Geometric analysis of a reaction-diffusion equation with nonlocal inhibition
by Joseph Boyd Raquepas

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics .
Montana State University

Abstract:
Presented are geometric results for a reaction-diffusion equation with nonlocal, inhibitory feedback. The equation corresponds to the limiting equation on the slow manifold for a singularly perturbed activator-inhibitor system. The existence of solutions and a global attractor is established using standard results of linear and nonlinear semigroup theory. The bifurcation of equilibria is studied and family of secondary bifurcations is demonstrated, by means of Lyapunov-Schmidt reduction. The -existence of an inertial manifold is verified as well as a family of approximate inertial manifolds. Evidence for the qualitative structure of the global attractor is provided using approximate inertial manifolds and simulations based on standard numerical methods.

GEOMETRIC ANALYSIS OF A REACTION-DIFFUSION EQUATION WITH NONLOCAL INHIBITION

by

Joseph Boyd Raquepas

A thesis submitted in partial fulfillment
of the requirements for the degree

of

Doctor of Philosophy

in

Mathematics

MONTANA STATE UNIVERSITY
Bozeman, Montana

July 1997

APPROVAL

of a thesis submitted by

Joseph Boyd Raquepas

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

7-16-97 — Date

Jack D. Dockery
Chairperson, Graduate Committee

Approved for the Major Department

7/16/97 — Date

John Lund
Head, Mathematics

Approved for the College of Graduate Studies

7/18/97 — Date

Robert Brown
Graduate Dean

STATEMENT OF PERMISSION TO USE

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

Presented are geometric results for a reaction-diffusion equation with nonlocal, inhibitory feedback. The equation corresponds to the limiting equation on the slow manifold for a singularly perturbed activator-inhibitor system. The existence of solutions and a global attractor is established using standard results of linear and nonlinear semigroup theory. The bifurcation of equilibria is studied and family of secondary bifurcations is demonstrated, by means of Lyapunov-Schmidt reduction. The existence of an inertial manifold is verified as well as a family of approximate inertial manifolds. Evidence for the qualitative structure of the global attractor is provided using approximate inertial manifolds and simulations based on standard numerical methods.

CHAPTER 1

INTRODUCTION

In this thesis, we investigate the dynamics of the reaction-diffusion equation

$$u_t = \epsilon^2 u_{xx} + f(u) + \gamma \int_0^1 g(x,s)u(s)dx, \quad u_x(0,t) = u_x(1,t) = 0. \tag{1.1}$$

This equation corresponds to the flow on the slow manifold for the singularly perturbed reaction-diffusion system

$$u_t = \epsilon^2 u_{xx} + f(u) - w, \quad u_x(0,t) = u_x(1,t) = 0, \tag{1.2}$$

$$\delta w_t = D w_{xx} + \gamma u - w, \quad w_x(0,t) = w_x(1,t) = 0, \tag{1.3}$$

where $\delta \ll 1$. Equation (1.1) is obtained by formally setting $\delta = 0$ and using a Green's function to solve (1.3) for w as a function of u. Systems such as (1.2)-(1.3) arise frequently as mathematical models in biology, physiology, morphogenesis in embryology, and chemical systems to name a few. Many examples can be found in the books by Murray [22] and Edelstein-Keshet [10]. The nonlinearity f is typically a cubic or cubic-like function, and the system is considered an activator-inhibitor system in that u activates the production of w while w inhibits u. Models of this type have attracted a great deal of attention since the celebrated work of Turing [30], who proved that diffusion can have a destabilizing effect resulting in spatial pattern formation.

In recent years these models have been shown to produce a variety of interesting phenomena and have been the impetus for many new and exciting mathematical results. Nishiura [23] and Sakamoto [27] have examined the existence and stability of large amplitude spatial patterns. The formation and propagation of transition layers has been studied by many authors (see [11] and references therein). An interesting case is the so called "breather solutions " which are solutions having oscillating layers ([24]). In terms of global dynamics, Hale [15] has demonstrated that as $D \to \infty$ the attractor of the system approaches the attractor for the so called "shadow system", which is obtained by spatially averaging (1.3). The underlying theme in each of these results is the use of perturbation techniques which exploit the extreme size of a model parameter, in particular, the cases $\epsilon \ll 1$, $\frac{1}{\delta} \ll 1$, and D large have been extensively examined. As far as we know, this thesis is the first study of the limiting equation for the case $\delta \ll 1$. Physically this corresponds to an activator-inhibitor system in which the rates of reaction and diffusion of the inhibitor greatly exceed those of the activator. The inhibition appears in the limiting equation (1.1) as nonlocal feedback.

In this work, we consider the particular model where f is the symmetric cubic polynomial $f(u) = u - u^3$, and $\gamma \geq 0$. Our goal is to gain a geometric or qualitative understanding of the global behavior of solutions to (1.1). This equation is a nonlocally perturbed version of the standard Chafee-Infante problem

$$u_t = \epsilon^2 u_{xx} + f(u), \quad u_x(0,t) = u_x(1,t) = 0, \tag{1.4}$$

for which a great deal is known (see [14] and [16]). As such, we expect the dynamics of (1.1) to inherit many of the properties of the dynamics for (1.4) when γ is sufficiently small, and we will verify this is so. However, when $\gamma = \mathcal{O}(1)$ we will see the dynamics differ considerably. In this case, the analysis is significantly complicated by the nonlocal perturbation. Classical results based on maximum principles and comparison arguments ([28]) do not apply and rigorous mathematical results are difficult to obtain. Thus, we must rely on approximation techniques and numerical simulations to gain insight.

This thesis is comprised of six chapters, the first being this introduction.

In Chapter 2, we prove the existence of local solutions for (1.1) using standard results from linear and nonlinear semigroup theory. We show (1.1) has a Lyapunov function which allows us to obtain a global existence result. We then establish the existence of a compact, connected, invariant set which attracts all solutions. This set, called a global attractor, contains much of the information regarding the large time dynamics. The necessary ingredients for the existence of a global attractor are compactness and dissipation. Since in general partial differential equations are posed on metric spaces which are not locally compact, the necessary compactness must come from the solution operator. We establish the compactness of the solution operator for (1.1) by the use of embedding theorems much like the standard Sobolev embedding results ([1]). The concept of dissipation essentially means there is a bounded set into which all solutions enter in a finite time and remain. These properties, along with the existence of a Lyapunov function, allow us to verify (1.1) falls into a special class of systems known as gradient systems. For gradient systems the attractor has a relatively simple description: it is the union of the equilibria and their unstable manifolds. Thus, a characterization of the attractor is complete when all equilibria and their unstable manifolds are found. The goal of characterizing the attractor provides the main theme for this thesis and motivates the work of the remaining chapters.

In Chapter 3,s we search for equilibria of (1.1) by examining local bifurcations from known solutions. This is accomplished using standard techniques of bifurcation theory for Fredholm operators. Due to the nonlocal perturbation, the bifurcation diagram differs significantly from that of (1.4), and the bifurcations have a dependence on both parameters, ϵ and γ. In particular, we demonstrate the existence of an interesting family of secondary bifurcations. This is accomplished by means of a projection method, known as Lyapunov-Schmidt reduction, which allows us to reduce the equation to a two-dimensional problem in the neighborhood of certain points in parameter space. The resulting two-dimensional system is simple and can be easily analyzed. This result generalizes a previous result obtained by Keener [20].

