv \langle 0|[\underline{\phi}(x), \underline{\phi}^*(x')]|0\rangle. \tag{86}$$

The retarded and advanced Green functions are defined by

$$G_R(x,x') \equiv -\Theta(t-t')G(x,x'), \tag{87}$$

and

$$G_A(x,x') \equiv \Theta(t'-t)G(x,x'). \tag{88}$$

With these definitions, it is straightforward to derive the relationship

$$G_F(x,x') = \overline{G}(x,x') - \frac{1}{2}iG^{(1)}(x,x'), \tag{89}$$

where $\overline{G}(x,x')$ is one-half the sum of the advanced and retarded Green functions. This relation shows that, were the Feynman Green function known, its imaginary part would be the Hadamard function from which the DS point-splitting expansions for $\langle\phi^2\rangle$, $\langle j^\mu\rangle$, and $\langle T^{\mu\nu}\rangle$ may be constructed. Determining the Feynman Green function requires determining the action of the differential operator of Eq.(71) on the Feynman Green function. To do this, we now make the transition from coordinate space to Schwinger's proper time space.

The transition from classical to quantum fields is made by imposing the equal time commutation relations for the quantum operators $\underline{\phi}(x)$ and $\underline{\pi}(x)$ [49],

$$\left[\underline{\phi}(t,\vec{x}), \underline{\phi}(t,\vec{x'})\right] = 0 \tag{90}$$

$$\left[\underline{\pi}(t,\vec{x}), \underline{\pi}(t,\vec{x'})\right] = 0 \tag{91}$$

$$\left[\underline{\phi}(t,\vec{x}), \underline{\pi}(t,\vec{x'})\right] = i\delta^3(\vec{x}-\vec{x}'), \tag{92}$$

where $\underline{\pi}(t,\vec{x})$ is the canonically conjugate variable to $\underline{\phi}(t,\vec{x})$ defined by

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi, \tag{93}$$

where x^0 is a time (temporal) coordinate. Using Eqs.(90)–(92), along with Eq.(71) and $\partial_t \Theta(t-t') = \delta(t-t')$, the action of the wave differential operator on $G_F(x,x')$ may be shown to be

$$g^{1/2}[D_\mu D^\mu - (m^2 + \xi R)]G_F(x,x') = -\delta^4(x-x'), \tag{94}$$

or

$$F(x)G_F(x,x') = -\delta^4(x-x'), \tag{95}$$

where $F(x) \equiv g^{1/2}[D_\mu D^\mu - (m^2 + \xi R)]$ operates with respect to the spacetime point x. Rewriting Eq.(95) as a matrix equation gives

$$\int \langle x|\underline{F}|x''\rangle\langle x''|\underline{G_F}|x'\rangle d^4x'' = -\langle x|\underline{1}|x\rangle, \tag{96}$$

where $\underline{F}$ and $\underline{G_F}$ are now matrix operators. Inserting two factors of $\underline{1} = \underline{g}^{-1/4}\underline{g}^{1/4}$ in Eq.(96) maintains the transformation properties of the bases $|x\rangle$ as in Eq.(28). Removing the projection operator, $\int |x''\rangle\langle x''|d^4x''$, along with the matrix elements $\langle x|(\ldots)|x'\rangle$ allows the matrix equation (96) to be written as,

$$\underline{g}^{1/4}\underline{G_F}\underline{g}^{1/4} = \frac{\underline{1}}{-(\underline{g}^{-1/4}\underline{F}\underline{g}^{-1/4} + i\epsilon\underline{1})} = i\int_0^\infty e^{-i\underline{H}s}ds = i\int_0^\infty e^{-i\underline{H}(s-0)}ds. \tag{97}$$

In Eq.(97), the factor $e^{-i\underline{H}0} = 1$ has been explicitly added for clarity below, and the matrix operator $\underline{H}$ is defined by

$$\underline{H} \equiv -(\underline{g}^{-1/4}\underline{F}\underline{g}^{-1/4} + i\epsilon\underline{1}) \tag{98}$$

The infinitesimal $+i\epsilon\underline{1}$ has been added to make this integral analytic in the upper half of the complex plane [2, 49]. Taking matrix elements of Eq.(97) and rearranging, yields,

$$G_F(x,x') = i\int_0^\infty g^{-1/4}(x)\langle x|e^{-i\underline{H}(s-0)}|x'\rangle g^{-1/4}(x')ds. \tag{99}$$

