Using iconic modeling and technology with American Indian Reservation students
by Christine Lynne Larson

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Education in
Secondary Curriculum and Instruction
Montana State University
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Abstract:
The primary purpose of this study was to investigate the effectiveness of using concrete, semi-concrete, and abstract pre-algebra learning models, utilizing multiple embodiments on the achievement of 6th and 7th grade students. The secondary purposes of the study were to determine if there is any difference in achievement between American Indians and other students, between male and female students, and between sixth grade and seventh grade students. The three dependent variables were the difference between the pretest and posttest scores on the Orleans-Hanna Algebra Prognosis test (OHPT), the difference between pretest and posttest scores on the criterion-referenced test (CRT), and the posttest scores on criterion-referenced test. The independent variables were method of instruction, ethnic background, and gender.

Students involved in the study were from two Montana schools located on American Indian Reservations. The study was conducted over a two week period. Classes were randomly assigned to one of three treatments: a multiple embodiment sequence utilizing the computer as one embodiment, another sequence without the computer, and a traditional textbook sequence. A preinstruction test was given before instruction began, and a post instruction test was administered at the completion of the two week session.

A nonequivalent control-group design was used as the experimental model. Analysis of covariance was used to determine whether the sample means were significantly different from one another by adjusting the mean group scores to remove bias. Factorial analysis of variance was also used to determine the effect of two or more independent variables, in interaction with each other, on the dependent variable.

A significant difference was found in the mean scores of treatment groups. Both non-traditional treatment groups scored higher than the traditional textbook treatment, on the mean scores of the post instruction test and on the mean scores of the difference between the pretest and post instruction OHPT scores and the CRT scores. A significant difference was also found in the mean scores of the difference between pretest and post instruction test scores of females and males. Females scored higher than males on the least square mean scores.

The conclusions of the study are that students learn to solve a linear equation or a system of linear equations better using a multiple embodiment sequence.
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A thesis submitted in partial fulfillment
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APPROVAL

of a thesis submitted by

Christine Lynne Larson

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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CHAPTER I

INTRODUCTION

Introduction

The transition from arithmetic to algebra is difficult. This transition, which requires the use of abstract representations, is very seldom taught in mathematics classes. Therefore, many students experience failure in algebra. "Algebra has long been the gatekeeper for access to study of mathematics beyond arithmetic" (Burrill, Choate, Phillips, & Westegaard, 1993, p. 4). Past practices in teaching algebra have not been successful for most students. Algebra has been dominated by a goal to train students to manipulate symbols. Teaching students traditional algebraic topics in a traditional manner will not suddenly produce students who have the algebraic literacy necessary to compete in a technological world. In Algebra for the Twenty-first Century (1993), organizers state "Evidence supports that the teaching of mathematics in the past decades with an emphasis on rote memorization of disconnected symbol manipulation has not worked" (Burrill, et.al., 1993, p. 50). According to Carole Lacampagne (1995a), algebra is a "gateway to academic development and full participation in citizenship" (p. 10). Therefore, algebra for all can no longer be an abstract hope, but has become a necessary goal.

There have been many attempts over the decades to deal with the
problems students encounter in mathematics and specifically in algebra. In the 1960's, technological competition with the former Soviet Union spurred mathematics educators to design a major reconstruction of the scope and sequence of the mathematics curriculum. The innovations implemented known as "new math" were primarily aimed at providing a high quality mathematics education to college bound students (National Advisory Committee on Mathematical Education [NACME], 1975). Concern in the 1970's over the increased number of students registering for mathematics classes and low test scores caused schools to restructure the mathematics curriculum in an attempt to increase these scores (National Council of Teachers of Mathematics [NCTM], 1980). But in the 1980's educators realized that improving test scores was not a solution to the problem. While students may have shown a slight increase in basic arithmetic skills, emphasis on computational skills left serious gaps in the students' understanding of underlying concepts according to the National Assessment Educational Progress [NAEP]. The conclusions of the fourth mathematics assessment stated that "nearly 1.5 million 17-year-old students across the nation appear scarcely able to perform the kinds of numerical operations that will likely be required of them in future life and work settings" (Swafford & Brown, 1989a, p.106). Results from the NAEP's 1992 mathematics assessment indicate that although student performance is improving nationally, there is still "just over 60% of the students in grades 4, 8, and 12 estimated to be at or above the Basic level on the 1992 mathematics assessment" (p. 1).

The National Council of Teachers of Mathematics (NCTM) responded to this problem in its Agenda for Action: Recommendations for School Mathematics
of the 1980's stating that more mathematics should be required for all students and specifically suggested "At least three years of mathematics should be required in grades 9 through 12" (p. 20). In order for students to meet this three year expectation, they must receive a thorough mathematics education background prior to the ninth grade. However studies provide evidence that most current elementary mathematics programs concentrate on teaching computational skills (Hill, 1982; NACME, 1975). In fact, a large amount of the content found in textbooks is skipped in favor of more time to develop computational skills that are comfortable to and valued by elementary teachers (NACME, 1975; Chandler & Brosnan, 1994). Therefore a reevaluation of what is emphasized in grades 5-8 is crucial. Since elementary mathematics instruction in the United States focuses on arithmetic, most students in grades 5-8 find the current mathematics curriculum unnecessary, boring, and routine (NCTM, 1989). Students in other countries are exposed to a wider variety of mathematical topics (Hill, 1982).

Examining several current textbook series reveals a repetition of topics, which indicates little change in the material being presented during these four years (Grades 5-8). In an examination of mathematics textbooks in grades 5-8 before and after 1989, Donald Chandler and Patricia Brosnan (1994) found that while mathematics textbooks are changing, they note that “perhaps textbooks have not changed as much as they have grown” (p. 8). Chapters that do contain new topics are found at the end of textbooks and are more likely to be skipped because of the time spent primarily on improving computational skills (NCTM, 1989; NACME, 1975). Remediation results in students receiving a negative
image of mathematics along with a background of mathematical topics which provide insufficient readiness for secondary school mathematics. In the Curriculum and Evaluation Standards for School Mathematics, major changes in content and emphasis are recommended for grades 5-8, featuring a broad range of topics and the use of technology. Suggested topics include estimation, measurement, geometry, statistics, probability, patterns, functions, and the fundamental concepts of algebra.

First year algebra students are required to solve linear equations using abstract manipulations of symbols and numbers. Many of these students have never seen a linear equation before, have never dealt with abstract symbolism, and have no idea why the manipulations they are required to perform result in either a correct or an incorrect answer. Since the students do not understand the concrete basis behind the abstractions confronting them, they are forced to rely on blind manipulations and memorization to satisfy the instructor and answer the questions. Most first year algebra students have never been exposed to the abstract language of mathematics even though they have been studying mathematics. Since algebra is a high school topic, instructors of elementary school students may not see the need for their students to explore algebraic topics. A change in the attitudes of these elementary teachers is needed because the majority of students that are confronted in a seventh, eighth, or ninth grade algebra classroom without any prior knowledge of algebra or algebraic topics are destined to struggle and many fail. Many first year algebra students receive D's and F's (Marquis, 1989). These students are likely to become frustrated which causes further deterioration in their ability and may result in the termination of
their mathematical careers. NCTM has acknowledged this deficiency and in the NCTM Curriculum and Evaluation Standards recommends that "in grades 5-8, students explore algebraic concepts in an informal way to build a foundation for the subsequent formal study of algebra" (p. 102).

Learning theorists such as Jean Piaget, Zoltan Dienes, and Jerome Bruner have long championed the need for students to be actively involved in the learning process. The most important result that can be derived from Piaget's work is that young children learn best from concrete activities (Post, 1988). In grades 5-8, most students are still in Piaget's concrete operational stage and so need to learn new mathematical topics using concrete examples and experimentation (Bell, 1978). For these students, equation solving is a new and important algebraic topic, so they should be introduced to the concept using concrete investigations. NCTM (1980) advises the use of manipulatives to illustrate a concept whenever appropriate. Manipulatives that demonstrate the process used in solving equations would satisfy both the recommendations by Piaget and NCTM. Dienes found learning to be the most effective when children are presented a concept using a variety of physical contexts or embodiments (Post, 1988). One result of Bruner's work is that students' ability to see a concept from a variety of perspectives using several models is needed for students to construct their own understanding of the concept. Bruner, relying heavily on both Piaget and Dienes's influences, developed a model depicting the different levels that students need to experience when faced with a new concept (Post, 1988). These levels are the enactive level, which involves hands-on, direct experience, the iconic level, which is based on the use of visual aids such as
pictures, and finally the symbolic level, which uses abstract symbols to represent reality (Post, 1988). The computer can facilitate learning at each of these levels since it can be used as an interactive device, ideally suited to providing different types of expressions for a particular concept (Clements, 1989). Specifically, the computer can be an effective tool in exploring relationships between expressions in a mathematical sentence. Exploration should help students gain confidence in their ability to use algebra in solving problems; unfortunately, there is little evidence to substantiate this in the research.

Increasing all students' ability to use algebra is an important goal, but some groups are in dire need of attention. Women and minorities will account for a large part the workforce in the future. Yet these groups are severely underrepresented in fields such as mathematics and other technical fields. American Indian students are finishing school without the mathematical ability necessary for productive lives. Students whose formal education will end with high school need mathematical competence to deal with consumer and career needs (NCTM, 1980). In addition, American Indian students are not receiving the mathematical education necessary for advanced study at the college level. "American Indian students suffer from a disproportionately high drop out rate when compared to the general population, resulting in an insufficient number of American Indian/Alaska Native role models in mathematics and science teaching positions, and decision-makers within the educational system" (American Indian Science and Engineering Society (AISES), 1995, p. 8). "Failure to study mathematics can close the doors to vocational, technical schools, college majors, and careers" (p. 68) according to the NCTM (1989). An integral part of
mathematics is knowledge of and success in algebra. Most colleges require students to pass a basic mathematics course to graduate, which includes knowledge of algebraic concepts.

**Statement of the Problem**

The primary purpose of this study was to investigate the effectiveness of using concrete, semi-concrete, and abstract pre-algebra learning models, utilizing multiple embodiments on the achievement of 6th and 7th grade students. The secondary purposes of the study were to determine if there is any difference in achievement between American Indians and other students, between male and female students, and between sixth grade and seventh grade students. The three dependent variables were the difference between the pretest and posttest scores on the Orleans-Hanna Algebra Prognosis test, the difference between pretest and posttest scores on the criterion-referenced test, and the posttest scores on the criterion-referenced test. The independent variables were method of instruction, gender, and ethnic background. Students should be able to solve mathematical sentences involving one equation with one unknown, one equation with two unknowns, and two equations with two unknowns using: i) manipulatives (concrete), ii) sequential pictures representing equations (semi-concrete), iii) symbolic manipulations (abstract). In one of the treatments, a modeling program on the computer was used as an additional embodiment to facilitate the movement from the concrete manipulative level to the pictorial level and finally to
the symbolic level.

The computer modeling program used was designed by Dr. Lyle Andersen. Students were able to solve problems similar to those given in class and on labs that represented a linear equation with one unknown or two unknowns, or a system of two linear equations with two unknowns. The modeling program was used since it allows the students to experience all embodiments, the concrete, semi-concrete, and abstract. A copy of the computer program is in Appendix IV along with examples of the types of problems in the program.

The population consisted of sixth and seventh grade students in school districts in Browning, Montana and Poplar, Montana. Browning is located on the Blackfeet Indian Reservation. Unemployment is a major problem on the Blackfeet Reservation with an unemployment rate of 31.1% (Tiller, 1996). Tribal members derive their income chiefly from farming, stock raising, timber and forest products, crafts, and oil activity. Poplar is located on the Fort Peck Indian Reservation. Unemployment is also a problem on the Fort Peck Reservation, at a rate of 27.7% (Tiller, 1996). The economy is based primarily on farming and ranching, with the extraction of oil a substantial secondary source.

Need for the Study

Curriculum materials and research data which involve the use of manipulatives and the computer along with research findings from psychology promoting an active, constructivist view of the learning process are limited.
Especially lacking are studies or classroom situations that integrated the use of the computer and the learning theories of Piaget and Bruner to explore algebraic topics with middle school children. One purpose of this study was to increase this research data base.

Several other factors were also influential in establishing a need for this study. The following paragraphs describe each factor and detail the importance of each of these factors.

Factor (1): Mathematics is important for a productive life. Mathematics is necessary not only for the individual but for the nation as a whole. Mathematical talent is important for the nation's welfare (Madison & Hart, 1990) and the United States is falling behind other countries such as Russia, Japan, and Germany in producing technically trained personnel. This is largely due to the fact that approximately half of the students in the U. S. opt out of mathematics study at an early stage of high school (Hill, 1982). "When compared with other nations, U. S. students lag far behind in level of mathematical accomplishment; the resulting educational deficit reduces our ability to compete in international arenas" (National Research Council [NRC], 1989, p.1). In Everybody Counts: A Report to the Nation on the Future of Mathematics Education, the NRC (1989) states, "Quality mathematics education for all students is essential for a healthy economy" (p. 1). In an increasingly technical society, an educated, informed electorate is an absolute necessity. The public needs mathematical literacy to deal with issues such as "acid rain, waste management, the greenhouse effect" (NRC, 1989, p. 8) "environmental protection, nuclear energy, defense spending, and taxation" (NCTM, 1989, p. 4). The general populace needs mathematical skill to
avoid allowing important national decisions involving mathematics and
technology to be made based on ignorance (Hill, 1982). In *A Framework for
Constructing a Vision of Algebra* (1995), researchers claim “the strongest
predictor of earnings nine years after graduation from high school is the number
of mathematics courses taken” (p. 7).