In Chapter 4, we show the existence of a finite-dimensional manifold containing the attractor which attracts all solutions of (1.1) at an exponential rate. This manifold, known as an inertial manifold, is the equivalent of a global center manifold for an infinite-dimensional system. The restriction of (1.1) to the inertial manifold yields a finite-dimensional system which exhibits all of the large time dynamics of (1.1). Although in general we cannot actually compute the inertial manifold or the equations on the manifold, the existence of these objects tells us that the dynamics of (1.1) are essentially finite-dimensional. In the case where ϵ is large, however, we provide proof that the large time dynamics are equivalent to the dynamics of a simple one-dimensional ordinary differential equation. We also prove the existence of a finite-dimensional manifold, known as a steady inertial manifold, which contains the equilibria for (1.1) and is, in a sense, close to the attractor. The steady inertial manifold is of low dimension and can be approximated by a sequence of manifolds, known as approximate inertial manifolds (AIMs), which can be explicitly computed. These approximations provide us with low-dimensional systems of equations which allow us to extend the local bifurcation results to larger regions in parameter space and locate the unstable manifolds of equilibria.

In Chapter 5, we use the results of the preceding chapters and simulations based on standard numerical methods to obtain a qualitative picture of subsets of the attractor in a limited parameter regime. We provide evidence for a global bifurcation picture and obtain approximations of the orbits which connect the equilibria. From the numerical evidence we obtain in this chapter, we conjecture the attractor contains a set of relatively simple two-dimensional structures.

Finally in Chapter 6, we summarize our results, especially noting the apparent differences between the dynamics for (1.1) and (1.4). We also provide directions for future investigations.

CHAPTER 2

PRELIMINARY RESULTS

Introduction

In this chapter, we gather some preliminary mathematical results for the reaction-diffusion equation with nonlocal inhibition,

$$u_t = \epsilon^2 u_{xx} + \gamma Bu + f(u), \quad u_x(0,t) = u_x(1,t) = 0, \tag{2.1}$$

where $\gamma \geq 0$. This equation corresponds to the limiting equation as $\delta \to 0$, with ϵ fixed for the singularly perturbed reaction-diffusion system,

$$\frac{du}{dt} = \epsilon^2 u_{xx} + f(u) - w, \quad u_x(0,t) = u_x(1,t) = 0, \tag{2.2}$$

$$\delta\frac{dw}{dt} = Dw_{xx} + \gamma u - w, \quad w_x(0,t) = w_x(1,t) = 0. \tag{2.3}$$

To see this, we note that

$$Dw_{xx} + \gamma u - w = 0 \tag{2.4}$$

can be solved for w as a function of u , $w = -\gamma Bu$, where B is the nonlocal operator $Bu(x) = \int_0^1 g(x,s)u(s)ds$ with the Green's function,

$$g(x,s) = \begin{cases} -\frac{\sqrt{D}cosh(\frac{x}{\sqrt{D}})cosh(\frac{s-1}{\sqrt{D}})}{sinh(\frac{1}{\sqrt{D}})}, & 0 \leq x < s \leq 1, \\ -\frac{\sqrt{D}cosh(\frac{x-1}{\sqrt{D}})cosh(\frac{s}{\sqrt{D}})}{sinh(\frac{1}{\sqrt{D}})}, & 0 \leq s < x \leq 1. \end{cases}$$

First, we introduce some standard concepts from linear functional analysis. Then we will prove the existence of solutions and a global attractor for equation (2.1).

Operator Theory and Function Spaces

In the following, we will let $(\cdot,\cdot)_0$ and $||\cdot||_0$ denote the usual inner-product and norm on the Hilbert space $L^2(0,1)$ and $||\cdot||$ the operator norm on $L^2(0,1)$. When referring to an arbitrary normed space or Hilbert space X, we will denote the norm, and inner-product by $||\cdot||_X$, and $(\cdot,\cdot)_X$, respectively. By H^k_N we denote the closure of the cosine functions $\{cos(n\pi x)\}_{n=0}^{\infty}$ in the Sobolev space $H^k(0,1)$. For $k>1$ this is simply the elements in $H^k(0,1)$ which satisfy the homogeneous Neumann boundary conditions in the classical sense (see [1]). For $\gamma>0$, let $A: D(A)\subset L^2(0,1)\to L^2(0,1)$ be defined by

$$D(A) = H^2_N, \tag{2.5}$$

$$Au = -\epsilon^2 u'' - \gamma Bu, \quad \forall u \in D(A). \tag{2.6}$$

We see that equation (2.1) can be written as the abstract evolution equation

$$u_t = -Au + f(u), \quad u(0) = u_0. \tag{2.7}$$

It follows from standard results (see [7]) that the operator A is closed, densely defined, positive, and self-adjoint. The spectrum of A consists solely of eigenvalues, which can easily be seen to be

$$\sigma_n = \epsilon^2(n\pi)^2 + \frac{\gamma}{D(n\pi)^2+1}, \quad n = 0,1,\ldots, \tag{2.8}$$

with normalized eigenfunctions

$$\phi_0(x) = 1, \quad \phi_n(x) = \sqrt{2}cos(n\pi x), \quad n = 1,2,\ldots.$$

From the spectral theorem (see [9]) it follows that

$$A^{-1}u = \sum_{n=0}^{\infty} \sigma_n^{-1}(u,\phi_n)_0\phi_n(x). \tag{2.9}$$

Thus, A^{-1} is the uniform limit of finite-dimensional operators and hence is compact. Now for any $\beta \in \mathbb{R}$ we define A^β by

$$D(A^\beta) = \left\{ u \in L^2(0,1) : \sum_{n=0}^{\infty} \sigma_n^{2\beta} |(u, \phi_n)_0|^2 < \infty \right\}$$

and

$$A^\beta u = \sum_{n=0}^{\infty} \sigma_n^\beta (u, \phi_n)_0 \phi_n(x).$$

Equipped with the graph norm, $||\cdot||_{2\beta} = ||A^\beta(\cdot)||_0$, $D(A^\beta)$ becomes a Banach space. We will need the following embedding result which is proven in [16].

Lemma 2.1 *The following embeddings hold:*

$$D(A^\alpha) \subset D(A^\beta), \quad when, \quad 0 \leq \beta \leq \alpha. \tag{2.10}$$

$$D(A^\alpha) \subset H^1(0,1), \quad when, \quad \frac{1}{2} < \alpha. \tag{2.11}$$

$$D(A^\alpha) \subset C[0,1], \quad when \quad \frac{1}{2} < 2\alpha. \tag{2.12}$$

Moreover, the embedding (2.10) is compact when the inequality is strict.