The operator $e^{-i\underline{H}(s-0)}$ was interpreted by Schwinger to act in a fictitious 4 + 1-dimensional Hilbert space where s is a proper time parameter. With the spacetime points x and x' understood to be associated with points s and s', respectively, in this Hilbert space, the operator $e^{i\underline{H}s}$ is interpreted to act on the kets $|x, s\rangle$ as,

$$e^{i\underline{H}s}|x_0, s = 0\rangle \equiv |x, s\rangle. \tag{100}$$

The ket $|x_0, s = 0\rangle$, which is associated with the spacetime point x_0 and the proper time parameter $s = 0$, has been taken to the ket $|x, s\rangle$, which is associated with the spacetime point x and the proper time parameter s. Thus,

$$G_F(x, x') \equiv i \int_0^\infty g^{-1/4}(x)\langle x, s|x', 0\rangle g^{-1/4}(x')ds. \tag{101}$$

The matrix element $\langle x, s|x', 0\rangle$ obeys a Schrödinger-like equation in both Hilbert space form,

$$i\frac{\partial}{\partial s}\langle x, s|x', 0\rangle = \langle x, s|\underline{H}|x', 0\rangle, \tag{102}$$

and coordinate space form,

$$i\frac{\partial}{\partial s}\langle x, s|x', 0\rangle = g^{-1/4}(x)g^{-1/4}(x')\left[(m^2 + \xi R) - D_\mu D^\mu\right]\langle x, s|x', 0\rangle, \tag{103}$$

where the infinitesimal factor $+i\epsilon$ has been dropped for brevity. Also, the reader should take care not to confuse the factor $g^{-1/4}(x)g^{-1/4}(x')$ in this equation with the two factors $g^{-1/4}(x)$ and $g^{-1/4}(x')$ in Eq.(101). Keeping track of these factors is very important in this derivation! In Eq.(103), the operator $\underline{g}^{-1/4}$ to the right of the derivatives has been allowed to operate on the ket $|x', 0\rangle$. Then, since $[(m^2 + \xi R) - D_\mu D^\mu$ is chosen to operate with respect to the point x, the factor of $g^{-1/4}(x')$ may be moved to the left of the derivative operator. The boundary condition,

$$\langle x, 0|x', 0\rangle = \langle x|x'\rangle = \delta(x - x'), \tag{104}$$

is imposed in accordance with Eq.(95).

At this point, solving either of the the Schrödinger-like equations exactly would completely determine G_F. Unfortunately, there is no known exact solution for G_F in a general spacetime, so an approximate solution is needed. This approximate solution is found by determining the action of the operator $\underline{H}$ in the proper time space of s.

To determine the action of the operator $\underline{H}$ in the $4+1$-dimension proper time space used by Schwinger [3], the operator $\underline{H}$ is rewritten in a flat space version where $g^{-1/4}(x) = 1$;

$$\underline{H} \equiv -\underline{F} = m^2\underline{1} + \underline{\pi}_\mu \underline{\pi}^\mu, \tag{105}$$

In Eq.(105), $\underline{\pi}_\mu \equiv (\nabla_\mu / i - eA_\mu)$ is the conjugate momentum derivative operator. The position operator in this proper time space is given by $\underline{x}^\mu$. The commutation relations of $\underline{\pi}_\mu$ and $\underline{x}_\mu$ are given by

$$[\underline{x}^\mu, \underline{x}^\nu] = 0, \quad [\underline{x}^\mu, \underline{\pi}^\nu] = i\delta^{\mu\nu}\underline{1}, \quad \text{and} \quad [\underline{\pi}^\mu, \underline{\pi}^\nu] = ieF^{\mu\nu}\underline{1}, \tag{106}$$

where $F^{\mu\nu} \equiv (\partial^\mu A^\nu - \partial^\nu A^\mu)$ is the familiar electromagnetic field tensor. Using these commutation relations, the equations of motion for the operators $\underline{x}^\mu$ and $\underline{\pi}^\mu$ are

$$\frac{d\underline{x}^\mu}{ds} = i\,[\underline{H}, \underline{x}^\mu] = 2\underline{\pi}^\mu, \tag{107}$$

and

$$\frac{d\underline{\pi}^\mu}{ds} = i\,[\underline{H}, \underline{\pi}^\mu] = 2eF^{\mu\nu}\underline{\pi}_\nu - ie\partial_\nu F^{\mu\nu}\underline{1}. \tag{108}$$