According to NRC (1989) "Mathematics is the worst curricular villain in
driving students to failure in school" and "filters students out of careers" (p. 7).
"Failure to study mathematics can close the doors to vocational, technical schools,
college majors, and careers" (p. 68) according to the NCTM (1989). Students
whose formal education will end with high school need mathematical competence
to deal with consumer and career needs (NCTM, 1980). Mathematical skill makes
a major contribution in allowing individuals the opportunity to become good
citizens (Kahn, 1983). Students who either elect to discontinue their
mathematical education or who finish high school without the mathematical
literacy necessary for an informed, productive life are creating problems not only
in their personal careers, but also for the United States as a competitive, healthy
nation.

Factor (2): Algebra is an important part of a mathematics education.
"Algebra is the language through which most of mathematics is communicated"
(NCTM, 1989, p. 150). The publication by NCTM of two resources concerning
the use, instruction, and necessity of algebra relays the message that algebra is a
concern for mathematics educators. These two publications, *Research Issues in
the Learning and Teaching of Algebra* (1989) and *The Ideas of Algebra, K-12*
(1988), convey to teachers from K-12 that the fundamental ideas of algebra are
not being understood by students and serious changes need to be considered in the learning and teaching of algebra. In addition, *Algebra for the Twenty-first Century* (1993) and *The Algebra Initiative Colloquium* (1995) continue to express the need for a reformation in algebra instruction and curriculum to meet the needs of a changing society. "Demands for change are being made by those in politics, the sciences, the mathematics community, and the schools themselves" (Burrill, et al., 1993, p. 2). Algebra is a requirement for graduation from high school and for entrance into many colleges. Algebra is not only necessary for high school and college, since according to *Everybody Counts* "Over 75 percent of all jobs require proficiency in simple algebra and geometry" (p. 4). Carole Lacampagne (1995b) suggests that algebra is the "academic passport for passage into virtually every avenue of the job market" (p. 3). The algebra that is being taught in a large number of today's mathematics classrooms is not the algebra that students will necessarily need after graduation. Workers will need quantitative skills that are very different from those that are taught in schools now (Burrill, et al., 1993). The NCTM Standards (1989) state that some aspects of mathematics have changed in the past decade and in areas such as life and social sciences "the fundamental mathematical ideas needed in these areas are not necessarily those studied in the traditional algebra-geometry-precalculus-calculus sequence" (p. 7). Peggy House (1988) insists that algebraic skill is far more than symbol manipulation, and yet that is what is the primary emphasis of many classrooms. Students need to understand what the algebraic statements they are manipulating represent (Booth, 1989). Algebra may be thought of as generalized arithmetic (Booth, 1988a; Post, Behr, & Lesh, 1988; Kieran, 1988), but this is not
the approach that is used to teach it. In fact, most students are not made aware
that a relationship exists between the mathematics of their elementary years and
the mathematics called algebra. If a lack of a bridge from the mathematics in
elementary school, principally arithmetic, to the mathematics in high school was
the only problem, it might be overcome by students without assistance from
instructors, but algebra is more than generalized arithmetic (Wheeler, 1989).
"Concepts and methods of algebra are essential to successful work in nearly
every branch of mathematics but the algebra necessary in the future will be
beyond the standard of today" (Fey, 1989, p. 213).

Factor (3): Student performance in algebra is poor. “Evidence that the
present organization of algebra is not working, even in the limited goal of
preparing students for the next level of study, can be seen both at the high school
and college level” (Algebra Working Group, 1995, p. 6). This evidence includes a
second year high school algebra course that repeats the content of the first year,
the number of students in remedial algebra courses in college, and the high failure
rate of students in a first year calculus course. Much of beginning algebra
concentrates on solving linear equations, so according to NAEP, students can do
the manipulations necessary to solve the equations, but have difficulty with the
technical vocabulary involved (1989). The ability to manipulate equations does
not imply ability to provide meaningful interpretations of those equations
(Chaiklin, 1989). Students don't understand the concepts underlying the skills
they are trying to learn (NAEP, 1989), and even though they may be able to
manipulate the variables they don't understand what those variables represent
(NAEP, 1989) or the meaning of the algebraic symbols (Tall, 1989). Students
have a poor understanding of mathematical relations and structures which are the basis of algebraic representation (Booth, 1989) and the internal representation they do possess of algebraic equations is often incorrect (Larkin, 1989). "For many students the first algebra course is full of mysteries that they never fully understand" (Demana & Leitzel, 1988, p. 61). According to the NAEP's 1992 mathematics assessment, only 6% of high school seniors and 1% of eighth graders demonstrated consistent success in algebra. “Algebra I has a particularly high failure rate and appears to be a filter to keep many students out of higher mathematics and limiting their access to higher education” (Risacher, 1994, p. 91). Students are not learning the algebra needed for further study, or for their future careers.

Factor (4): Exploration of algebraic topics in earlier grades is one proposed solution. One of the primary reasons algebra is not an elementary topic is the belief that students ages 10-13 do not have the mental processes necessary for the abstract nature of algebra. Piaget refers to students in this age group as being at the concrete operational stage (Bell, 1980). However, several sources argue that students in this stage can learn the basic concepts of algebra, with only knowledge of the four basic operations with whole numbers, if allowed first to use concrete materials to explore the properties involved (Chalouh & Herscovics, 1988; Demana & Leitzel, 1988; Kaput, 1995a; Thompson, 1989). The terminology and symbolism used in algebra is understood by young children when associated with physical materials (Thompson, 1989). Introducing algebraic ideas informally to elementary students would enhance beginning algebra students' ability to make sense of procedures taught in high school (Kieran, 1988), would give these
students a foundation for success in their first year algebra course (Demana & Leitzel, 1988), and would be a step toward the goal of higher mathematics achievement for all students (DeBower & DeBower, 1990). A first year algebra course is not mysterious for students who have an understanding of basic algebraic concepts before they begin formal study (Demana & Leitzel, 1988). “Students should begin to learn algebraic concepts in elementary school, and these should be built on and reinforced as students move throughout their mathematical experience” (Burrill, et. al., 1993, p. 7).

Factor (5): Instructional materials in algebra using Piaget and Bruner’s theory involving the use of concrete materials, manipulatives, pictures, and abstract symbolism are limited. The major conclusion of the learning theories of Piaget is that each child learns at his/her own developmental pace, progressing sequentially from the concrete operational level to the abstract level (Thompson, 1989). Leslie Booth (1988b), in Research Issues in the Learning and Teaching of Algebra, recommends devising new learning activities to assist students in developing recognition and use of the structure of algebra. The traditional method of instruction that involves showing students the manipulations needed to solve a particular problem and then providing them with a number of similar problems to practice these manipulations, is apparently not effective (Chaiklin, 1989). Models of instruction in algebra which are more explicit are necessary (Wagner & Kieran, 1989). New material is needed that is organized into a constructivist teaching sequence which takes into consideration the students' cognitive development (Herscovics, 1989). A constructivist believes that "construction should always precede analysis" (Post, 1988, p. 10). Children
should be allowed to develop and build concepts using their own experiences. There is a need for multiple representations of a concept that will facilitate student development of the relationship between actions being performed on the object and the changes those actions produce (Thompson, 1989). According to Kieran and Wagner (1989), there is a need for studies that explore the effects of working with multiple representations and whether this enhances students’ abilities to make appropriate generalizations and solve problems. Students can then construct for themselves generalizations and concepts based on this knowledge (Thompson, 1989). With this active involvement in the learning process, students construct their own meaning of algebraic expressions (Chalouh & Herscovics, 1988) and actually recognize the need for general procedures (Booth, 1988a). Use of variables becomes a natural extension of the use of a placeholder and is explicitly linked to the number being represented (Chalouh & Herscovics, 1988). DeBower and DeBower (1990) suggest extensive use of manipulatives before moving onto abstract algebraic axioms. In the publication *An Agenda for Action*, NCTM (1980) recommends "Teachers should use diverse instructional strategies, materials, and resources, such as the use of manipulatives, where suited, to illustrate or develop a concept or skill" (p. 12). Since the overall performance of students in algebra does not meet the desired standard, formal instruction may not be beneficial to beginning algebra students with no understanding of algebraic concepts. Time may be better spent in exploring the concepts involved using concrete contexts which will build a foundation for formal definitions later. Students need to be aware of the visible correspondence between objects and actions (Kaput, 1989). New ideas should be introduced first via informal and
exploratory experiences (Davis, 1989). Once the basis behind the concept is understood, the symbolism used should then be a natural extension of the students' knowledge. Appropriate notations can be created as the purpose for the symbolism is clear to the solver (Davis, 1989). According to Thorpe (1989) "It is time for a major reevaluation of the content of the algebra curriculum and of the instructional strategies that are used in teaching algebra" (p.16).

Factor (6): Research pertaining to the results of computer-assisted instruction in algebra is limited. Education at all levels needs to respond to the growing availability of the computer. According to Hill (1982), educational programs need to keep pace with computer technology. The NCTM, in the Agenda for Action, expresses the need for "curricular materials that integrate and require use of the computer in diverse and imaginative ways" (p. 9) to be developed and made available. Wagner and Kieran (1989) request models for integrating computers into the algebra classroom. Kaput (1989) agrees that the computer can be an important connection between "multiple, linked notations" (p. 177) in algebra instruction. Tall (1989) asks, "How does the computer environment change the nature of the mathematical concepts, the development of students' conceptualizations, and the related cognitive obstacles?" (p. 91) All of these educators are interested in the development of instruction utilizing the vast potential of the computer. The NCTM recommends that educators take "full advantage of the immense and vastly diverse potential" (p. 9) of the computer environment. After many years of recommending integration of technology into the mathematics classroom, questions such as "How does technology enhance algebraic reasoning?" (Algebra Working Group, 1995, p. 13), are still being asked.
"The impact of use of technology on teaching and learning and its integration into the curriculum is not yet even conceived" (Burrill, 1995. p. 59). In order to take advantage of the potential of technology, instructors need proven methods using the computer in the algebra classroom.

Factor (7): Improving the algebraic competency of American Indians students is necessary. American Indians are seriously underrepresented in careers involving mathematics. Since sample sizes of Indians in most national surveys are relatively small, not much information is available on the education of this minority group. In their report, NAEP (1989) compared the performances of white, black, and Hispanic students but found that sample sizes were insufficient for the analysis of American Indian populations. American Indians comprise 9.2% of the students enrolled in Montana public schools according to A Plan for American Indian Education in Montana. American Indians have the highest dropout rate of any minority group, which in 1988 was 35.5% (Hodgkinson, 1990). More recent statistics place the dropout rate for American Indian students at 29.2%, the highest of any ethnic group (Bowker, 1993). Montana is the eleventh ranking state in the number of American Indians so it is especially vital to Montana's citizens that the Indian population be acknowledged and their problems be addressed. Although Montana does not report statewide data, reports from individual school districts within the state range from 14 to 85% (Bowker, 1993). The avoidance of mathematics by American Indian youth is a primary contributor to their under representation in higher education and fields involving quantitative skills. As stated before, algebra is an important part of a mathematics education. To increase the number of Indians in fields involving
math and science, studies need to expose the problem and propose solutions. Hopefully, increasing the mathematical ability of American Indian students will begin to break the chain of poverty, unemployment, and suicide currently facing a large percentage of Indians in the United States.

Questions to be Answered:

1. Was there a significant difference in student achievement among treatments?
2. Was there a significant difference in student achievement between American Indian students and other students?
3. Was there a significant difference in student achievement between 6th and 7th grade students?
4. Was there a significant difference in student achievement between gender groups?
5. Was there a significant interaction between treatment and gender?
6. Was there a significant interaction between treatment and ethnicity?
7. Was there a significant interaction between treatment and grade level?
8. Was there a significant interaction between gender and ethnicity?
General Procedure

The experimental-control model used involved teaching sequences that included: (a) the multiple embodiment instructional sequence with the computer as one embodiment, (b) the multiple embodiment instructional sequence without the computer, (c) a traditional textbook instructional sequence. The multiple embodiments utilized were manipulatives consisting of film boxes and counters, worksheets involving a pictorial view of the mathematical sentences to be solved, and an abstract presentation of the problems using algebraic notation. See Appendix III for a full description of the treatments.