The spaces $(D(A), ||\cdot||_2)$, $\left(D(A^{1/2}), ||\cdot||_1\right)$ and $(L^2(0,1), ||\cdot||_0)$ provide us a convenient setting to analyze the nonlocal reaction-diffusion equation. The following lemma gives Poincaré type inequalities for these spaces. Note the stated form of the eigenvalues of A in equation (2.8) does not guarantee the smallest positive or principle eigenvalue of A is σ_0. In the following, σ_p will denote the principle eigenvalue.

Lemma 2.2 *The following inequalities hold:*

$$\sqrt{\sigma_p}||u||_0 \leq ||u||_1, \quad \forall u \in D(A^{1/2}). \tag{2.13}$$

$$\sqrt{\sigma_p}||u||_1 \leq ||u||_2, \quad \forall u \in D(A). \tag{2.14}$$

Proof: Let $u \in D(A^{1/2})$, then $u = \sum_{n=0}^{\infty} u_n \phi_n$ and $A^{1/2}u = \sum_{n=0}^{\infty} \sqrt{\sigma_n} u_n \phi_n$, hence

$$||u||_1^2 = \sum_{n=0}^{\infty} \sigma_n u_n^2 \geq \sigma_p \sum_{n=0}^{\infty} u_n^2 = \sigma_p ||u||_0^2,$$

and the first inequality is proved. To establish (2.14), let $u \in D(A)$ and $Au = \sum_{n=0}^{\infty} a_n \phi_n$. Then $u = \sum_{n=0}^{\infty} \frac{a_n}{\sigma_n} \phi_n$ and

$$||u||_1^2 = \sum_{n=0}^{\infty} \frac{a_n}{\sigma_n}^2 \leq \frac{1}{\sigma_p} \sum_{n=0}^{\infty} a_n^2 \leq \frac{1}{\sigma_p} ||u||_2^2.$$

□

Definition 2.3 *An* **analytic semigroup** *on a Banach space X is a family of continuous linear operators on X, $\{T(t)\}_{t \geq 0}$, satisfying:*

1. $T(0) = I$, $T(t)T(s) = T(t+s)$ *for* $t, s \geq 0$.
2. $T(t)u \to u$ *as* $t \to 0^+$ *for each* $u \in X$.
3. *The map* $t \to T(t)u$ *is real analytic on* $0 < t < \infty$ *for each* $u \in X$.

The **infinitesimal generator** *of a semigroup $T(t)$ is the linear operator L defined by*

$$Lu = \lim_{t \to 0^+} \frac{1}{t}(T(t)u - u),$$

with domain $D(L)$ consisting of all $u \in X$ for which this limit exists. It is standard notation to write $T(t) = e^{Lt}$.

The following theorem, which is proven in [25], provides sufficient conditions for L to be the generator of an analytic semigroup.

Theorem 2.4 *Let L be a closed, densely defined, linear operator on a Banach space X. If there exist constants $\theta \in (0, \frac{\pi}{2})$, and $M > 0$ such that*

$$\Sigma = \left\{ \lambda \in \mathbb{C} : |arg\lambda| \leq \frac{\pi}{2} + \theta \right\} \cup \{0\} \subset \rho(L), \tag{2.15}$$

were $\rho(L)$ is the resolvent set of L, and

$$||(\lambda I - L)^{-1}|| \leq \frac{M}{|\lambda|}, \quad \forall \lambda \in \Sigma, \quad \lambda \neq 0. \tag{2.16}$$

Then L is the infinitesimal generator of an analytic semigroup on X.

An operator which satisfies the hypotheses of the theorem is called a sectorial operator. To see that $-A$ generates an analytic semigroup on $L^2(0,1)$, note that the spectrum of $-A$ consists of real, negative eigenvalues. Thus, (2.15) holds for any $\theta \in (0, \frac{\pi}{2})$. To verify (2.16), consider the spectral representation of the resolvent operator,

$$(\lambda I + A)^{-1}u = \sum_{n=0}^{\infty} \frac{(u, \phi_n)_0}{\lambda + \sigma_n} \phi_n(x).$$

From this we see

$$||(\lambda I + A)^{-1}u||_0 \leq \sum_{n=0}^{\infty} \frac{|(u, \phi_n)_0|}{|\lambda + \sigma_n|}, \tag{2.17}$$

but for all $\theta \in (0, \frac{\pi}{2})$ and $\lambda \neq 0$ such that $|arg\lambda| < \frac{\pi}{2} + \theta$,

$$\frac{1}{|\lambda + \sigma_n|} \leq \frac{csc(\frac{\pi}{2} + \theta)}{|\lambda|}, \quad \forall n.$$

Therefore,

$$||(\lambda I + A)^{-1}|| \leq \frac{csc(\frac{\pi}{2} + \theta)}{|\lambda|}, \quad \lambda \in \Sigma. \tag{2.18}$$

The semigroup e^{-At} has a simple representation

$$e^{-At}u = \sum_{n=0}^{\infty} e^{-\sigma_n t}(u, \phi_n)_0 \phi_n(x)$$

from which we obtain the uniform decay estimate, $||e^{-At}|| \leq e^{-\sigma_p t}$, for all $t \geq 0$.

We will need the following standard result, which is proven in [25].

Lemma 2.5 *Let $-A$ be the infinitesimal generator of an analytic semigroup e^{-At} on a Hilbert space H with $0 \in \rho(A)$. Also, assume $||e^{-At}|| \leq Me^{-\omega t}$ for some $M > 0$ and $\omega \in \mathbb{R}$, where $||\cdot||$ is the operator norm on H. Then:*

1. *$e^{-At} : H \mapsto D(A^\alpha)$ for every $t > 0$ and $\alpha \geq 0$.*

2. *For every $u \in D(A^\alpha)$, $A^\alpha e^{-At}u = e^{-At}A^\alpha u$.*

3. *For every $t > 0$ the operator $A^\alpha e^{-At}$ is bounded and $||A^\alpha e^{-At}|| \leq M_\alpha t^{-\alpha} e^{-\omega t}$, for some $M_\alpha > 0$.*

Existence of Solutions and the Global Attractor

To discuss the existence of solutions and a global attractor for equation (2.7) we need the notion of a nonlinear semigroup, which acts as the solution operator for (2.7). The results here essentially follow the presentation in [14].

Definition 2.6 *Let X be a complete metric space. A family of mappings $S(t) : X \to X$, $t \geq 0$, is said to be a C^r-***semigroup***, $r \geq 0$, provided :*

1. $S(0) = I$.

2. $S(t+s) = S(t)S(s), \quad s,t \geq 0.$

3. *$S(t)u$ is continuous in t and u together with Frechét derivatives in u through order r for $(t,u) \in [0,\infty) \times X$.*

For completeness we recall the definition of the Frechét derivative.

Definition 2.7 *Let X and Y be Banach spaces. The* **Frechét derivative** *of an operator $T : X \to Y$ at a point $x \in X$ is a continuous linear operator $L : X \to Y$ such that*

$$T(x+h) - T(x) = Lh + R(x,h), \tag{2.19}$$

where $\frac{||R(x,h)||_Y}{||h||_X} \to 0$ as $||h||_X \to 0$.

To fix our notation, throughout this work we will denote the Frechét derivative with respect to a variable x of a mapping T at point x_1 by $D_x T(x_1)$.