Now consider the case of zero electromagnetic fields, or $F^{\mu\nu} = 0$. The equations of motion become

$$\frac{d\underline{\pi}^\mu}{ds} = 0 \Rightarrow \underline{\pi}^\mu(s) = \underline{\pi}^\mu(0). \tag{109}$$

and

$$\frac{d\underline{x}^\mu}{ds} = 2\underline{\pi}^\mu(0) = \frac{\underline{x}^\mu(s) - \underline{x}^\mu(0)}{s - 0}. \tag{110}$$

Taking the matrix element $\langle x,s|\underline{H}|x',0\rangle$ requires all $\underline{x}^\mu(s)$ be moved to the left of the $\underline{x}^\mu(0)$:

$$\begin{aligned}\underline{H} &= m^2\underline{1} + \frac{(\underline{x}^\mu(s)-\underline{x}^\mu(0))(\underline{x}_\mu(s)-\underline{x}_\mu(0))}{4s^2} \\ &= m^2\underline{1} + (4s^2)^{-1}\left(\underline{x}^\mu(s)\underline{x}_\mu(s) - 2\underline{x}^\mu(s)\underline{x}_\mu(0) + \underline{x}^\mu(0)\underline{x}_\mu(0)\right) + \\ &\quad (4s^2)^{-1}\left[\underline{x}^\mu(s),\underline{x}_\mu(0)\right]. \end{aligned} \tag{111}$$

Using

$$\left[\underline{x}^\mu(s),\underline{x}_\mu(0)\right] = \left[2s\underline{\pi}^\mu(0)+\underline{x}^\mu(0),\underline{x}_\mu(0)\right] = 2s\left(-i\delta^\mu{}_\mu\right) = -8is, \tag{112}$$

and taking matrix elements of Eq.(111) yields

$$\langle x,s|\underline{H}|x',0\rangle = \left[m^2 + (4s^2)^{-1}(x^\mu(s)-x^\mu(0))^2 - \frac{2i}{s}\right]\langle x,s|x',0\rangle. \tag{113}$$

With $x^\mu(s)$ and $x^\mu(0)$ associated with the spacetime points x and x', respectively, the coordinate space Schrödinger-like equation (103) now reads

$$i\frac{\partial}{\partial s}\langle x,s|x',0\rangle = \left[m^2 + (4s^2)^{-1}(x^\mu - x^{\mu\prime})^2 - \frac{2i}{s}\right]\langle x,s|x',0\rangle, \tag{114}$$

which is readily integrated to give

$$\langle x,s|x',0\rangle = C\Phi(x,x')s^{-2}e^{-i\left[m^2s-\frac{(x-x')^2}{4s}\right]}. \tag{115}$$

Eq.(115) must behave as a delta function as $s\to 0$ according to Eq(104). This will be true so long as

$$\lim_{x'\to x}\Phi(x,x') = 1, \tag{116}$$

and

$$C\ s^{-2}\int e^{\left[\frac{ix^2}{4s}\right]}\,d^4x = 1. \tag{117}$$

Eq.(116) is required to be true by definition. Eq.(117) then shows that $C = -i(4\pi)^{-2}$.

DeWitt used this flat space example, culminating in Eq.(115), to make an ansatz for the solution of Eq.(103), namely,

$$\langle x,s|x',0\rangle = -\frac{i}{(4\pi)^2}\frac{D^{\frac{1}{2}}(x,x')}{s^2}\exp\left[i\frac{\sigma(x,x')}{2s} - im^2s\right]\Omega(x,x',s), \tag{118}$$

where $D(x,x')$ is the Van Vleck-Morette determinant. The unknown function $\Omega(x,x',s)$ is yet to be determined, yet DeWitt [38] proposed the flat space solution suggests the boundary condition

$$\lim_{x'\to x}\Omega(x,x',s)=1, \tag{119}$$

in agreement with Eq.(116).

Geometric quantities are unaffected by the presence of the gauge field so no gauge field information can arise in the point-splitting procedure through the quantities $\sigma(x,x')$ and $D^{\frac{1}{2}}(x,x')$ in Eq.(118). The only way the gauge field can explicitly appear is through the unknown function $\Omega(x,x',s)$, which carries information about the gauge field. Whenever derivatives of this function are taken, they must be gauge covariant derivatives which obey the commutation relation

$$f(x,x')_{|\mu\nu}-f(x,x')_{|\nu\mu}=ieF_{\mu\nu}f(x,x'), \tag{120}$$

where $f(x,x')$ is a function that explicitly carries information about the gauge field.