A standardized test of mathematical ability, the Orleans-Hanna Algebra Prognosis test, was administered to attempt to adjust the mean group scores to remove bias and to test for an overall increase in algebraic ability. A criterion-referenced test (CRT) was developed to test both concept and skill attainment. Both examinations were used as a post-instruction test and as a pretest.

The study took place in Poplar sixth and seventh grade classrooms and in Browning sixth grade classrooms in Montana. The selection of the schools involved in this study was based on the criterion that: (1) the teachers recently participated in AIM, (2) the teachers were interested in teaching algebra readiness in their classrooms, and (3) the administration at these schools was willing to cooperate with the researcher. The instructor in Browning sixth grade, is an American Indian. He taught two sections of each treatment. The instructors in Poplar were not American Indians. They each taught one section of each treatment.
The study involved three different treatment groups. The first treatment group involved students working with i) manipulatives (concrete) ii) pictures (semi-concrete) and iii) symbolic manipulations (abstract) in the solving of algebraic equations involving one equation with one unknown, one equation with two unknowns, and two equations with two unknowns. This model was developed by the researcher, Dr. Lyle Andersen, and Dr. Warren Esty (Andersen & Esty, 1990). The computer was used to facilitate the change from the manipulative level to the pictorial level and finally to the abstract level. The second treatment group involved students working with the same manipulatives (concrete), pictures (semi-concrete), and symbolic manipulations (abstract), but without the utilization of the computer. The manipulatives were film containers and paper clips which were used to demonstrate concretely the equations to be solved. The pictures were worksheets that had the same types of equations to be solved as the manipulatives except in picture form, as opposed to physical, concrete objects. The abstract symbolism was the traditional method of writing equations involving variables and other algebraic notations. The third treatment group used a traditional method of instruction for learning to solve algebraic equations involving one equation with one unknown, one equation with two unknowns, and two equations with two unknowns. The computer modeling program was designed by Dr. Lyle Andersen. The computer program provided students with problems to be solved that were similar to the problems using the manipulatives. The students were allowed to solve these problems using the manipulatives, and then were asked to write a pictorial description of the problem presented to them along with the solution. The program was also used to
facilitate the change from the pictorial representation of the problems to the abstract symbolic representation.

Since this study was done in a natural school setting, random assignment of students to treatment groups was impossible. Therefore a nonequivalent control group design which uses a pretest and posttest to measure the effect of the treatments was utilized. A nonequivalent control group design is effective for classrooms where groups are as similar as possible but where use of a pretest is necessary as when working with intact groups (Gay, 1981). An advantage of this design is classes can be left as they are, and possible effects from reactive interference from variables is minimized (Gay, 1981).

**Limitations and Delimitations**

Limitations of this study included:

1. Classrooms were limited to teachers who volunteered to participate in the study and were involved in the AIM project.
2. Students were from sixth and seventh grade classrooms in Montana located on two American Indian reservations.
3. Instruction was limited to ten 50 minute class periods.
4. The subject of the study was pre-algebra problem solving skills involving equations and unknowns.

Delimitations of this study included:

1. The population of the study was sixth and seventh grade students in
Montana attending a school located on an American Indian reservation.

**Definition of Terms**

For the purpose of this study, the following definitions are used:

**Concrete learning model** - model using manipulatives

**Semi-concrete learning model** - model using pictorial representations

**Pre-algebra skills** - ability to solve mathematical sentences involving one equation with one unknown or two equations with two unknowns

**Embodiment** - concrete or semi-concrete situation representing a mathematics concept (Indelicate, 1979)

**Multiple embodiment** - two or more experiences of a mathematical concept using a variety of materials (Indelicate, 1979)

**Problem solving** - interpreting a mathematical statement involving equations with unknowns resulting in an appropriate conclusion

**Computer-assisted instruction (CAI)** - a data processing application using a computer to assist with instruction (Rosenberg, 1989)

**Manipulatives** - objects operated on by learners or teachers (Indelicate, 1979)

**American Indians in Mathematics (AIM)** - a Dwight D. Eisenhower National Project award whose primary mission is to increase the representation and success in mathematics of American Indians living on reservations in rural settings (The AIM Project, 1991)
CHAPTER II

REVIEW OF LITERATURE

Introduction

The primary purpose of this study was to investigate the effectiveness of using concrete, semi-concrete, and abstract prealgebra learning models, utilizing multiple embodiments on the achievement of 6th and 7th grade students. The secondary purposes of the study were to determine if there is any difference in achievement between American Indians and other students, between male and female students, and between sixth grade and seventh grade students. The three dependent variables were the difference between the pretest and posttest scores on the Orleans-Hanna Algebra Prognosis test, the difference between pretest and posttest scores on the criterion-referenced test, and the posttest scores on the criterion-referenced test. The independent variables were method of instruction, gender, and ethnic background. The population consisted of sixth and seventh grade students in school districts in Browning, Montana, located on the Blackfeet Indian Reservation and Poplar, Montana, located on the Fort Peck Indian Reservation. The researcher studied the effect of a multiple embodiment instructional sequence including the computer as one embodiment on student achievement. The study also included the effect of the sequence without the computer embodiment. The control group was taught with a traditional textbook.
method.

The review of literature will first focus on the failure of students to achieve algebraic literacy and solutions to this problem, including theories of how and when algebra should be taught. Then, a review of studies which examine teaching algebraic concepts to middle school students, specifically, methods of solving linear equations, using a constructivist model of instruction, using a multiple embodiment instructional sequence, and using the computer in instruction will follow. Finally, research describing the importance of increased attention to the needs of American Indian students and females will be addressed.

**Failure to Achieve Algebraic Literacy**

Research indicates that students are failing to achieve algebraic literacy. There are several reasons that students are not succeeding in algebra. Some students are dropping out of school and some are dropping out of mathematics classes. Other students are taking algebra courses but are receiving failing grades. Finally, other students who pass algebra, still do not have the fundamental understanding of the algebraic concepts necessary to continue in mathematics or other fields of study requiring mathematical knowledge. Nicolas Herscovics and Liora Linchevski (1994) state “We are thus concerned with two main groups of students: those who enter Algebra I and fail or encounter major difficulties, and those who do not even enroll in an initial course” (p. 60). The reasons stated in the research as to why students are not succeeding in algebra
vary. According to James Kaput (1995b) algebra courses are redundant and isolated from other subject matter, introduced abruptly to students who are not prepared, and are repeated as remedial mathematics at the post secondary level. “The net effect is a tragic alienation from mathematics for those who survive this filter and an even more tragic loss of life-opportunity for those who don’t” (Kaput, 1995b, p. 71). Herscovics and Linchevski (1994) agree with one of Kaput’s assertions, that students are not prepared for algebra. “The pupils’ lack of readiness may explain the dismal results achieved in algebra in our secondary schools” (Herscovics & Linchevski, 1994, p. 59).

Other researchers claim that students are unable to cope with the abstraction they encounter in a first year algebra course. Carolyn Kieran and Louise Chalouh (1993) claim that the transition from arithmetic to algebra requires use of abstract representations and this transition is seldom directly covered in mathematics classes. Herscovics and Linchevski (1994) also feel that a possible explanation for the difficulties students experience in algebra involve the formal approach used in its presentation. Lyn English and Patrick Sharry (1996) found that students often use other mathematical symbols as referents for algebraic symbols. Therefore “it is not surprising that students frequently manipulate symbols without regard for their referents or for the effect that the arrangement of symbols has on these referents” (p. 140). So students are failing to construct meaning for the symbols they use, and are reduced to performing meaningless operations on symbols they do not understand (Herscovics & Linchevski, 1994). Lesley Booth (1984) reported the results of an investigation referred to as "Strategies and Errors in Secondary Mathematics (SESM)". One
possible basis found for the errors in students' conceptions was that "the development of understanding in algebra may correspond to a progression in the ways in which letters are interpreted" (Booth, 1984, p.4). Further analysis in the report substantiated the theory that "Children have difficulty grasping the notion of letter as generalized numbers" (83) and "Children have difficulty in representing formal mathematical methods" (83). According to Sigrid Wagner and Sheila Parker (1993), research indicates that the majority of students' obstacles to understanding algebra stem from difficulties with notational concepts or problems with using letters as variables. A lack of readiness for algebra instruction, an inability to deal with the abstract aspect of algebra, and difficulty with the concept of variable are reasons for students' failure to achieve success in algebra. This study intends to investigate the introduction of algebraic concepts to students prior to a formal first year algebra course, using a concrete, multiple embodiment instructional model with the computer as one embodiment, to give students a basic understanding of the use of variables and linear equations.

Solving Systems of Equations with Middle School Students using Multiple Embodiments

Since the lack of readiness for algebra instruction is one problem for students, a possible solution is to introduce these concepts earlier. Kaput (1995b) claims that children are capable of handling formal symbolism and building formal abstract structures. However, introducing these concepts using the same
methodology as used with older students in a formal algebra course would have the same effect: lack of success. So instruction for younger students, specifically students in middle school, would have to use a different technique. The instruction of algebraic topics will follow a model using i) manipulatives (concrete), ii) sequential pictures (semi-concrete), iii) symbolic manipulations (abstract). This model, suggested by the learning theories of Jean Piaget, Jerome Bruner, and Zoltan Dienes, could help students understand algebraic concepts. Several studies have been conducted which indicate this is true. This progression, from concrete to semi-concrete to abstract, is useful in the introduction of the concept of variable, and in the solving of linear equations. W.W. Sawyer (1988) used a method of visualization to introduce children to the concept of variable. Sawyer asked students to think of a number and then imagine placing that number of stones in a bag. The class was then asked to visualize adding three stones to the number of stones in the bag, without placing them in the bag. The lesson progressed from there, including a visualization of doubling the number of stones, taking away some stones, and dividing the number of stones by 2. Students were asked to draw their bags and eventually a need for abbreviation became obvious. Although the first level, using a concrete, physical object to describe the activity was not used, this could have been easily remedied by introducing an actual bag with stones into the sequence.

In a study in New South Wales, designed as a part of an integrated mathematics curriculum, algebra was introduced to twelve-year olds first using concrete experiences (Pegg & Redden, 1990). The goal of the study was to develop algebra in a logical fashion that avoided early use of symbol
manipulation so that students would see a need for algebraic notation to arise as a natural and useful consequence of expressing generality. The concrete materials used included matches, counters, and straws. After concrete experiences, the next major focus was on clarifying both verbal and written language through communication with both the teacher and fellow students. Symbolism developed as a natural extension of the work the students were doing. These techniques are still being used by teachers at the school and students seem more positive in their approach to algebra and are better able to develop algebraic generalizations from practical investigations.

Barbara Kinach (1985) developed material used to provide opportunities for students to investigate solving linear equations using both a physical and pictorial model and then discover a method to solve the equations at the symbolic level. The materials designed included strips and squares which the students used to represent first-degree polynomials physically and pictorially, and balance scales which were used to solve linear equations physically. A similar idea was used by Midlands Mathematical Experiment (MME) in which balancing mobiles succeeded in giving middle school pupils a concrete base to use while developing a familiarity with algebraic symbolism (Eagle, 1986). This study will investigate these concepts further by examining two equations with two unknowns.

Luciano Meira (1995) used videotaped interactions of 5th grade students to discover how students actually use concrete materials to explore concepts. Meira found that whatever tools teachers use in their classroom students should be allowed to use these tools to communicate their ideas and develop their own
representations. Jorge Tarcisio (1995) conducted a study of 11 students in Recife, Brazil, ages 12 - 17. Using four sets of activities including a balance scale, Tarcisio found that the use of "represent first, try to solve later" (p. 71) was an appropriate model for these students.

Barbara Berman and Fredda Friederwitzer (1989) claimed that "Algebra can be elementary when its concrete" (21). They used small containers such as medicine bottles along with buttons to simulate linear equations. Using this model follows a concrete to abstract continuum that they found could be used by younger elementary students as well as older, middle school or high school students. Harry Borenson (1987) developed the "Hands-On Equations Learning System", which was used to solve algebraic linear equations, and provide young children with a successful introduction of this algebraic topic through a concrete approach. The children physically represented given equations using pawns and numbered cubes and solved the equation using a number of legal moves. Borenson found that "Elementary school children using a successful, hands-on, experience in algebraic linear equations can develop a positive mind-set and expectation for success in later formal, algebraic studies" (p. 56).

Frances Thompson (1988) developed and implemented several instructional sequences based on the learning phases of Bruner. Using one-inch square pieces of construction paper to concretely represent the idea of variable in a linear equation, Thompson then moved to a pictorial representation using drawings closely resembling the concrete representations. Thompson demonstrated this instructional technique could be used to successfully teach simple algebraic concepts to children in grades 3-6. Several of these researchers
used a sequence which started with the concrete and moved to the abstract, which they found to be beneficial. However, statistical studies proving the success of such an approach are lacking.

A study conducted by Elizabeth Warren (1995) explored the relationships between general reasoning processes and understanding basic algebraic concepts. Warren used a correlational research design with eighth and ninth grade students to find that logical reasoning and patterning had some bearing on the students' success in algebra. The students' ability to generalize from tables was a strong predictor of success in understanding the concept of variable. These studies indicate that the use of various representations of variable are beneficial to students.