In the following definitions, we assume $\{S(t), t \geq 0\}$ is a semigroup on a Banach space X.

Definition 2.8 *The* ω-**limit set** *of a point $u_0 \in X$ and a subset $B \subset X$ are defined as*

$$\omega(u_0) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)u_0},$$

$$\omega(B) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)B},$$

where $\overline{\cdot}$ *denotes the closure of the set.*

Definition 2.9 *A subset* $B \subset X$ *is said to be* **invariant** *if*

$$S(t)B = B, \quad \forall t \geq 0.$$

Example 2.10 *A point* $u \in X$ *is an equilibrium point for a semigroup* $S(t)$ *on* X *if and only if* $S(t)u = u$ *for all* $t \geq 0$. *Thus we see equilibria are invariant sets.*

Other examples of invariant sets include the stable and unstable manifolds of equilibria, whose definition we now provide.

Definition 2.11 *The* **stable manifold**, $W^s(u)$, *and the* **unstable manifold**, $W^u(u)$, *of an equilibrium point* u *for* $S(t)$ *are defined as*

$$W^s(u) = \{x \in X : S(t)x \;\; is \;\; defined \;\; \forall t \geq 0, \;\; and \;\; S(t)x \rightarrow u \;\; as \;\; t \rightarrow \infty\},$$

$$W^u(u) = \{x \in X : S(t)x \;\; is \;\; defined \;\; \forall t \leq 0, \;\; and \;\; S(t)x \rightarrow u \;\; as \;\; t \rightarrow -\infty\}.$$

Definition 2.12 *A set* $\mathcal{B} \subset X$ *is said to* **attract** *a set* $U \subset X$ *under* $S(t)$ *if* $dist(S(t)U, \mathcal{B}) \rightarrow 0$ *as* $t \rightarrow \infty$.

Definition 2.13 *An invariant set* $\mathcal{A}$ *is said to be the* **global attractor** *for a semigroup* $S(t)$ *if* $\mathcal{A}$ *is a maximal, compact, invariant set which attracts each bounded set* $U \subset X$.

If a system has a global attractor, then the ω-limit set of each bounded set is contained in the global attractor. Therefore, much of the dynamic behavior of the system is determined by the restriction of the semigroup to the attractor. Although transient behavior may be important, understanding the dynamics on the attractor is a natural place to begin exploring the global dynamics of the evolution equation.

The notion of dissipation is an important property of a dynamical system which roughly corresponds to the lack of a conservation law for the system. The particular form of dissipation we will use, point dissipation, is defined as follows:

Definition 2.14 *A semigroup $S(t)$ is said to be* **point dissipative** *if there is a bounded set $\mathcal{B} \subset X$ that attracts each point of X under $S(t)$.*

For finite-dimensional systems, point dissipation is sufficient to guarantee the existence of a global attractor. This is due to the fact that finite-dimensional Banach spaces are locally compact. Since infinite-dimensional Banach spaces are not locally compact, we need a degree of compactness from the semigroup itself to obtain the desired results. Many semigroups generated by evolution equations possess a "smoothing" property which is equivalent to the notion of compactness of the operator.

Definition 2.15 *The semigroup $S(t)$ is said to be* **eventually compact** *if for each bounded set B, there exists a T such that $\bigcup_{t\geq T} S(t)B$ is relatively compact in X.*

The following theorem, whose proof can be found in [14], provides sufficient conditions for the existence of a global attractor for point dissipative systems.

Theorem 2.16 *Let $S(t)$ be an eventually compact, point dissipative C^r-semigroup on a complete metric space X such that for each bounded set $U \in X$, the set*

$$\{S(t)u : t \geq 0, u \in U\}$$

is bounded. Then there exists a global attractor $\mathcal{A}$. If X is a Banach space, then $\mathcal{A}$ is connected. If, in addition, $S(t)$ is one-to-one on $\mathcal{A}$ then $S(t)|_{\mathcal{A}}$ is a C^r-group.

Before proceeding to verify the hypotheses of the above theorem, we need to show the existence of a semigroup for equation (2.7). Let $f^e : D(A) \to L^2(0,1)$ be defined by $f^e(u)(x) = f(u(x))$. By (2.12), given $r_1 > 0$ there exists an $r_2 > 0$ so that for $u, v \in D(A^{1/2})$ with $||u||_1, ||v||_1 \leq r_1$ we have $|u|, |v| \leq r_2$ for all $x \in [0,1]$. Now

$$||f(u) - f(v)||_0^2 = \int_0^1 [f(u(x)) - f(v(x))]^2 \, dx, \tag{2.20}$$

$$= \int_0^1 \left[\int_{v(x)}^{u(x)} f'(s) ds \right]^2 dx. \tag{2.21}$$

But there exists a constant c which depends on r_2 such that

$$\left| \int_{v(x)}^{u(x)} f'(s) ds \right| \leq c|u(x) - v(x)|, \quad \forall x \in [0,1]$$

and therefore by Lemma 2.2,

$$||f^e(u) - f^e(v)||_0^2 \leq c^2 \int_0^1 [u(x) - v(x)]^2 \, dx \leq \frac{c^2}{\sigma_p} ||u - v||_1^2.$$

Thus, f^e is locally Lipschitz continuous. In the following, we will not distinguish between f as a function on the real numbers and f^e as a mapping between function spaces, rather we will denote both by f. The equation is now locally well posed in $D(A^{1/2})$.

Lemma 2.17 *For initial data, $u_0 \in D(A^{1/2})$, the evolution equation (2.7) has a unique local solution in $C((0,T), D(A)) \cap C([0,T), D(A^{1/2}))$. Moreover, either $T = \infty$ or $||u(t)||_1 \to \infty$ as $t \to T$. Also, the mapping $t \mapsto \frac{du(t)}{dt} \in D(A^{1/2})$ is locally Hölder continuous on $(0,T)$.*

Proof: The proof follows from Theorems 3.3.3, 3.3.4, and 3.5.2 of [16]. □

To show the solution is globally defined, it suffices to show that solutions remain bounded in $D(A^{1/2})$ as $t \to T$. To see this, consider the function $V : D(A^{1/2}) \to \mathbb{R}$ defined by

$$V(u) = \frac{1}{2}||u||_1^2 + \int_0^1 \left(\frac{u^4}{4} - \frac{u^2}{2} \right) dx.$$

If $u(t)$ is a local solution on $[0,T)$, then

$$\frac{dV(u)}{dt} = \left(A^{1/2}u, A^{1/2}\frac{du}{dt}\right)_0 - \left(f(u), \frac{du}{dt}\right)_0, \tag{2.22}$$

$$= \left(Au, \frac{du}{dt}\right)_0 - \left(f(u), \frac{du}{dt}\right)_0, \tag{2.23}$$

$$= \left(Au - f(u), \frac{du}{dt}\right)_0, \tag{2.24}$$

$$= -||\frac{du}{dt}||_0^2 \leq 0. \tag{2.25}$$

Thus, V is a nonincreasing function of t along local solutions. Now for each $k > 0$ there exists a C_k so that $\frac{s^2}{2} - \frac{s^4}{4} < ks^2 + C_k, \quad \forall s \in \mathbb{R}$. Therefore,

$$V(u) \geq \frac{1}{2}||u||_1^2 - k||u||_0^2 - C_k, \tag{2.26}$$

and by Lemma 2.2,

$$V(u) + C_k \geq \frac{1}{2}||u||_1^2 - \frac{k}{\sigma_p}||u||_1^2. \tag{2.27}$$

Choosing $k < \frac{\sigma_p}{2}$ we have

$$V(u(0)) + C_k \geq V(u(t)) + C_k \geq (\frac{1}{2} - \frac{k}{\sigma_p})||u||_1^2 \geq 0, \quad t \in [0,T), \tag{2.28}$$

and the solution is uniformly bounded on $[0,T)$. Therefore, by Lemma 2.17, solutions are defined for all time and the equation generates a nonlinear semigroup $\mathcal{S}(t)$ on $D(A^{1/2})$.