Substituting Eq.(118) into Eq.(103), and using Eq.(57), yields a differential equation for $\Omega(x,x')$

$$i\frac{\partial\Omega}{\partial s}+\frac{i}{s}\Omega_{|\mu}\sigma^{|\mu}=-D^{-1/2}(D^{1/2}\Omega)_{|\mu}{}^{\mu}+\xi R\Omega. \tag{121}$$

The function Ω is given the *asymptotic* series representation [2],

$$\Omega(x,x')=\sum_{0}^{\infty}a_n(x,x')(is)^n. \tag{122}$$

This may be done, so long as the gravitational and electromagnetic fields, upon which the coefficients $a_n(x,x')$ depend, are slowly varying over the infinitesimal distance between the points x and x'. Substituting Eq.(122) into Eq.(121), defining

$$\Delta(x,x')\equiv g^{-1/2}(x)D(x,x')g^{-1/2}(x'), \tag{123}$$

and equating like powers of s in the infinite series that result, yields the recursion relations which will be used to determine the a_n:

$$\sigma^{|\mu} a_{0|\mu} = 0, \tag{124}$$

and

$$\sigma^{|\mu} a_{n+1|\mu} + (n+1)a_{n+1} = \Delta^{-1/2}(\Delta^{1/2} a_n)_{|\mu}{}^{\mu} - \xi R a_n. \tag{125}$$

These relations are of the same form as those derived by Christensen [30]. The new feature here is that the presence of the gauge field requires that all derivatives be gauge covariant when appropriate. As mentioned previously, objects such as $\sigma^{;\mu\nu}, \Delta^{1/2;\mu\nu}, R^{;\mu\nu\rho}$, etc., will only require partial derivatives and the Christoffel connections in their covariant derivatives. Only the coefficients $a_n(x, x')$, which carry information about the gauge field, will require the gauge connection, A^{μ}, in addition to partial derivatives and the christoffel connections, in their derivatives are gauge covariant.

Substituting Eq.(122) and Eq.(118) into Eq.(101) yields

$$G_F(x, x') = \frac{\Delta^{1/2}}{(4\pi)^2} \int_0^{\infty} s^{-2} e^{-i\left[m^2 s - \frac{\sigma}{2s}\right]} \sum_{n=0}^{\infty} a_n(x, x')(is)^n ds. \tag{126}$$

Note how the two factors $g^{-1/4}(x)$ and $g^{-1/4}(x')$ have been grouped with the factor $D^{1/2}(x, x')$ to form $\Delta^{1/2}(x, x')$ defined in Eq.(123). The summation and integration may be exchanged, and Eq.(126) may be rewritten as

$$G_F(x, x') = \frac{\Delta^{1/2}}{(4\pi)^2} \sum_{n=0}^{\infty} a_n(x, x') \left(-\frac{\partial}{\partial m^2}\right)^n \int_0^{\infty} s^{-2} e^{-i\left[m^2 s - \frac{\sigma}{2s}\right]} ds. \tag{127}$$

Using the identity [39]

$$\frac{1}{(4\pi)^2} \int_0^{\infty} s^{-2} e^{-i\left[m^2 s - \frac{\sigma}{2s}\right]} ds = -\frac{m^2}{8\pi} \frac{H_1^{(2)}\left[(-2m^2\sigma)^{1/2}\right]}{(-2m^2\sigma)^{1/2}}, \tag{128}$$

yields

$$G_F(x, x') = -\frac{\Delta^{1/2}}{8\pi} \sum_{n=0}^{\infty} a_n \left(-\frac{\partial}{\partial m^2}\right)^n \frac{m^2 H_1^{(2)}\left[(-2m^2\sigma)^{1/2}\right]}{(-2m^2\sigma)^{1/2}}, \tag{129}$$

where $H_1^{(2)}(x)$ is the Hankel function of the second kind of order one;

$$H_1^{(2)}(x) = J_1(x) - iY_1(x). \tag{130}$$

The Hankel function $H_1^{(2)}$ may be expanded in an asymptotic series in its small argument $(-2m^2\sigma)^{1/2}$,