In one study designed to investigate the use of manipulative materials and symbolic representations in teaching the concept of variable, Edmund Antosz (1989) used four treatment groups. One treatment group was taught to solve linear equations with integer solutions using standard symbolic notation and traditional instruction. The second treatment group was taught using Tac-Tiles (a set of tiles which represent the mathematical ideas used in learning to solve equations and describing the concept of variable). The third treatment group used Tac-Tiles and was also asked to develop a symbolic notation for the operations used. The fourth treatment group received instruction using a balance bar as a manipulative activity. The study involved twenty-six seventh grade middle school students. Using a one-way analysis of variance, the subjects in the Tac-Tile Symbolic group performed significantly better than all the other groups with a p-value less than .05. This is one study that indicates use of concrete
materials to teach students is appropriate for students in middle school. Additional questions to be answered include the use of this technique with minority students and whether the computer could be used to facilitate an even better understanding of variable and equations.

The multiple embodiment principle was developed by Zoltan Dienes in the 1960s. Dienes claimed that mathematical concepts can only be properly understood if first presented to children through a variety of concrete, physical representations (Bell, 1980). Multiple experiences using a variety of materials promotes the abstraction of a mathematical concept (Post, 1988). Conceptual learning is maximized when children are exposed to a concept through a variety of physical contexts or embodiments. Children are frequently required to abstract mathematical ideas before they have the opportunity to experience them in concrete form. Thomas Post (1988) claims mathematical learning requires a very active type of physical and mental involvement on the part of the learner. According to Patrick Thompson (1989), although the idea of multiple representational systems appears to be powerful, there is little known about the actual effect their use has on students' cognition. Reys (1972) found that successful instruction using multiple embodiments depended on the teacher's ability to prepare appropriate lessons, which accurately represented the concept. Indelicato (1979) reviewed several doctoral dissertations on mathematics instruction using multiple embodiments, and found that use of physical embodiments reinforced concept development. According to James Kaput, ongoing Educational Technology Center work suggests that appropriate experience in a multiple, linked representation environment may provide
connections in mathematics that are normally missed by students, and provide methods of using these connections to learn the power of mathematics (1989). Behr (1976) used several treatments involving the manipulative, pictorial, and symbolic phases with multiple embodiments to teach addition and subtraction. The multiple embodiments were found to increase achievement, but only one classroom participated in the study. Jerome Bruner (1966) suggests that a child should experience and think about a particular idea or concept on three different levels: enactive or concrete, iconic or pictorial, and symbolic or abstract (11). This involves a hands-on manipulative level, a visual or pictorial level, and a symbolic, abstract level. Jean Piaget adds to this theory by claiming children in a concrete-operational stage of mental development may have trouble applying intellectual processes to abstract ideas, but could use their intellect to manipulate concrete, physical objects (Bell, 1980). Wheeler (1971) used a random sample of second grade students to measure the amount of exposure to concrete materials on the effectiveness of instruction on multi-digit addition and subtraction problems. Wheeler's conclusion was the exposure had a positive effect on achievement. These studies indicated that instruction involving multiple embodiments were effective in improving achievement so children should construct their own representations of ideas, beginning with concrete representations.

Mary Brenner and Bryan Moseley (1994) conducted a project which introduced seventh grade prealgebra students to the basic algebraic concepts of variables, expressions, and one variable equations. Twenty-four students were given an oral pretest and both an oral and a written posttest. One teacher taught two experimental classes and one control class with a traditional textbook
prealgebra curriculum. Students in the experimental classes worked in cooperative groups to solve problems based on the theme of running a candy store. Algebraic concepts and notation were introduced after the students had spent time exploring a situation in which algebra was appropriate. Each group presented its results to each problem to the class using a variety of representations. The conclusions were that the students in the experimental classes were able to interpret the meanings of variables in equations and expressions, whereas the students in the control class were unable to give meaning to variables unless they had the answer to the equation. The experimental group was also able to use other forms of problem representations such as drawings, graphs, tables, and number lines to explain their solutions. The control group was only able to use tables and drawings. Students who are exposed to multiple representations of a concept can in turn use these to make sense of basic algebraic topics. This study explored the use of multiple representations.

A Need for Early Explorations

Students in middle school need to experience algebraic concepts before a formal algebra course. According to James Kaput (1995a), “the elementary and middle school curricula will need to do far more for children than our one curriculum does now” (p. 34). Kaput suggest that students experience informal representations that lead to more formal experiences. Carole Lacampagne
(1995b) recommends that all students have significant experiences in algebra before the end of eighth grade, and these experiences should flow throughout the K-8 mathematics curricula.

Lyn English and Patrick Sharry (1996) conducted a study of 10 students. Five students were in grade ten and five students were in grade twelve. Early algebraic experiences of all the students included an introduction to linear equations via activities with balance scales and exploration of meaningful problems. Individually, students were presented cards with equations written on them. The equations consisted of linear equations in the general form, linear equations with unknowns on both sides of the equal sign, quadratic equations, and equations with two or more variables. Students were instructed to sort the cards into “groups of equations that you think are the same type” (p. 140). The students were allowed to form as many groups as they wished, and were asked to explain their groupings. English and Sharry found that not all students were capable of algebraic abstraction, even after five years of algebraic study. The students were unable to look beyond solving equations and computational processes to detect underlying relational properties. They concluded that students require experiences that focus on exploring algebraic processes that will lead to meaningful acquisition of abstract algebraic concepts. “The development of mathematical abstraction or structural thought, begins with the exploration and use of processes or operations performed on lower level mathematical constructs” (p. 137).

Traditional algebraic instruction involves the introduction of an abstract concept, the variable. The variable is used to represent a problem situation, which
is then solved using a set of abstract rules to obtain an answer. "Middle school mathematics textbooks typically use equation solving to introduce the concept of variable" (Briggs, Demana, & Osborne, 1986). According to Carolyn Kieran (1992), research indicates that teachers tend to follow the chapters in a textbook when teaching algebra, using a lecture style which includes explaining some examples and then assigning practice problems. However, research also indicates that students experience difficulty with learning algebra in this way, so teachers are forced to find alternative sources, but little guidance is available (Kieran, 1992). A review of mathematics textbooks before and after 1989 was conducted by Donald Chandler and Patricia Brosnan (1994). The researchers found that many teachers use their mathematics textbook as their curriculum guide. Therefore, it is important to know if the content in textbooks reflect the guidelines of the NCTM Standards. Chandler and Brosnan found that publishers have attempted to be sure their textbooks conform to current professional expectations of what textbooks should contain. Although there was an increase in all content areas, arithmetic remained the largest content area and algebra the smallest content area. Chandler and Brosnan concluded that "textbooks have not changed as much as they have grown" (p. 8).

Patrick Thompson and Tommy Dreyfuss (1988) investigated whether elementary school students could construct operations of thought for integers crucial for understanding of basic algebra. Two sixth graders were taught for six weeks in eleven 40 minute sessions using a constructivist teaching model. The two students used a microworld called INTEGERS which presented them with integers in the form of a turtle which could be manipulated to perform operations
such as negation, addition, and composition. Their conclusions were that traditional instruction needed to focus on operations of thought that generalize to elementary algebra.

Reviewing the research on the transition from arithmetic to algebra, Carolyn Kieran and Louise Chalouh (1993) decided that algebraic notation should be a natural way to express arithmetic for middle school students. If this is accomplished, the students would have a strong foundation for future algebra instruction. Carolyn Kieran (1992) reviewed the research available in teaching algebra and concluded that “the amount of research that has been carried out with algebra teachers is minimal” (p. 413). Since Kieran and Chalouh (1993) found that students need a strong sense of variables and how they are used, and Kieran (1992) reported a lack of research on this topic, this study, which emphasizes one use of variables, would be beneficial to teachers.

The Algebra Working Group (1995) consists of ten people with a variety of mathematical backgrounds. Through reviewing papers and reports about algebra, members developed A Framework for Constructing a Vision of Algebra. This document was designed to generate discussion, motivate continuing research, and stimulate change in the teaching and learning of algebra. Throughout the framework, the use of physical models and applications are emphasized. On using variables, The Algebra Working Group suggests “Concrete experiences enable children to develop a sense of the need for variables and symbolic notation” (p. 34). In addition, Anne Raymond (1994) states that “Research has shown that the use of manipulatives with students in the middle school greatly increases their understanding of the concept being
taught. Such experiences with concrete materials helps them connect their understanding to their world” (p. 72). This philosophy directly reflects the stance taken in this study.

Using Computers in Algebra Instruction

Research indicates that computer assisted instruction (CAI) in mathematics education is effective and yet the use of computers in classrooms is still limited. Although computers are well established in American schools, the mathematics curriculum remains relatively untouched by computer technology (McConnell, 1988). Specifically, the focus of algebra curriculum in many schools is still symbol manipulation instead of a deeper understanding of algebraic concepts (Dugdale, Thompson, Harvey, Demana, Waits, Kieran, McConnell, & Christmas, 1995). Dugdale, et.al. (1995) found that much attention has been directed toward using computers to enhance or support the traditional algebra curriculum and more attention should be paid to how the curriculum should change and how computers can facilitate that change. "Despite the rich variety of proposals and developmental efforts seeking to use computers for mathematics teaching, the actual content of most secondary mathematics programs today is remarkably similar to that of thirty years ago" (Fey & Heid, 1984, 21). Not only is the content of mathematics programs similar to the content several years ago, but mathematics textbooks have not changed significantly. According to Susan Smith, textbooks in seventh and eighth grade mathematics classes look like sixth grade textbooks.
Topics covered are review and practice of material covered in grades K-6, and time is spent on drill and practice of computational skills (Smith, 1984). In a review of textbooks before and after 1989, Donald Chandler and Patricia Brosnan (1994) found that sixth, seventh, and eighth grade textbooks are changing and are moving in the direction of the NCTM Standards. However, they discovered that the textbooks have not changed as much as they have grown. While algebra instruction increased by 71%, algebra content is still only between 3.4% and 12.4% of the content in textbooks. They also found that although the number of pages with calculator usage increased by .55%, the use of computers was still missing.

There is potential for the computer to change the teaching and learning of algebra. CAI programs could bring major changes to the mathematics curriculum and to algebra in particular (Thorpe, 1989; Leitzel, 1989; McConnell, 1988; Heid, Sheets, & Matras, 1990). The potential for change in the curriculum is due to the low cost of microcomputers and the public expectation that students will be able to use them (Gawronski, 1982). Carpenter and Lindquist (1989) explain "The rapid growth in technological tools and procedures provides an impetus for change in what mathematics is to be learned and the way in which it is taught" (p. 167). The changes in algebra prompted by the computer include decreased emphasis on symbolic manipulations such as factoring, simplifying complicated rational and radical expressions and equations, and analytic solutions to polynomial equations (Fey & Heid, 1984; Fey & Good, 1985; Coxford, 1985; and McConnell, 1988). Topics that could be eliminated from the algebra curriculum are trinomial factoring and polynomial long division (Steffani, 1986). Computer
graphing tools could be used to increase the importance of graphing, functions, variables and real-life problems in the algebra curriculum (McConnell, 1988) and topics which require using visualization skills will need to be re-examined to take advantage of the computer's unique ability to provide this visualization (Mathis, 1986). This research indicates the possibility of increasing students understanding of algebraic topics, when the computer is an instructional tool.

Research also suggests that the use of computers in middle school would be especially beneficial, since computing technology allows students to explore abstract concepts in a more concrete manner. Geometric representations of algebraic problems can be displayed on the computer so the power of visualization can be used to study mathematical concepts and ideas (Demana & Waits, 1990b). As James Fey and Richard Good explain, "Technology opens effective ways to teach important new topics and new ways to approach traditional topics. It offers both a means and a reason to change" (1985, 44). Franklin Demana and Bert Waits agree that "Technology available today should dramatically change the way mathematics is both taught and learned" (1990a, 212). Thus, the introduction of the computer into algebra classrooms can affect what material is taught and the approach used to teach this material. This study is interested in using the computer to facilitate the introduction of algebraic topics traditionally not found in middle school mathematics curricula. According to this research, this is precisely the goal of CAI.

Studies on the use of computers in algebra demonstrate positive results including increased comprehension of concepts and improved attitudes toward the necessity of algebra instruction. A project begun by the University of
Maryland's Department of Mathematics in 1984 funded by the National Science Foundation (NSF) was designed to create a computer-intensive curriculum in algebra. As a result of three years of fieldwork, a shift in curricular emphasis was designed to focus on concepts, relationships, structures, and problem solving and away from manipulative expertise. Results included an increased appreciation for the purpose of algebra along with greater initiative in problem-solving (Lynch, Fischer, & Green, 1989).

In 1985 a computer-based algebra curriculum was developed and implemented into an inner city school and a rural school over a two year period. The curriculum "Algebra with Computers" used a computer-enriched curricula that encouraged the use of the computer as a tool in exploring ideas. The focus of algebra shifted toward problem solving and students developed a larger view of concepts, procedures and strategies, while being allowed to experience independent thought (Heid, Sheets, & Matras, 1990).