The function V defined above is known as a Lyapunov function, and for completeness we provide the following definition.

Definition 2.18 *A* **Lyapunov function** *for a semigroup $S(t)$ on a metric space X is a continuous real valued function $V : X \to \mathbb{R}$ such that*

$$\frac{dV(u_0)}{dt} = \limsup_{t\to 0+} \frac{1}{t}\left\{V(S(t)u_0) - V(u_0)\right\} \leq 0, \quad \forall u_0 \in X.$$

To show the existence of a global attractor, it remains to verify that $\mathcal{S}(t)$ is compact and point dissipative. To verify the compactness of $\mathcal{S}(t)$, it suffices to show that orbits remain bounded in $D(A^{\beta})$, for $\frac{1}{2} < \beta < 1$. The relative compactness of the orbits follows from the compactness of the embedding $D(A^{\beta}) \subset D(A^{1/2})$. To see that solutions of (2.7) are bounded in $D(A^{\beta})$, note that if $u(t)$ is a solution with $u(0) = u_0$, then

$$\begin{aligned} A^{\beta}u(t) &= A^{\beta}e^{-At}u_0 + \int_0^t A^{\beta}e^{-A(t-s)}f(u(s))ds, && (2.29)\\ &= A^{\beta-1/2}e^{-At}A^{1/2}u_0 + \int_0^t A^{\beta}e^{-A(t-s)}f(u(s))ds, && (2.30)\end{aligned}$$

and by Lemma 2.5,

$$||A^{\beta}u(t)||_0 \leq M_{\beta-1/2}t^{\beta-1/2}e^{-\sigma_p t}||u_0||_1 + M_{\beta}\int_0^t (t-s)^{-\beta}e^{-\sigma_p(t-s)}||f(u(s))||_0 ds.$$

Since $u(t)$ is bounded in $D(A^{1/2})$ and $f : D(A^{1/2}) \to L^2(0,1)$ is locally Lipschitz, there is a constant C depending on u_0 such that $||f(u(t))||_0 \leq C$ for all $t \geq 0$. Therefore,

$$||A^{\beta}u(t)||_0 \leq M_{\beta-1/2}t^{\beta-1/2}e^{-\sigma_p t}||u_0||_1 + M_{\beta}C\int_0^t (t-s)^{-\beta}e^{-\sigma_p(t-s)}ds,$$

which is bounded for all $t > 0$. Hence, the nonlinear semigroup is compact.

To verify the semigroup is point dissipative, we need the following result which can be found in [16].

Lemma 2.19 *Let $S(t)$ be a semigroup on a Banach space X. Suppose $u_0 \in X$ and $\{S(t)u_0, t \geq 0\}$ lies in a compact set in X, then $\omega(u_0)$ is nonempty, compact, connected, invariant and $dist(S(t)u_0, \omega(u_0)) \to 0$ as $t \to +\infty$.*

Now define $E = \left\{u \in D(A^{1/2}) : \frac{dV(S(t)u)}{dt} = 0\right\}$ and let $\mathcal{M}$ be the maximal invariant set of E. The set $\mathcal{M}$ will be our candidate for a bounded attracting set. To see that $\mathcal{M}$ is bounded in $D(A^{1/2})$, first note that by equation (2.25), $u \in E$ if and only if u is an equilibrium solution. Therefore,

$$||u||_1^2 = (A^{1/2}u, A^{1/2}u)_0 = (u, Au)_0 = (u, f(u))_0 = \int_0^1 \left(u^2 - u^4\right) dx.$$

But $s^2 - s^4 \leq \frac{1}{4}$, for all $s \in \mathbb{R}$ and thus $||u||_1^2 \leq \frac{1}{4}$ for all $u \in E$. By the estimate of equation (2.27) with $k < \frac{\sigma_p}{2}$, we see that V is bounded below. Hence, $V(\mathcal{S}(t)u_0)$ is a continuous nonincreasing function that is bounded below and therefore $l(u_0) \equiv \lim_{t\to\infty} V(\mathcal{S}(t)u_0)$ exists. If $y \in \omega(u_0)$, then $V(y) = l(u_0)$ and so $V(\mathcal{S}(t)y) = l(u_0)$ for all $t \in \mathbb{R}$. Thus, $y \in E$ and $\omega(u_0) \in \mathcal{M}$. It follows that $\mathcal{M}$ is globally attracting and the semigroup is point dissipative. We have proven the following theorem:

Theorem 2.20 *The nonlocal reaction-diffusion equation has a global attractor $\mathcal{A}$ which is connected in $D(A^{1/2})$.*

The semigroup for the evolution equation falls into a special class of systems, known as gradient systems, for which the flow on the attractor can be described with some detail.

Definition 2.21 *A C^r-semigroup $S(t)$ on a Banach space X is said to be a gradient system if:*

1. *Each bounded positive orbit is relatively compact.*
2. *There is a Lyapunov function $V : X \to \mathbb{R}$ for $S(t)$ such that,*
   - *(a) V is bounded below.*
   - *(b) $V(u) \to \infty$ as $||u||_X \to \infty$.*
   - *(c) $V(S(t)u)$ is nonincreasing in t for each $u \in X$.*
   - *(d) If u is such that $V(S(t)u) = V(u)$ for all $t \in \mathbb{R}$ then u is an equilibrium.*

Since the semigroup $\mathcal{S}(t)$ generated by (2.7) is a gradient system, it is known (see [14]) the attractor consists of the unstable manifolds of the set E, that is

$$\mathcal{A} = \left\{ u \in D(A^{1/2}) : \mathcal{S}(-t)u \quad exists \quad for \quad t \geq 0 \quad and \quad \mathcal{S}(-t)u \to E \quad as \quad t \to \infty \right\}.$$

Definition 2.22 *An equilibrium u_0 for a semigroup $S(t)$ is said to be* **hyperbolic** *if the spectrum of $D_u S(t)u_0$ does not intersect the unit circle centered at zero in $\mathbb{C}$.*

For $\mathcal{S}(t)$ this definition is equivalent to the condition that zero is not in the spectrum of the linear operator $\mathcal{L} : D(A) \to L^2(0,1)$ defined by

$$\mathcal{L}\phi \equiv -A\phi + f'(u_0)\phi, \quad \phi \in D(A). \tag{2.31}$$

If E consists of hyperbolic equilibria then it is necessarily finite and

$$\mathcal{A} = \bigcup \{W^u(\phi) : \phi \in E\}.$$

Away from bifurcations all equilibria of equation (2.7) are hyperbolic, hence a characterization of the attractor is complete when all of the equilibria and unstable manifolds are found.