$$\frac{m^2 H_1^{(2)}\left[(-2m^2\sigma)^{1/2}\right]}{(-2m^2\sigma)^{1/2}} = \frac{1}{i\pi}\Big(\frac{1}{\sigma+i\epsilon} +$$
$$2m^2\Big\{\Big[\gamma + \frac{1}{2}ln(\frac{1}{2}m^2) + \frac{1}{2}ln(\sigma+i\epsilon)\Big]\Big[\frac{1}{2} + \frac{2m^2\sigma}{2^2\times 4} + \frac{(2m^2\sigma)^2}{2^2\times 4^2\times 6} + \cdots\Big]$$
$$-\frac{1}{4} - \frac{2m^2\sigma}{2^2\times 4}\Big(1+\frac{1}{4}\Big) - \frac{(2m^2\sigma)^2}{2^2\times 4^2\times 6}\Big(1+\frac{1}{2}+\frac{1}{6}\Big) - \cdots\Big\} - \cdots\Big), \tag{131}$$

where γ is Euler's constant and the infinitesimal $+i\epsilon$ has been explicitly included. The differentiations and summations of Eq.(129) may be performed on this series. Using the identities,

$$\frac{1}{\sigma+i\epsilon} = \frac{1}{\sigma} - i\pi\delta(\sigma) \quad , \quad \ln(\sigma+i\epsilon) = \ln|\sigma| + i\pi\Theta(-\sigma), \tag{132}$$

allows the identification of the real and imaginary parts of the Feynman Green function $G_F(x,x')$. Using Eq.(89), the DeWitt-Schwinger point-splitting expansion for the biscalar $G^{(1)}(x,x')$ is obtained:

$$G^{(1)}(x,x') = \frac{\Delta^{1/2}(x,x')}{4\pi^2}\times$$
$$\Big\{a_0(x,x')\Big[\frac{1}{\sigma(x,x')} + m^2(\gamma + \frac{1}{2}\ln|\frac{1}{2}m^2\sigma(x,x')|)(1+\frac{1}{4}m^2\sigma(x,x')+\cdots)$$
$$-\frac{1}{2}m^2 - \frac{5}{16}m^2\sigma(x,x') + \cdots\Big]$$
$$-a_1(x,x')\Big[(\gamma+\frac{1}{2}\ln|\frac{1}{2}m^2\sigma(x,x')|)(1+\frac{1}{2}m^2\sigma(x,x')+\cdots) - \frac{1}{2}m^2\sigma(x,x') - \cdots\Big]$$
$$+a_2(x,x')\sigma(x,x')\Big[(\gamma+\frac{1}{2}\ln|\frac{1}{2}m^2\sigma(x,x')|)(\frac{1}{2}+\frac{1}{8}m^2\sigma(x,x')+\cdots) - \frac{1}{4} - \cdots\Big]$$
$$+\frac{1}{2m^2}\left[a_2(x,x')+\cdots\right] + \frac{1}{2m^4}\left[a_3(x,x')+\cdots\right] + \cdots\Big\}. \tag{133}$$

In this form, $G^{(1)}$ appears to have at least a quadratic divergence in the infinitesimal separation σ^μ due to the presence of the term proportional to $1/\sigma = 2/(\sigma_\mu\sigma^\mu)$.

This is in agreement with the discussion of the degree of divergence of scalar QED. The task now is to determine the true form of the biscalars $\Delta^{1/2}(x,x')$, $\sigma(x,x')$ and $a_n(x,x')$ in Eq.(133). With $\langle j^\mu \rangle$ and $\langle T_{\mu\nu} \rangle$ being constructed from $G^{(1)}$ and its derivatives, it will also be necessary to determine the form of derivatives of these biscalars.

As discussed in the next chapter, it will not be possible to exactly determine either the biscalars appearing on the right hand side of Eq.(133), nor to exactly determine the derivatives of these biscalars. Yet we may construct approximations to the biscalars and their derivatives, or any bitensor, which will satisfy the requirements of the point-splitting procedure. As explained more fully at the beginning of the next chapter, each bitensor is expanded in a Taylor series about the stationary point x in terms of the infinitesimal separation vector σ^μ. This expansion is given by

$$a^{\mu\nu\cdots}(x,x') = a0^{\mu\nu\cdots}(x) + a1^{\mu\nu\cdots}{}_{\alpha}(x)\sigma^\alpha + \frac{1}{2!}a2^{\mu\nu\cdots}{}_{\alpha\beta}(x)\sigma^\alpha\sigma^\beta + \cdots, \qquad (134)$$

where the coefficients $a0^{\mu\nu\cdots}(x), a1^{\mu\nu\cdots}{}_{\alpha}(x), \ldots$, are functions of the stationary point x only. The next chapter is devoted to evaluating these coefficients and construction of the expansions of all of the bitensors required by Eq.(133) and its derivatives.