Memphis State University established a survey course on the impact of computers on upper-level nontechnical students (Ordman, 1984). A calculator and computer precalculus project was funded by NSF, British Petroleum, Ohio Board of Regents, and Ohio State University to explore two important technologically driven instructional strategies. One strategy involved students in classrooms with a single computer and so an interactive lecture-demonstration instructional model was used. The other strategy used computer labs, with a guided-discovery instructional model. Students from both groups developed strong intuition and understanding about functions. The power of visualization helped the students question, conjecture, and discover important mathematical
concepts (Demana & Waits, 1990b). These studies explore the possible changes that could be made in the algebra curriculum to increase understanding and reevaluate the content. Since this study is interested in exploring the effect of changing the way algebra is introduced, these studies provide necessary information about how best to implement CAI into mathematics instruction.

Studies indicate several positive results when using the computer in the instruction of algebraic topics. Results of the three-year project at the University of Maryland in 1984 included increased student ability to express mathematical ideas orally and in writing, flexibility and willingness to take risks in problem-solving situations, confidence and initiative in problem-solving, and appreciation of algebra (Lynch et al., 1989). Ronald Wenger and Morris Brooks (1984) examined the performance of approximately 1200 students per year on paper-based, multiple choice tests in a precalculus course at the University of Delaware's Mathematical Sciences Teaching and Learning Center. During more than five years of instruction using computer-based materials to serve as an "intelligent diagnostic system to provide instructional support for students studying algebra and precalculus topics" (219), Wenger and Brooks found that the computer-based environment was appropriate for student interaction and "The task of helping students acquire the extremely important ability to perceive an algebraic expression in a variety of ways may well be easier to teach dynamically with the help of computers than by using a more static text-based mode" (p. 229).

At the New York Institute of Technology Invitational Seminar on Computer-Based Education, Abraham Jeger and Robert Slotnick in January of 1985, presented a paper describing a project conducted during the 1983-84
academic year. CAI was introduced into two math courses, Developmental Math and College Algebra. CAI was used in the form of drill and practice programs and Logo based problem solving. Changes observed in the students as a result of the CAI included a better conceptual grasp of subject matter, a more active, participatory role in learning, greater perseverance in solving problems, and more positive attitudes overall. Statistically, a quasi-experimental design was used and a significant difference in pretest and posttest scores indicated a positive gain in knowledge; however there was no significant difference in attitude (Jeger & Slotnick, 1985). In addition to providing impetus for the use of CAI, this study also indicated the use of a quasi-experimental design was effective.

The initial introduction of a prototype intelligent computer-based algebra tutor in five first year algebra classes in a California high school resulted in the conclusion that positive attitudes towards learning with computers and actual achievement in algebra are unrelated (Stasz, 1988). However, although the study consisted of eighty participants and both a pre- and post- algebra achievement test was administered, the study did not have a comparison control group.

Robert Tilidetzke compared four sections of a college algebra course at the University of North Carolina at Greenboro during spring semester of 1988. Two instructors each taught one control and one experimental group. A pretest established the similarity of all four classes on the topics taught. After both a posttest and a delayed posttest, no significant differences were found between the groups using the computer materials and the groups not using the computer materials. Tilidetzke concluded that use of software was as effective as classroom instruction on the algebraic topics studied.
While some studies focused on teaching general algebraic topics, other studies explored the use of computers to teach equation solving and the concept of variable. In a year long computer coordinated curriculum for algebra, a truly student-computer interactive curriculum was discovered which allowed the students to use the computer to reinforce concepts and skills (Hughes, 1988). Hughes (1988) found students needed only enough theory to be able to write equations and understand how to solve them. Once this was accomplished, the computer could be used to solve the equations. Hughes (1988) concluded that "A mathematics curriculum is needed which requires students to use computers to reinforce concepts and apply skills" (p. 23).

Thomas Kieran and Alton Olson (1989) used novice seventh and tenth grade teachers to discover what contributions the computer can make in algebra. The computer was used by students to represent problems and situations with equations. The researchers concluded that students developed a more intuitive understanding of mathematics. A computer environment for learning algebra was illustrated by Alan Hoffer (1987) who stated an "Environment that provides learners with control over a technology to aid in the mastery of skills and concepts" (p. 24) was needed.

Although the following investigation uses a different teaching strategy, the design used was similar to the research design proposed for this study and so provided validation for the design to be used in this research study. A CAI experiment by Vicki Payne (1989) involving 44 students enrolled in a first year algebra course at a private high school in Texas used an experimental-control group model to study the effects of instruction on solving equations graphically
prior to instruction on solving linear equations algebraically. Multiple regression was used to delineate differences in students learning. The experimental group used computer generated graphs to solve equations and the control group used traditional methods to solve the equations. Payne discovered no significant differences between the students using computer-generated graphs to solve equations and the students solving equations using traditional methods.

Ninth grade students in three CAI classes and two traditional classes were compared using a written posttest and two sets of interviews to discover their acquisition and use of conceptual knowledge of variables (Boers-van Oosterum, 1990). Conceptual knowledge was established if students knew the way symbols were used to represent elements in a domain, interpreting information on variables, using variables to model problems, and using variables in a proof. The conclusion was that students taught traditionally had a limited concept of variable while four of six computer taught students developed an understanding of the concept of variable as generalizer as well as missing number. This reinforced the theory proposed in this research study that a non-traditional teaching sequence may be more effective than the traditional textbook approach.

In a pilot study in 1989, David Tall and Michael Thomas used a combination of activities, including programming exercises in BASIC and games involving the physical storage of a number in a box, designed to give children the ability to visualize the concept of an algebraic variable (1989). With matched pairs of 11 and 12 year olds, these students learned basic concepts of algebra using a traditional method of instruction or an instructional method with games and the computer. The experimental students performed at a higher conceptual
level when using algebraic notation with a p-value of less than .0005. More than one year later, the students from the computer environment still performed better, at a p-value of less than .005. Tall and Thomas concluded the "Traditional approaches lead to a narrow symbolic interpretation of variable, while use of the computer gives a visual framework for mental manipulation of higher order concepts" (p.119). A second experiment was performed on a larger scale with 57 matched pairs in 1991. The experimental group received the Dynamic Algebra Module, which consisted of a combination of programming exercises in BASIC, physical activities which simulated computer storage and manipulation of variables, and software designed to evaluate expressions in standard mathematical notation, for four weeks while the control group received traditional instruction on similar material (Thomas and Tall, 1991). The results indicated no significant difference between groups on the post test but a significant difference on the second post test given two months later, with a p-value less than .025. It was concluded that students using the Dynamic Algebra Module were more versatile in their thinking, better able to cope with obstacles, and significantly better at interpreting symbolism.

Carolyn Rhoads (1986) studied the effects of computer instruction on curricular presentation. The goal of the microcomputer instruction was to discover the effect of variation in slope and y-intercept on graphs of linear equations in two variables. The instruction consisted of high and low degree of guidance. With 114 students in five algebra I classes in a suburban high school using nineteen Apple Ile computers for two days, a multiple analysis of covariance indicated no significant difference, but a t-test did indicate a statistical
significance with a p-value less than .001. The study supports the idea that
discovery learning is an effective method for microcomputer instruction and also
discovered that learning can become increasingly self-directed as microcomputer
instruction progresses (Rhoads, 1986).

Norman Blackett (1989) developed a method using computers to teach
solving linear equations. In a Warwickshire comprehensive school, three groups
of fourteen and fifteen year olds were introduced to linear equations using a
graph plotter (Supergraph). The children taught using the experimental model
produced significantly better results on questions requiring understanding of
linear equations and their graphs than those children taught using a more
traditional method. "Appropriate software can change the learning environment
so that teachers can encourage discussion as well as demonstrate ideas, with
pupils being free to explore, formulate conjectures, and build up concepts in a far
richer way than the more traditional exposition followed by a step by step
process through text books or worksheets" (Blackett, 1989, p. 17).

Gregg Cox (1990) studied the effects of curriculum specific computer
aided instruction on student achievement in a college algebra course. College
algebra students were randomly assigned to one of four groups. At a p-value of
.05 there was a significant difference in achievement in learning mathematics
between college algebra students who studied linear inequalities using
microcomputer drill and practice in combination with traditional lecture and
college algebra students who studied linear inequalities using a traditional
lecture-homework approach.

Two hundred twenty-three eighth grade algebra students attending a
junior high school in Seoul, Korea were taught BASIC programming one class per week for five weeks (Kang, 1988). During the next ten weeks, during one period per week, 112 students studies mathematics using computer programs while 111 control students were taught similar content in a traditional manner. The experimental students scored significantly higher than the control students on a posttest composed of number and operation items, function items, and fundamental operations items (Kang, 1988).

In a study designed to determine the effect of a microcomputer based, interactive, graphics program on retention and conceptual learning of algebra students, Herbert Gesshel-Green (1987) used two of three intact algebra II classes. One of the two classes was randomly assigned to a treatment group which interacted with a microcomputer graphics program during classroom demonstrations and in a computer laboratory. Gesshel-Green found the interactive computer graphics had no significant effect on achievement based on scores of formative tests.

Ten suburban Detroit teachers directed half the students in twenty introductory algebra classes in writing, running, and modifying BASIC computer programs to complete some practice exercises on beginning algebra topics, while the other half completed all exercises with paper and pencil methods (Marchionini, 1981). Using analysis of variance, Gary Marchionini found that the use of computer enhanced practice led to less student achievement of particular algebraic topics, which supports other studies concluding writing, running, and modifying computer programs does not promote the learning of the mathematical topics involved.
A constructivist approach designed by Jonathon Rosenberg (1989) was used to compare two methods of CAI on systems of equations and composition of functions. Two treatments consisted of a standard formal textbook approach to teaching these algebraic concepts and a constructivist model using a symbol manipulator to manipulate and probe the mathematical constructs. The symbol manipulation and standard textbook groups performed equally well on computation tests, but the symbol manipulation group gained greater ability to understand and apply abstract concepts from the material. Use of the computer in the instruction of these algebraic concepts resulted in a deeper understanding of the concept as opposed to rote manipulation.

The computer was also used to demonstrate the ability of younger students to explore algebraic topics. Traditionally it has been assumed that students could not learn these algebraic concepts before eighth or ninth grade. Betty Collis and Geoffrey Mason (1983) developed an instructional sequence of thirteen forty-five minute lessons over a three week period using Apple II Plus microcomputers to address the problems of computer availability, minimal experience by teachers, and instruction of traditional topics using computers. Twenty-four fourth grade students explored coordinate graphing. Pretest mean scores were twenty-eight while posttest mean scores were seventy-seven. Lack of a control group and knowledge of test-item difficulties for larger groups made statistical analysis impossible, but enthusiasm and confidence were noted by the researchers (Collis & Mason, 1983).

A mathematics resource teacher along with seven classroom teachers provided instruction using a computer lab to 152 fifth and sixth grade students
three hours a week for eight months while a control group consisting of 72 students received no computer instruction (Salem, 1989). Both groups were predominantly Hispanic with fifteen percent Vietnamese. The purpose of the study was to determine how well these students could learn and apply specific mathematical skills as they learned to program a microcomputer in graphics mode with Logo and BASIC. The pretests and posttests were constructed from the Los Angeles Mathematics Project Test. Student using computer programming in mathematics did significantly better than those not using computers with a p-value of less than .001 (Salem, 1989).

Both Theodore Haynie (1989) and Richard Shumway (1984) used computer instruction with elementary school children. Haynie used CAI on math achievement with students in Calvert County, Maryland and discovered that participation in a CAI program resulted in a significant increase in mathematics achievement test scores on standardized tests for all students except those in the low-ability group. Shumway taught students in grade one to program computers. Concepts students encountered as a result of the programming were variables, sequences, number names, and recursion. Shumway concluded that "The richness of the mathematics encountered will be multiplied dramatically; more young children will see mathematics as a dynamic, creative activity; and teaching and learning mathematics will be a lively, active sport indeed" (134). Students can be successfully introduced to mathematical concepts long before the traditional sequence recommends using computer instruction.
American Indians in Mathematics

The largest minority group in Montana are American Indians. In "The Demographics of American Indians: One Percent of the People; Fifty Percent of the Diversity" (Hodgkinson, 1990), it is estimated that 34,000 American Indians are living on or near a reservation in Montana. Unemployment among American Indians in Montana range from 41% to 72% and education is a key that could be valuable in changing these dismal statistics. The dropout rate for American Indian students nationally is estimated between 30% and 50% (Backes, 1993). Although statewide data is not reported in Montana, individual school districts report a dropout rate from 14% to 85% (Bowker, 1993). Among the most serious problems confronting American Indian educators and tribal groups is that American Indian children have the highest dropout rate among all ethnic minority groups in the nation. “Current statistics suggest that 50 percent of all American Indian students now enrolled in school will not graduate” (Bowker, 1993, p. xv). Many studies have explored the factors that influence this appalling dropout rate. Some theories include factors such as personal crisis, socioeconomic status, substance abuse, legal problems, peer pressure, health, and the perceived value of education by students and parents (Backes, 1993). In addition, the level of funding for Indian education tends to be inadequate. All of these facts contribute to the lack of educational support necessary if American Indians are to break out of the unemployment, poverty cycle.