CHAPTER 3

BIFURCATION RESULTS

Introduction

Due to the gradient structure of the equation (2.1), we know that when the equilibria are hyperbolic the attractor consists of the equilibria and their unstable manifolds. Therefore, the characterization of the attractor is complete when the equilibria and connecting orbits are found. Our equation is a nonlocal perturbation of the standard Chafee-Infante problem

$$u_t = \epsilon^2 u_{xx} + f(u), \quad u_x(0,t) = u_x(1,t) = 0, \tag{3.1}$$

for which a great deal is known. In fact, for any value of ϵ a complete description of the attractor can be given and a summary of results can be found in [14] and [16]. The steady state equation is a planar system which can easily be analyzed and all equilibria located. Spatially heterogeneous solutions occur as bifurcations from the zero solution at $\epsilon = \frac{1}{n\pi}$, for $n = 1, 2, \ldots$. For each n, the bifurcation is a pitchfork which yields a pair of solutions which are $n\pi$-periodic and continue to large amplitude patterns having n internal transition layers as $\epsilon \to 0$. These solutions are unstable with the dimension of the unstable manifolds being equal to the number of internal transition layers. It has been shown that for two equilibria u^+, u^- a heteroclinic connection between them exists if and only if $dim\,(W^u(u^-)) \neq dim\,(W^u(u^+))$ and the flow proceeds in the direction as to decrease the number of transition layers. Moreover, if u^- and u^+ are hyperbolic equilibria and $dim\,(W^u(u^-)) > dim\,(W^u(u^+))$, then $W^u(u^-)$ and $W^s(u^+)$ intersect transversally. For small values of γ we should expect the structure of the attractor for (2.1) to be similar to that of (3.1). To see this, note that for ϵ fixed

$$\mathcal{F}(u, \epsilon, \gamma) = \epsilon^2 u'' + f(u) + \gamma Bu \tag{3.2}$$

is a C^2 map from $D(A) \times \mathbb{R}$ into $L^2(0,1)$. It is in fact analytic. If u_0 is a hyperbolic equilibria of (3.1) then $\mathcal{F}(u_0, \epsilon, 0) = 0$, $D_u\mathcal{F}(u, \epsilon, \gamma)$ is continuous in u and γ in a neighborhood of $(u_0, 0)$ and $D_u\mathcal{F}(u_0, \epsilon, 0)$ is nonsingular. Therefore, by the implicit function theorem (see [31]) there is a solution curve $(u(\gamma), \gamma)$ passing through $(u_0, 0)$ for γ small. This curve is analytic in γ and can be written as $u(x, \gamma) = u_0(x) + \gamma u_1(x, \gamma)$. Thus, the equilibria perturb smoothly. Now since $\mathcal{F}$ is analytic in γ, so is the nonlinear semigroup $\mathcal{S}(t)$ (see [16]). Assume $u_0^-(x)$ and $u_0^+(x)$ are hyperbolic equilibria of (3.1) and a heteroclinic connection exists between them. Let $u^-(x, \gamma) = u_0^-(x) + \gamma u_1^-(x, \gamma)$ and $u(x, \gamma)^+ = u_0^+(x) + \gamma u_1^+(x, \gamma)$ be solutions of (3.2), through $(u_0^-, 0)$ and $(u_0^+, 0)$, respectively. Assume also that for all γ sufficiently small these equilibria remain hyperbolic. Then the heteroclinic connection, being the transverse intersection of the stable and unstable manifolds, perturbs continuously for all γ sufficiently small. That is, the connection perturbs to a connection between the equilibria of (2.1).

Unfortunately, when γ is large the nonlocal perturbation γB complicates matters significantly. No longer can we locate the equilibria by phase plane analysis. The results concerning heteroclinic connections for the Chafee-Infante problem rely on the strong maximum principle which does not hold for the operator A when $\gamma > 0$. The goal of this work is to begin a program of characterizing the global attractor for the case where γ is not necessarily small. In this chapter, we perform local bifurcation analysis in order to locate equilibria. As we will see, the global bifurcation picture for (2.1) promises to be much more interesting than that of the Chafee-Infante problem in that it includes secondary bifurcations.

Bifurcations from Simple Eigenvalues

The main bifurcation result we will use is a standard theorem of Crandall and Rabinowitz [6] concerning bifurcations from a simple eigenvalue of Fredholm operators.

Definition 3.1 *Let X and Y be Banach spaces. A bounded linear operator $L : X \to Y$ is called* **Fredholm** *if the following two conditions hold:*

1. *The null space of L, $N(L)$, is a finite-dimensional subspace of X.*

2. *The range of L, $R(L)$, is closed and has finite codimension.*

The **Fredholm index** *of L is the integer*

$$ind(L) = dim\,(N(L)) - codim\,(R(L))\,. \tag{3.3}$$

We will need the following result, whose proof can be found in [31]. The first two statements of the theorem are known as the Fredholm alternative, which concerns the existence of solutions of the equation

$$Lx = y \tag{3.4}$$

for Fredholm operators. Although the result can be stated for operators on Banach spaces, we will use a version for operators on Hilbert spaces X and Y.

Theorem 3.2 *For a Fredholm operator $L : X \to Y$ the following are true:*

1. *If $ind(L) = 0$ and $N(L) = 0$ then (3.4) has exactly one solution for every $y \in Y$ and $L^{-1} : Y \to X$ is a bounded linear operator.*

2. For $y \in Y$, (3.4) has a solution if and only if $(y^*, y)_Y = 0$ for all $y^* \in N(L^*)$, where $L^* : Y \to X$ is the adjoint of L.

3. The adjoint operator L^* is also Fredholm and

$$dim(N(L^*)) = codim(R(L)), \tag{3.5}$$

$$codim(R(L^*)) = dim(N(L)), \tag{3.6}$$

$$ind(L^*) = -ind(L). \tag{3.7}$$

4. If $K : X \to Y$ is a linear operator that is compact or bounded, then $L + K$ is Fredholm and

$$ind(L + K) = ind(L). \tag{3.8}$$

Example 3.3 *It is easy to verify that the second derivative operator $\epsilon^2 \frac{d^2}{dx^2} : D(A) \to L^2(0,1)$ is a Fredholm operator with zero index. By the previous theorem, for any $a \in C[0,1]$ and compact operator $K : D(A) \to L^2(0,1)$, the operator $L : D(A) \to L^2(0,1)$ defined by*

$$Lu = \epsilon^2 u'' + a(x)u + Ku, \quad u \in D(A), \tag{3.9}$$

is a zero index Fredholm operator.

The following theorem of Crandall and Rabinowitz, provides sufficient conditions for bifurcations from known solutions. The proof of this result can be found in [6] and [31].

Theorem 3.4 (Crandall and Rabinowitz) *Let X and Y be Hilbert spaces, U an open neighborhood of $(0,0)$ in $X \times \mathbb{R}$ and $G : U \subset X \times \mathbb{R} \to Y$ a function which is C^2 at $(0,0)$. Assume also that $G(0,\alpha) = 0$ for all $\alpha \in (-\delta, \delta)$ for some $\delta > 0$, and $L \equiv D_x G(0,0)$ is a Fredholm operator with index zero. If*

$$dim(N(L)) = 1, \quad N(L) = span\{x_0\}, \quad and \tag{3.10}$$

$$L_1 x_0 \equiv D_\alpha D_x G(0,0) x_0 \notin R(L), \tag{3.11}$$

then $(0,0)$ *is a bifurcation point and there exists a curve* $s \mapsto (x(s), \alpha(s))$ *through* $(0,0)$ *for* $|s|$ *sufficiently small so that*

$$G(x(s), \alpha(s)) = 0. \tag{3.12}$$

In a sufficiently small neighborhood of $(0,0)$ *every solution is either on the curve or of the form* $(0,\alpha)$. *On the curve,* x *is given by* $x(s) = sx_0 + sy(s)$, *where* $(y(s), x_0)_X = 0$. *Moreover, if* G *is analytic at* $(0,0)$ *then the curve can be written as an absolutely convergent power series in* s,

$$x(s) = sx_0 + \sum_{k=2}^{\infty} s^k x_k, \quad (x_0, x_k)_X = 0, \quad k \neq 0, \tag{3.13}$$

$$\alpha(s) = \sum_{k=0}^{\infty} s^k \alpha_k. \tag{3.14}$$

Condition (3.10) means zero is a simple eigenvalue of L. The condition (3.11), known as the transversality condition, says the line $L_1(N(L))$ and the hyperplane $R(L)$ intersect transversely at the origin, meaning together they span the range space Y. This condition guarantees that at the bifurcation point the critical eigenvalue passes through zero with nonzero speed.

We will use this theorem to find bifurcations from the constant solutions of the steady state equation

$$\mathcal{F}(u, \epsilon, \gamma) = \epsilon^2 u'' + f(u) + \gamma B u = 0, u \in D(A). \tag{3.15}$$

The restriction of (3.15) to the subspace of constant functions is simply the polynomial equation

$$f(u) - \gamma u = 0. \tag{3.16}$$

Real solutions of (3.16) are the only constant solutions of (3.15). When $\gamma > 1$, the only real solution of (3.16) is the zero solution and as γ passes through 1 the zero solution under goes a pitchfork bifurcation which yields a pair of solutions , $m_0^{\pm} = \pm\sqrt{1-\gamma}$.

We will look for bifurcations from the constant solutions by fixing one parameter while allowing the other to play the role of the bifurcation parameter. First, consider bifurcations as ϵ is varied. Assume γ is fixed and u is a solution of (3.16). It follows that u is a solution for all ϵ. Now suppose at ϵ_0 , zero is a simple eigenvalue of the linear operator $\mathcal{L} : D(A) \to L^2(0,1)$ given by

$$\mathcal{L}\phi \equiv \epsilon_0^2\phi'' + f'(u)\phi + \gamma B\phi \tag{3.17}$$

and let $N(\mathcal{L}) = span\{u_0\}$. From Example 3.3, we see that $\mathcal{L}$ is a zero index Fredholm operator. Now define $G : D(A) \times \mathbb{R} \to L^2(0,1)$ by

$$G(v,\alpha) = (\epsilon_0 + \alpha)^2 v'' + f(u+v) + \gamma B(u+v). \tag{3.18}$$

Then G is analytic in both variables and

$$D_vG(0,0)\phi = \epsilon_0^2\phi'' + f'(u)\phi + \gamma B\phi = \mathcal{L}\phi, \tag{3.19}$$

$$\mathcal{L}_1\phi \equiv D_\alpha D_v G(0,0)\phi = 2\epsilon_0\phi''. \tag{3.20}$$

The eigenfunctions of $\mathcal{L}$ are $\{cos(n\pi x)\}_{n=0}^{\infty}$, and therefore u_0 is a nonzero scalar multiple of $cos(n\pi x)$ for some n. From this we see that

$$\mathcal{L}_1 u_0 = -2\epsilon_0(n\pi)^2 u_0 \in N(\mathcal{L}), \tag{3.21}$$

and the transversality condition holds provided $n \neq 0$. When $n = 0$, u_0 is a constant function and $\mathcal{L}u_0 = 0$ implies

$$0 = \mathcal{L}u_0 = f'(u)u_0 - \gamma u_0, \tag{3.22}$$

from which we have $f'(u) = \gamma$ and $u = \pm\sqrt{\frac{1-\gamma}{3}}$. However, $u = \pm\sqrt{\frac{1-\gamma}{3}}$ is a solution of (3.2) if and only if $\gamma = 1$ and $u = 0$. In this case, the constant functions are in $N(\mathcal{L})$ for all values of ϵ and Theorem 3.4 does not apply. This case will be considered later when we examine secondary bifurcations. Thus, we see under the assumption $\gamma \neq 1$, a sufficient condition for a bifurcation to occur at ϵ_0 is that $\mathcal{L}$ has a zero eigenvalue which is simple.

To consider the case where γ plays the role of the bifurcation parameter, let $u(\gamma)$ be a solution curve of (3.15) for γ in some open interval $S = (a, b)$, and ϵ fixed. Suppose at $\gamma_0 \in S$ zero is a simple eigenvalue of $\mathcal{L} : D(A) \to L^2(0,1)$ defined by

$$\mathcal{L}\phi \equiv \epsilon^2\phi'' + f'(u(\gamma_0))\phi + \gamma_0 B\phi. \tag{3.23}$$

Define $G : D(A) \times S \to L^2(0,1)$ by

$$G(v, \alpha) = \epsilon^2 v'' + f(u(\gamma_0 + \alpha) + v) + (\gamma_0 + \alpha)B(u(\gamma_0 + \alpha) + v). \tag{3.24}$$

Then

$$D_v G(0,0) = \mathcal{L}, \tag{3.25}$$

and $\mathcal{L}_1$ is given by

$$\mathcal{L}_1\phi \equiv f''(u(\gamma_0))\frac{\partial u(\gamma_0)}{\partial \gamma}\phi + \gamma_0 B\phi = \begin{cases} 3\phi + \gamma_0 B\phi, & u(\gamma_0) \neq 0, \\ \gamma_0 B\phi, & u(\gamma_0) = 0. \end{cases} \tag{3.26}$$

As in the previous case we see $\mathcal{L}_1 N(\mathcal{L}) \subset N(\mathcal{L})$ and thus the transversality condition holds.

Bifurcations from the Zero Solution

The natural place to begin looking for solutions is to search for those which bifurcate from the zero solution. We have already examined the case of the constant solutions, $m_0^{\pm} = \pm\sqrt{1-\gamma}$, bifurcating from zero. We will now consider bifurcations which result in nonconstant solutions. We shall soon see, as in the case for the Chafee-Infante problem, modal bifurcations from the zero solution occur in a predictable way as ϵ is decreased with γ fixed. However, we must also consider the dependence on the parameter γ.