If improving the overall education of American Indians is a first step towards employment, the first step in education should be encouraging students
to stay in school and enroll in mathematics and science courses. In a study by the National Science Foundation (1988) data showed that a consistent 10 percent less American Indian students take algebra, geometry, algebra II, and trigonometry which means American Indian students are underrepresented in math courses. One outcome of a conference sponsored by the American Indian Science and Engineering Society’s (AISES) Research and Evaluation Department was that “American Indians suffer from a disproportionately high dropout rate when compared to the general population” (1995, p. 8). The University of New Mexico College of Engineering developed precollege and undergraduate programs directed at recruiting and retaining Hispanic and American Indian students in mathematics classes. This was a longitudinal program organized to increase awareness and motivation of minority students in algebra I classes. It was also developed to provide support to these students through high school and during college. Students are provided with orientation programs, tutoring, counseling, employment programs, and scholarships (Cummings, 1984).

Tribal Resource Institute in Business, Engineering, and Science (TRIBES) is a summer institute for American Indian High School graduates who plan to attend college in the fall. Students receive instruction in math and other subjects, along with help with study skills. This program has operated for seven years at Colorado College and is sponsored by the Council of Energy Resource Tribes (CERT) (TRIBES, 1988). The American Indian Science Engineering Society (AISES) provides students with academic and social support, along with sponsoring summer programs for Indian grade school students and teachers of Indian students. Included in these camps are Math Camps which utilize hands-on
approaches, computers, problem solving tasks, and culturally relevant materials (AISES News, 1988).

A three year project developed by the Center for Native American Studies and Montana State University proposed to establish a training program for American Indian students and their mathematics teachers from reservation schools. The project American Indians in Mathematics (AIM) planned to increase the participation of college-bound American Indian students in mathematics courses and to strengthen the quality of mathematics education provided to these students while still in high school. The population of the schools that participated in the project was predominantly American Indian. The project incorporated several components such as a four-week summer course for students, a six-week summer course for junior high and high school teachers, a training program for parents, activities for Indian reservation schools, and ongoing inservice training programs for other teachers (The AIM Project, 1991).

The Mathematics, Engineering, and Science Achievement (MESA) / Minority Engineering Program (MEP) was designed to recruit Black, Hispanic, and American Indian students and provide assistance in fields of mathematics (Lee, 1990). Although many studies and projects have studied the problems of American Indians staying in school and have developed programs to encourage and support students who choose to continue their education, few studies existed which focused on learning styles and sequences of instruction which would be beneficial for the learning of mathematic topics.

One study by John Backes (1993) did explore the relationship between learning styles and the learning success or failure of American Indian students. In
the study, subjects were graduates and dropouts from community high schools in North Dakota and Minnesota. Backes found the dominant personal learning style of the American Indian Chippewa was abstract random, while the dominant learning style of the non-American Indian subjects was concrete sequential. The important result of these findings was that the dominant learning style of the American Indian students coincides with a deductive, holistic instructional methodology which does not match the teaching style of many classrooms. This mis-match is a determining factor in their learning success or failure.

Recommendations include moving away from instructional strategies that focus primarily on the teacher controlling and dictating the learning environment to strategies that force students to take an active role in the learning process. This study examined the use of such a teaching strategy.

In a document developed from a three-day conference in May 1994, research indicated that American Indian students usually do not respond to a verbal, abstract style of mathematics learning and instead prefer a visual, tactile style. However, a verbal, abstract mode is often used when a visual/spatial mode would be more appropriate (American Indian Science and Engineering Society, 1995). They also found that there were other barriers to mathematics education for American Indian students such as institutional barriers, lack of role models, parental expectations, and language barriers. Institutional barriers include the lack of preparation by teachers, lack of support, outdated materials, and poor equipment.

David Davison (1992) identifies difficulties American Indians experience when learning mathematics which include content and contexts that are not
culturally relevant, a lack of English language fluency, and excessively abstract instruction. Through reviewing a number of studies, Davison (1992) reports that “A consistent picture emerges of the American Indian learner as one who prefers that mathematics be presented through physical or visual stimuli” (p. 246). Vera Preston (1991) agrees that Native students are most successful at tasks that use visual and spatial abilities. American Indian students who use hands-on manipulatives are better able to bridge the gap from concrete to pictorial to symbolic representations (Preston, 1991). This validates the techniques used in this study.

**Gender**

While the dropout rate of American Indian students is appalling, the dropout rate of American Indian females is even worse. Ardy Bowker (1993) found that American Indian females are more likely to drop out than Indian males. In *Growing Smart What’s Working for Girls in School* (1995) researchers state “Gaps in the body of data available to researchers underscore the need for more studies that include gender, race, and socioeconomic status in their data collection, analysis, and reporting” (p. vii). They suggest that “Girls benefit from greater access to computers” (p. 4) and “Incorporating plenty of hand-on activities and offering girls ample opportunities to manipulate often unfamiliar equipment are key principles of successful programs to spur girls’ enthusiasm for math and science” (p. 10). Both of these recommendations have been
incorporated into this study.

Most of the research done recently suggested that gender differences are not a factor in the learning of algebraic topics. "The research that has been done in relation to students in algebra seems to indicate that the differences are inconsequential in most instances" (Dessart & Suydam, 1983, 49). Swafford (1980) studied 320 female and 294 male students in a first-year algebra class in several different schools. No significant difference was found between boys and girls with regard to arithmetic skill or skill demonstrated by a standardized algebra test. A study by Wolleat, Pedro, Becker, and Fennema (1980) included 647 girls and 577 boys in an algebra or geometry class. Differences were found in the attitudes of girls and boys as related to their reasons for success or failure in mathematics. Fennema and Sherman (1977) found small differences between 644 male and 589 female students, and the differences which did occur were a result of sociocultural factors. Other studies which found no significant differences in attitude or achievement with respect to gender are Keller (1974), Barrick (1980), and Broomes (1971). Although all of these studies indicated few differences in mathematics achievement exist due to gender, the fact that women were still underrepresented in mathematics-related careers was well documented (Fennema, Wolleat, Pedro, & Becker, 1981, Tartre & Fennema, 1995). They concluded the reason for the lack of women in these careers was inadequate preparation in high school mathematics. In addition, in a paper presented at the Annual Meeting of the National Association of Developmental Education, a study which explored the relationship between mathematics achievement, mathematics anxiety, and attitude of American Indian students was discussed (Bretscher, 1989). The study
found significant differences between American Indian males and American Indian females in mathematics anxiety and attitude. Bowker (1993) states that "Despite the research addressing this issue, females and minorities have not been encouraged to take traditionally male courses such as math, science, and computer education courses" (p. 8). Further research, although not indicating differences, could find ways to encourage women to continue to study mathematics.

Conclusion

This review of the literature indicated that although research existed which explored the use of computers in the instruction of algebraic topics, few studies used the computer in addition to other embodiments in the instruction of solving systems of linear equations. Many of the studies described methods used to find the solution to a single equation with one unknown, but this concept was not extended to include the solution of two equations with two unknowns. The literature indicated the possible usefulness of several concrete experiences in learning algebraic concepts, yet these models were not studied for actual effectiveness. While many projects have been developed in an attempt to increase the participation of American Indians in mathematics classes, actual studies exploring methods that are successful in teaching these minority students are lacking.
CHAPTER III

PROCEDURES

Introduction

The primary purpose of this study was to investigate the effectiveness of using concrete, semi-concrete, and abstract pre-algebra learning models, utilizing multiple embodiments on the achievement of 6th and 7th grade students. The secondary purposes of the study were to investigate whether or not American Indian students would do as well as other students, whether or not 6th grade students would do as well as 7th grade students and whether female students would do as well as male students when using concrete, semi-concrete, and abstract pre-algebra learning models utilizing multiple embodiments.

This chapter includes a description of the population, a description of the treatments, the research design, the methods of data collection, the statistical hypotheses, and an analysis of the data.

Population Description

The population in this study was sixth and seventh grade students enrolled in two school districts in Montana. The sixth and seventh grades were
chosen because NCTM recommends that students in grades 5-8 explore algebraic concepts (NCTM, 1989). In addition, several sources recommend the introduction of algebraic concepts prior to a formal algebra course (Davis, 1989; Demana and Leitzel, 1988; Kaput, 1995b; Thompson, 1988). Montana students are generally introduced to algebra in the ninth grade. Teachers involved in the AIM project were contacted in the summer of 1991. Additional information was provided to those teachers who indicated an interest in participating in the study. Each teacher chosen was provided with manipulative materials, a complete set of instructions, tests, computer software, and student worksheets. After completing the study, all tests were returned for evaluation.

Browning and Poplar, two Montana schools located on American Indian reservations, participated in the study. Classes were randomly assigned to one of the three treatment groups. One teacher from each school was provided with a demonstration of the materials to be used, including the manipulatives and the computer modeling program while participating in a six week workshop on the Montana State University campus. During this six week period, the teachers were involved in a pilot study with American Indian students. Then all the teachers were provided with a video tape, which described in detail how the manipulatives and the computer program were to be used, the methodology for each treatment, and how the testing would be done.
Description of Treatments

Instruction consisted of teaching the concept of solving equations with unknowns. The instruction period was two weeks or ten class days. The study consisted of three treatments:

Treatment I - a multiple embodiment instructional sequence including manipulatives (concrete), pictorial activities (semi-concrete), and symbolic manipulations (abstract) with computer modeling.

Treatment II - a multiple embodiment instructional sequence including manipulatives (concrete), pictorial activities (semi-concrete), and symbolic manipulations (abstract), but without the computer modeling.

Treatment III - a sequence based on a traditional textbook sequence.

Ten fifty-minute periods were used to complete the entire sequence and address all the instructional objectives, yet fit into the time allowed for the topic in the school systems. Since most sixth grade textbooks do not include instruction on solving equations, the material in the study was not a part of the general mathematics curriculum to be covered by these teachers. A complete description of the treatments can be found in Appendix III.

The objectives of the sequences were:

1. Develop an ability to solve systems of equations with one equation and one unknown and two equations and two unknowns using abstract symbolism.

2. Develop an understanding of systems of equations and the concept of variable in that circumstance.
Treatment I

An instructional sequence was developed by Dr. Lyle Andersen and Dr. Warren Esty, Professors in the Department of Mathematical Sciences at Montana State University, to introduce solving systems of equations. The sequence progressed from concrete to semi-concrete to abstract levels using multiple embodiments. Student materials were developed and a pilot study involving seventh grade students was performed in the summer of 1991. Several modifications were made as a result of the pilot such as the removal of duplicated materials, better written instructions for both instructor and student, and the addition of exploratory activities. The instructional sequence consisted of ten fifty minute periods.

The lessons began with students working with film boxes and paperclips in a lab setting. A minimal amount of instruction was provided to the students prior to exposure to the manipulatives. Students were told that the film containers in each lab problem contained the same number of paperclips and that the same number of paperclips were on each side of the lab sheet, separated by a heavy black line. They were then asked to find the hidden number of paperclips without opening the container. Each student was to record their answers on a lab worksheet. Interaction between students was encouraged, but students were required to complete their own worksheet. After the students had worked on the labs for ten minutes, the worksheets were collected and students were brought back together as a group. The instructor then gave the students, via the overhead projector, a problem similar to the problems presented in the lab, but involving more than just one container. Discussion on a procedure to discover
the number of paperclips in the containers was encouraged. After this discussion, students were once again asked to complete a set of lab problems of a more complex nature. These concrete sets of activities were the first embodiment.

The procedure was repeated in a semi-concrete format, with students drawing pictures to represent the problems presented to them. Once again, the instructor provided them with an example, and then the students were asked to complete a set of lab problems, this time representing each of the problems with a pictorial representation. This set of activities was the second embodiment.

The modeling program was used to facilitate the students' transition from the concrete manipulatives, to the semi-concrete pictures and finally, to the abstract symbolic representation. The computer program, which simulated systems of equations with one equation and one unknown, was the third embodiment. Students were required to solve the problems presented to them on the screen.

The final embodiment involved helping the students to develop a symbolism to represent the problems, and the ability to use the symbolism to solve the problems. By using algebraic notation, the students made the final transition from the semi-concrete or pictorial to the abstract.

The same sequence was used to develop the concept using two equations and two unknowns. The students began again with the concrete manipulatives, moved to the semi-concrete pictures, and finally to the abstract symbolism.

Treatment II

Treatment II was identical to Treatment I except for the use of the
computer modeling program. Students were not allowed to use the modeling program.

**Treatment III**

A traditional textbook instructional sequence was developed for solving systems of equations. The model was found in a mathematics pre-algebra textbook *Holt Introductory Algebra I* (1988) by R. F. Jacobs. This textbook is a popular text that is used in a number of algebra classrooms. The instruction and activities were teacher directed, beginning with the abstract representation of equations, and using a demonstration to describe the method to be used. Students were asked to emulate the instructor's actions.