Modal Bifurcations from the Zero Solution

Let

$$\mathcal{E} = \{(\gamma, \epsilon) \in \mathbb{R}^2 : \gamma \geq 0, \epsilon \geq 0\} \tag{3.27}$$

denote the parameter space for our bifurcation analysis. Linearizing (2.1) about the zero solution we obtain the eigenvalue problem

$$\mathcal{L}\phi \equiv \epsilon^2\phi'' + \phi + \gamma B\phi = \mu\phi, \quad \phi'(0) = \phi'(1) = 0, \tag{3.28}$$

which has solutions

$$\mu_n = 1 - \epsilon^2(n\pi)^2 - \frac{\gamma}{D(n\pi)^2 + 1}, \tag{3.29}$$

$$\phi_n = cos(n\pi x), \quad n = 0, 1, \ldots. \tag{3.30}$$

The equations, $\mu_n = 0$, define a family of curves $\{M_n\}_{n=0}^{\infty}$ in parameter space along which $\mathcal{L}$ has a zero eigenvalue. We will refer to these curves as the primary bifurcation curves. Specifically

$$M_n = \{(\gamma, \epsilon) \in \mathcal{E} : \gamma = (1 - (\epsilon n\pi)^2)(D(n\pi)^2 + 1)\}. \tag{3.31}$$

The curves M_0 through M_3 with $D = 1$ are plotted in Figure 1. For $m \neq n$, $M_n \cap M_m$ is nonempty and consists of a single point $C_{m,n}$ given by $C_{m,n} = (\gamma_{m,n}, \epsilon_{m,n})$ where

$$\gamma_{m,n} = \frac{(D(n\pi)^2 + 1)(D(m\pi)^2 + 1)}{D\pi^2(n^2 + m^2) + 1}, \tag{3.32}$$

$$\epsilon_{m,n} = \sqrt{\frac{D}{D(n\pi)^2 + D(m\pi)^2 + 1}}. \tag{3.33}$$

On the curve segments

$$M_n / \bigcup_{n=0}^{\infty} C_{m,n} \tag{3.34}$$

zero is a simple eigenvalue of $\mathcal{L}$ with $N(\mathcal{L}) = span\{cos(n\pi x)\}$. Theorem 3.4 holds and bifurcations occur as these curves are crossed transversally by varying one of the two parameters, ϵ or γ.

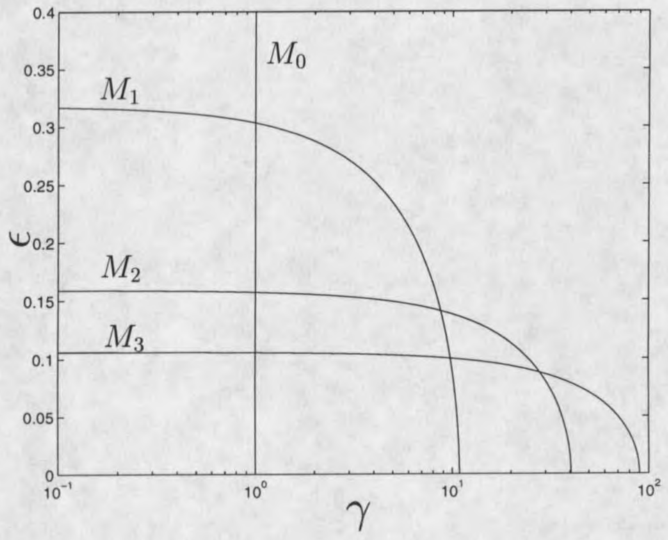

Figure 1: Primary bifurcation curves for modes 0, 1, 2 and 3.

At the intersection points, $C_{m,n}$, $N(\mathcal{L}) = span\{cos(n\pi x), cos(m\pi x)\}$ and we cannot use the Crandall-Rabinowitz theorem. As we shall see in a later section, this condition suggests the existence of secondary bifurcations.

Computation of the Local Solution Branch

From the bifurcation curves (3.31) we see that the zero solution of (3.15) can undergo a sequence of bifurcations which depends on the path taken in parameter space. To determine the local structure and stability of the bifurcating solutions, we will consider crossings of the bifurcation curves as ϵ is varied while fixing γ in such a way that we remain away from the intersection points. A similar analysis can be performed by fixing ϵ and allowing γ to vary. Consider a bifurcation of the n^{th} mode which occurs at $(\gamma, \epsilon_0) \in M_n / \bigcup_{n=0}^{\infty} C_{m,n}$. We will obtain the local solutions branch as a power series in a variable s by assuming $u = \sum_{k\geq 1} s^k u_k$, and $\epsilon = \epsilon_0 - \sum_{k\geq 1} s^k \epsilon_k$ for $|s| << 1$.

The $\mathcal{O}(s)$ equation,

$$\epsilon_0^2 u_1'' + u_1 + \gamma B u_1 = 0, \tag{3.35}$$

has an infinite number of solutions consisting of the space spanned by $cos(n\pi x)$. We will take $u_1 = cos(n\pi x)$, since without loss of generality we can simply rescale s. Collecting the $\mathcal{O}(s^2)$ terms yields the equation,

$$\epsilon_0^2 u_2'' + u_2 + \gamma B u_2 = 2\epsilon_0\epsilon_1 u_1'' = -2\epsilon_0\epsilon_1 n^2\pi^2 cos(n\pi x). \tag{3.36}$$

By the Fredholm alternative, this equation has a solution if and only if the right-hand side is orthogonal to $cos(n\pi x)$, from which we obtain $\epsilon_1 = 0$. With the additional requirement $(u_2, u_1)_0 = 0$ we have $u_2 = 0$. The $\mathcal{O}(s^3)$ equation is

$$\epsilon_0^2 u_3'' + u_3 + \gamma B u_3 = 2\epsilon_0\epsilon_2 u_1'' + u_1^3. \tag{3.37}$$

Expanding the u_1^3 term we obtain

$$2\epsilon_0\epsilon_2 u_1'' + u_1^3 = (\frac{3}{4} - 2\epsilon_0\epsilon_2 n^2\pi^2)cos(n\pi x) + \frac{1}{4}cos(3n\pi x), \tag{3.38}$$

and from the solvability condition we find

$$\epsilon_2 = \frac{3}{8}\frac{1}{n^2\pi^2\epsilon_0}.$$

Therefore, we have

$$u = s\cos(n\pi x) + \mathcal{O}(s^3), \tag{3.39}$$

$$\epsilon = \epsilon_0 - \frac{3}{8}\frac{1}{n^2\pi^2\epsilon_0}s^2 + \mathcal{O}(s^3), \tag{3.40}$$

for $|s|$ sufficiently small.

At the bifurcation value ϵ_0 the zero solution undergoes a pitchfork bifurcation. This bifurcation yields a pair of solutions $m_n^{\pm}$ such that $m_n^- = -m_n^+$ as depicted in Figure 2. When $0 \leq \gamma < 1$, for each mode there is an ϵ value at which a bifurcation