**The Software**

A program was written by Dr. Lyle Andersen, professor at Montana State University, to simulate the semi-concrete portion of the teaching sequence. The programming language used to create the program was Logowriter. The program gave the students problems similar to the problems in the labs, and the students were allowed to use the manipulatives to solve the problems presented to them. In order for software to be effective, Douglas Clements (1989) suggests it must meet certain criteria:

1. The content should be educationally and mathematically significant and match the mathematics curricula.
2. The mathematics, spelling, and grammar should be accurate.
3. The software should be consistent with the rules of an educational
4. The feedback should be high-quality. "Immediate feedback is most useful for younger students or those just being introduced to the material" (44).

5. Graphics, color, and sound should help illustrate concepts.

6. The program should be interesting to children, containing elements of challenge and curiosity.

7. Programs should use graphics that motivate children.

8. The program should be easy to use for the student and the teacher.

Harley Flanders also claims the software should be interactive, as "Users should feel they are a part of the process" (1988, p.149). Teachers from the AIM (American Indians in Mathematics) project in the summer of 1989 evaluated the software to determine if it was effective. They determined that the computer program met all eight of these criteria. For a sequence of pictures from the software as well as a copy of the program used, see Appendix IV. One computer was at the front of the class, to provide problems for the students to work in groups.

Advantages of using Logowriter to write the program include: (a) the language can be used with a variety of brands of computers; (b) the graphics available are good; (c) the language is friendly and therefore the program is easily debugged or changed; (d) both graphics and text can appear on the screen.
Research Design

A nonequivalent control-group design was used as the experimental model. This is one of the most widely used quasi-experimental designs in educational research since the experimental and control groups are not formed randomly (Borg & Gall, 1989). The distinguishing features of this design are the administration of a pretest and posttest to all treatment groups and nonrandom assignment of subjects to the groups. The classes were randomly assigned one of three treatment groups: (a) the multiple embodiment instructional sequence with the computer as one embodiment, (b) the multiple embodiment instructional sequence without the computer, (c) a traditional textbook instructional sequence. Pretests were administered from one to two weeks before instruction began. The posttest was given the day following the last day of instruction. The instruction occurred over a period of ten school days. All tests were used to evaluate the hypotheses of the study.

Nonequivalent Control Group Design

The nonequivalent control-group design was used to measure the effect of treatments. This design is recommended for use with intact groups, such as assembled classrooms, since the ability to completely randomize was limited. Gay (1981) encourages this design when working with intact groups. Campbell and Stanley (1963) add that this design controls the main effects of six contaminating variables that include history, maturation, testing, instrumentation, selection, and mortality. An internal validity problem with this design is the control of
regression. Regression should not be a concern if treatment groups have not been selected based on extreme scores on the pretest (Campbell & Stanley, 1963). Since classrooms were assigned to treatments prior to testing, this factor is minimally controlled.

Methods of Data Collection

Orleans-Hanna Algebra Prognosis Test (OHPT)

The OHPT provides students with an indication of how they might perform in a first year algebra course. The test contains 60 items grouped under nine lessons. The student reads the lesson and then answers corresponding questions. These items are intended to test aptitude which is described as the ability to handle new learning situations that are similar to ones they will encounter in a first year algebra course (Kuchemann, 1985). The test was designed to reflect contemporary language and concepts in teaching algebra.

One method suggested for calculating the reliability of a test is the coefficient of equivalence and stability. This is computed by preparing two parallel forms of the test and administering them to a group of individuals with a time interval between the administration of the first and second test (Borg & Gall, 1989). The scores on the tests are then correlated to obtain a reliability coefficient. "The reliability coefficient reflects the extent to which a test is free of error variance" (Borg & Gall, 1989, p. 257), which is the chance differences between persons that arise from factors associated with a particular measurement,
such as the wording of the test or the ordering of test items. The reliability coefficient is expressed as a value between .00 and 1.00 with 1.00 indicating perfect reliability and .00 indicating no reliability (Borg & Gall, 1989).

In the spring of 1980, two forms of the OHPT were administered to three schools approximately two weeks apart. The reliability coefficient obtained was 0.94 (Hanna & Orleans, 1982), indicating the test is highly reliable. In addition, the Kudor-Richardson Formula 20 was used to estimate the internal consistency of the OHPT. The data was based on 1195 seventh grade students who participated in the standardization of the examination. The Kudor-Richardson Formula 20 reliability coefficient was 0.95 (Kuchemann, 1985).

A variety of longitudinal investigations were undertaken to provide information on the validity of the OHPT. One study in particular was undertaken from the spring of 1981 to the spring of 1982 using 21 school districts. The corresponding validity coefficients for the end of the year grades with the 1982 editions' scores was 0.75 (Hanna & Orleans, 1982). Therefore, the OHPT does provide a reasonably good indication of how well a student might do in an algebra course (Secolsky, 1985).

Criterion-Reference Test

Since this study was a test of solving systems of equations and not general math ability or achievement, other examinations of ability would not be as effective at testing students ability as a criterion-referenced test. A criterion-referenced test (CRT) was used which tested both concept and skill knowledge. The test used the same format as the OHPT, with a lesson and then six questions
relating to the lesson. The students were to solve systems of equations given in both pictorial and symbolic form. The same format was chosen to be consistent with the OHPT.

A complete description of the test can be found in Appendix II.

Pilot Study

The pilot study was conducted at Montana State University in the summer of 1991 with students and teachers participating in the American Indians in Mathematics Project (AIM). Five teachers volunteered to use the materials and activities developed with the twenty-six American Indian students participating in the AIM workshop. During the study, the students were divided randomly into two groups. One group used the activities and worksheets for the treatment without a computer component and the other group used the activities and worksheets along with the modeling program designed for this treatment. Teachers kept a daily log of the success of the activities, the time completion, and any suggested changes in the instruction for the teachers. Based on the input of the teachers from their logs and personal comments, along with the students' responses to the activity worksheets, revisions were made in time allotments, teacher instruction, student worksheets, and the modeling program.

Further testing of the modeling program and manipulative activities used was conducted in the fall of 1991. Graduate students and faculty members attending a mathematics education seminar at Montana State University reviewed
the changes made in the activities and modeling program as recommended in the pilot study. The changes were compared to the comments and suggestions of the teachers participating in the pilot study for accuracy in their interpretation. Further modifications were made in the student worksheets and instructions to the students, along with minor changes in the modeling program. Members of the seminar agreed the changes were consistent with the recommendations of the pilot study instructors.

Modifications made included more explicit instructions for the participating teachers. The lesson plans were designed to be easier for instructors to follow and implement. Labs were numbered in a numerical sequence, along with introduction materials so that the sequence was easier to follow. The labs were modified slightly, changing the number of labs requiring completion by students, and the way the lab using two equations with two unknowns was presented. Time allotments were changed to allow more time for some activities and less time for others and lessons were changed from 50 minutes to 40 minutes. The general order of the lessons was unchanged. Instructions for students on worksheets were modified to be more explicit and easier to understand.

Statistical Hypotheses

1A. There was no significant difference in student achievement among treatments as measured by the difference of pre-instruction and post-instruction OHPT scores.
1B. There was no significant difference in student achievement among treatments as measured by the difference of pre-instruction and post-instruction CRT scores.

1C. There was no significant difference in student achievement among treatments as measured by items on the post-instruction CRT.

2A. There was no significant difference in student achievement between American Indian students and other students as measured by the difference of pre-instruction and post-instruction OHPT scores.

2B. There was no significant difference in student achievement between American Indian students and other students as measured by the difference of pre-instruction and post-instruction CRT scores.

2C. There was no significant difference in student achievement between American Indian students and other students as measured by items on the post-instruction CRT.

3A. There was no significant difference in student achievement between sixth and seventh grade students as measured by the difference of pre-instruction and post-instruction OHPT scores.

3B. There was no significant difference in student achievement between sixth and seventh grade students as measured by the difference of pre-instruction and post-instruction CRT scores.

3C. There was no significant difference in student achievement between sixth and seventh grade students as measured by items on the post-instruction CRT.

4A. There was no significant difference in student achievement between
gender groups as measured by the difference of pre-instruction and post-instruction OHPT scores.

4B. There was no significant difference in student achievement between gender groups as measured by the difference of pre-instruction and post-instruction CRT scores.

4C. There was no significant difference in student achievement between gender groups as measured by items on the post-instruction CRT.

5A. There was no interaction between treatment and gender as measured by the difference of pre-instruction and post-instruction OHPT scores.

5B. There was no interaction between treatment and gender as measured by the difference of pre-instruction and post-instruction CRT scores.

5C. There was no interaction between treatment and gender as measured by items on the post-instruction CRT.

6A. There was no interaction between treatment and ethnicity as measured by the difference of pre-instruction and post-instruction OHPT scores.

6B. There was no interaction between treatment and ethnicity as measured by the difference of pre-instruction and post-instruction CRT scores.

6C. There was no interaction between treatment and ethnicity as measured by items on the post-instruction CRT.

7A. There was no interaction between treatment and grade level as measured by the difference of pre-instruction and post-instruction OHPT scores.

7B. There was no interaction between treatment and grade level as measured by the difference of pre-instruction and post-instruction CRT scores.

7C. There was no interaction between treatment and grade level as measured
by items on the post-instruction CRT.

8A. There was no interaction between gender and ethnicity as measured by
the difference of pre-instruction and post-instruction OHPT scores.

8B. There was no interaction between gender and ethnicity as measured by
the difference of pre-instruction and post-instruction CRT scores.

8C. There was no interaction between gender and ethnicity as measured by
items on the post-instruction CRT.

Analysis of Data

Independent variables to be used in the analysis of data were:

1. Treatment - experimental with computer, experimental without computer,
   and textbook control;

2. Gender - male and female.

3. Grade level - 6th and 7th.

4. Ethnicity - American Indian and non-American Indian.

Dependent variables were:

1. The difference of pre-instruction and post-instruction achievement scores
   on the Orleans-Hanna Algebra Prognosis test;

2. The difference of pre-instruction and post-instruction achievement scores
   on the criterion-referenced test;

3. The post-instruction achievement scores on the criterion-referenced test.

Statistical Analysis System (SAS) was used to analyze the data.
Equivalent Groups

The main threat to the internal validity of the nonequivalent control group design was the possibility that group differences on the posttest were due to preexisting group differences rather than to a treatment effect (Borg & Gall, 1989). Walter R. Borg and Meredith D. Gall (1989) suggest the use of analysis of covariance to reduce the effect of initial group differences statistically by making adjustments to the posttest means of the groups. "Analysis of covariance analyzes the differences between experimental groups on the dependent variable, after taking into account either initial differences between the groups on the pretest, or differences between the groups in some pertinent independent variable or variables, substantially correlated with the dependent variable" (Kerlinger, 1986, p. 339). Since the study involved intact groups, there was a possibility that the scores could be different based on criterion apart from the treatments received. Analysis of covariance (ANCOVA) was used to determine whether the sample means were significantly different from one another by adjusting the mean group scores to remove bias. This would enhance the power of the statistical tests.

Factorial Analysis of Variance

Kerlinger states that "Factorial analysis of variance is the statistical method that analyzes the independent and interactive effects of two or more independent variables on a dependent variable" (1986, p. 228). A factorial experiment determines not only the effect of two or more independent variables, each by itself but also in interaction with each other, on a dependent variable (Borg &
Gall, 1989). The main effect is the effect of each independent variable on the dependent variable, but of equal interest is the interaction effect, the effect of the interaction of two or more independent variables on the dependent variable. Since there were an unequal number of subjects per cell, a general linear model was used.

Factorial analysis has several advantages such as the:

1. Ability to manipulate and control two or more variables simultaneously;
2. Ability to study the interactive effects of independent variables on dependent variables;
3. Precision is better than with one-way analysis;
4. Variables not manipulated can be controlled by building them into the design.

The effect of each variable was evaluated after all other factors had been accounted for.

**Alpha Level**

A rejection of the null hypothesis in this study when it is indeed correct could result in teachers spending time and money developing an instructional sequence which results in no improvement in student success. However, failure to reject the null hypothesis when there really is a difference may result in students not being exposed to an educational sequence that may increase achievement. As a result of these considerations, an alpha level of .05 was chosen.
CHAPTER IV

ANALYSIS OF THE DATA

Introduction

The primary purpose of this study was to determine if there is any difference in achievement regarding the effectiveness of using concrete, semi-concrete, and abstract pre-algebra learning models on the achievement of 6th and 7th grade students. The secondary purposes of the study were to determine if there is any difference in achievement between American Indians and other students, between male and female students, and between sixth and seventh grade students. Subjects in the study were sixth and seventh grade students in Montana. The schools used in Montana were in Poplar and Browning and are located on the Fort Peck Indian Reservation and the Blackfeet Indian Reservation respectively. The student population at these schools is primarily American Indian. The distribution of the students in the study is recorded in Table 1.
The study consisted of two tests administered prior to instruction, three instructional treatments, and the same two tests administered after instruction. The two tests consisted of the Orleans-Hanna Algebra Prognosis test (OHPT) and a locally created criterion-referenced test (CRT). For a complete description of these tests see Appendix II and for a complete description of the treatments see Appendix III. The students pretest results for the two tests (OHPT and CRT) are found in Appendix I. The OHPT is designed to predict success in learning algebra. The OHPT contains 60 multiple choice items based on nine lessons and scores are on a scale of 0 - 60. The student reads through each lesson before attempting the corresponding items. The score on the test is intended to test aptitude or the ability to handle new learning situations similar to those encountered in an algebra course. To test the validity of the OHPT, a variety of longitudinal investigations have been conducted. For example, a study conducted from the spring of 1980 to the spring of 1981 with 21 school districts.
found the correlation coefficient between the OHPT and the end of the year grades for eighth grade students was .75 (Orleans & Hanna, 1982). The CRT was a locally developed test used to test the specific content addressed by the treatments: solving a single linear equation with one unknown, a linear equation with two unknowns, and a system of linear equations with two unknowns. The scale of scores was 0 - 18.

The OHPT was used to adjust the mean group scores to remove bias. Some adjustment was necessary since the groups were intact which made random assignment impossible. The OHPT was chosen because it is well known and has been extensively studied. Because the treatment groups were intact classes, the number of subjects in each category could not be adjusted and the design was unbalanced with unequal numbers of subjects in the cells. Therefore, a general linear model was used (Borg and Gall, 1989). The data was analyzed using analysis of covariance (ANCOVA) with pretest scores on the Orleans-Hanna Algebra Prognosis test as a covariate. Three dependent variables were considered: A) the difference between the pretest and posttest scores on the OHPT (Hypotheses using this variable are labeled “A”); B) the difference between the pretest and posttest scores on the CRT (Hypotheses using this variable are labeled “B”); C) the posttest score on the CRT (Hypotheses using this variable are labeled “C”). Because the OHPT pretest scores was a covariate, the fourth possible combination, the posttest score on the OHPT, yields precisely the same probabilities and results as the first dependent variable.

The Orleans-Hanna Algebra Prognosis test is designed to predict success in learning algebra. Therefore the difference between the pretest and posttest
scores on the OHPT (dependent variable A) should relate to the success of the treatments in preparing students for algebra in general. The difference between the pretest and posttest scores on the CRT (dependent variable B) and the posttest CRT scores (dependent variable C) should relate to the success of the treatments in teaching the particular content material about equations and unknowns discussed in the treatments. The three different dependent variables were recommended by Dr. John Borkowski, Associate Professor of Mathematics and Statistics at Montana State University, to test not only the amount of overall mean difference between the pretest and posttest scores on both the OHPT and the CRT, but the overall mean score difference on the CRT posttest. In Tables 2 and 3, the means and standard deviations of raw scores on the Orleans-Hanna Algebra Prognosis Test are given. In Tables 4 and 5, the means and standard deviations of raw scores on the criterion-referenced test are given. The complete data is in Appendix I.
### TABLE 2.
**Means of Raw Scores on the OHPT (sample sizes in parenthesis)**

<table>
<thead>
<tr>
<th></th>
<th>Treatment I (manipulatives with computer)</th>
<th>Treatment II (manipulatives only)</th>
<th>Treatment III (textbook only)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>pre diff</td>
<td>post</td>
<td>pre diff</td>
</tr>
<tr>
<td><strong>Poplar 7th Grade</strong></td>
<td>15.3</td>
<td>16.1</td>
<td>14.4</td>
</tr>
<tr>
<td>mean diff.</td>
<td>0.8</td>
<td>1.3</td>
<td>-0.4</td>
</tr>
<tr>
<td>(sample size)</td>
<td>(10)</td>
<td>(24)</td>
<td>(17)</td>
</tr>
<tr>
<td><strong>Poplar 6th Grade</strong></td>
<td>19.2</td>
<td>20.3</td>
<td>14</td>
</tr>
<tr>
<td>mean diff.</td>
<td>1.1</td>
<td>2.1</td>
<td>-2.5</td>
</tr>
<tr>
<td>(sample size)</td>
<td>(12)</td>
<td>(15)</td>
<td>(13)</td>
</tr>
<tr>
<td><strong>Browning 6th Grade</strong></td>
<td>16.1</td>
<td>18.8</td>
<td>14</td>
</tr>
<tr>
<td>mean diff.</td>
<td>2.7</td>
<td>2.6</td>
<td>0.1</td>
</tr>
<tr>
<td>(sample size)</td>
<td>(34)</td>
<td>(37)</td>
<td>(34)</td>
</tr>
</tbody>
</table>

### TABLE 3.
**Standard Deviations of Raw Scores on the OHPT (sample sizes are the same as in Table 2)**

<table>
<thead>
<tr>
<th></th>
<th>Treatment I (manipulatives with computer)</th>
<th>Treatment II (manipulatives only)</th>
<th>Treatment III (textbook only)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>pre</td>
<td>post</td>
<td>pre</td>
</tr>
<tr>
<td><strong>Poplar 7th Grade</strong></td>
<td>6.75</td>
<td>4.77</td>
<td>5.71</td>
</tr>
<tr>
<td><strong>Poplar 6th Grade</strong></td>
<td>7.86</td>
<td>6.25</td>
<td>4.80</td>
</tr>
<tr>
<td><strong>Browning 6th Grade</strong></td>
<td>7.30</td>
<td>7.13</td>
<td>6.62</td>
</tr>
</tbody>
</table>
TABLE 4.
Means of Raw Scores on the CRT (sample sizes in parenthesis)

<table>
<thead>
<tr>
<th>Treatment I</th>
<th>Treatment II</th>
<th>Treatment III</th>
</tr>
</thead>
<tbody>
<tr>
<td>manipulatives with computer</td>
<td>manipulatives only</td>
<td>textbook only</td>
</tr>
<tr>
<td>pre</td>
<td>diff</td>
<td>post</td>
</tr>
<tr>
<td>Poplar 7th Grade</td>
<td>3.70</td>
<td>1.70</td>
</tr>
<tr>
<td>Poplar 6th Grade</td>
<td>3.83</td>
<td>2.17</td>
</tr>
<tr>
<td>Browning 6th Grade</td>
<td>3.94</td>
<td>1.53</td>
</tr>
</tbody>
</table>

TABLE 5.
Standard Deviations of Raw Scores on the CRT
(sample sizes are the same as in Table 2)

<table>
<thead>
<tr>
<th>Treatment I</th>
<th>Treatment II</th>
<th>Treatment III</th>
</tr>
</thead>
<tbody>
<tr>
<td>manipulatives with computer</td>
<td>manipulatives only</td>
<td>textbook only</td>
</tr>
<tr>
<td>pre</td>
<td>post</td>
<td>pre</td>
</tr>
<tr>
<td>Poplar 7th Grade</td>
<td>2.28</td>
<td>2.01</td>
</tr>
<tr>
<td>Poplar 6th Grade</td>
<td>2.11</td>
<td>2.41</td>
</tr>
<tr>
<td>Browning 6th Grade</td>
<td>1.80</td>
<td>2.74</td>
</tr>
</tbody>
</table>
The three models all have the same form and use treatment, ethnicity, grade level, gender, school, and pretest OHPT scores as independent variables.

\[
Y_{ijklmn} = \mu + T_i + E_j + L_k + G_l + S_m + P_n + TE_{ij} + TL_{ik} + TG_{il} + TS_{im} + TP_{in} + EL_{jk} + EG_{jl} + ES_{jm} + EP_{jn} + LG_{kl} + LP_{ln} + GS_{lm} + GP_{ln} + SP_{mn} + \varepsilon_{ijklmn}
\]

where \( T_i \) is the main effect due to treatment, 
\( E_j \) is the main effect due to ethnicity, 
\( L_k \) is the main effect due to grade level, 
\( G_l \) is the main effect due to gender, 
\( S_m \) is the main effect due to school, 
\( P_n \) is the main effect due to pretest scores on OHPT, 
and the rest are interactions symbolized with the same letters.

The main effects that are of interest are treatment, ethnicity, grade level, and gender. The effects of school are not of interest and the pretest scores are used as covariates. The interactions that are of interest are treatment with ethnicity, treatment with grade level, treatment with gender, treatment with
pretest OHPT scores, and gender with ethnicity.

Post-Instruction Analysis

Hypotheses 1A, 1B, and 1C refer to the treatment effect. Hypotheses 2A, 2B, and 2C refer to the effect of ethnicity. Hypotheses 3A, 3B, and 3C refer to the effect of grade level. Hypotheses 4A, 4B, and 4C refer to the effect of gender. Three different dependent variables were used to discuss these hypotheses. The first dependent variable was the difference of pretest and posttest OHPT scores. This was used in Hypotheses 1A, 2A, 3A and 4A and the ANCOVA is reported in Table 6. The second dependent variable was the difference of pretest and posttest CRT scores. This was used in Hypotheses 1B, 2B, 3B, and 4B and the ANCOVA is reported in Table 7. The third dependent variable was the posttest CRT scores. This was used in Hypotheses 1C, 2C, 3C, and 4C and the ANCOVA is reported in Table 8.
TABLE 6.
Difference of Pre Instruction and Post Instruction OHPT Scores
ANCOVA of Test Scores (N = 196):

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>Sum of squares</th>
<th>F value</th>
<th>p &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treat</td>
<td>2</td>
<td>371.3</td>
<td>5.01</td>
<td>0.0076</td>
</tr>
<tr>
<td>Ethnic</td>
<td>1</td>
<td>20.3</td>
<td>0.55</td>
<td>0.4602</td>
</tr>
<tr>
<td>Grade</td>
<td>1</td>
<td>17.6</td>
<td>0.48</td>
<td>0.4911</td>
</tr>
<tr>
<td>Gender</td>
<td>1</td>
<td>47.2</td>
<td>1.27</td>
<td>0.2605</td>
</tr>
<tr>
<td>School</td>
<td>1</td>
<td>25.6</td>
<td>0.69</td>
<td>0.4065</td>
</tr>
<tr>
<td>Residual</td>
<td>177</td>
<td>6554.3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 7.
Difference of Pre Instruction and Post Instruction CRT Scores
ANCOVA of Test Scores (N = 196):

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>Sum of squares</th>
<th>F value</th>
<th>p &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treat</td>
<td>2</td>
<td>45.85</td>
<td>3.12</td>
<td>0.0468</td>
</tr>
<tr>
<td>Ethnic</td>
<td>1</td>
<td>2.79</td>
<td>0.38</td>
<td>0.5392</td>
</tr>
<tr>
<td>Grade</td>
<td>1</td>
<td>7.14</td>
<td>0.97</td>
<td>0.3258</td>
</tr>
<tr>
<td>Gender</td>
<td>1</td>
<td>57.1</td>
<td>7.76</td>
<td>0.0059</td>
</tr>
<tr>
<td>School</td>
<td>1</td>
<td>1.08</td>
<td>0.15</td>
<td>0.7018</td>
</tr>
<tr>
<td>Residual</td>
<td>177</td>
<td>1302.4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 8.
Post Instruction CRT Scores ANCOVA of Test Scores (N = 196):

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>Sum of squares</th>
<th>F value</th>
<th>p &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treat</td>
<td>2</td>
<td>72.8</td>
<td>8.01</td>
<td>0.0005</td>
</tr>
<tr>
<td>Ethnic</td>
<td>1</td>
<td>1.28</td>
<td>0.28</td>
<td>0.5969</td>
</tr>
<tr>
<td>Grade</td>
<td>1</td>
<td>5.72</td>
<td>1.26</td>
<td>0.2634</td>
</tr>
<tr>
<td>Gender</td>
<td>1</td>
<td>18.4</td>
<td>4.05</td>
<td>0.0456</td>
</tr>
<tr>
<td>School</td>
<td>1</td>
<td>6.69</td>
<td>1.47</td>
<td>0.2268</td>
</tr>
<tr>
<td>Residual</td>
<td>177</td>
<td>804.8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Effect Due to Treatment

Hypothesis 1A: There was no significant difference in student achievement among treatments as measured by the difference of pre-instruction and post-instruction OHPT scores.

Decision: Reject the null hypothesis at $p = 0.0076$.

Hypothesis 1B: There was no significant difference in student achievement among treatments as measured by the difference of pre-instruction CRT and post-instruction CRT scores.

Decision: Reject the null hypothesis at $p = 0.0468$.

Hypothesis 1C: There was no significant difference in achievement among treatments as measured by items on the post-instruction CRT.

Decision: Reject the null hypothesis at $p = 0.0005$.

There was a significant difference in mean scores of treatment groups on the difference between pretest and post-instruction CRT scores (Hypothesis 1B). Both the computer-manipulative group and the manipulative group (Treatments I and II) scored higher than the traditional textbook treatment (Treatment III). The least square means for Treatment I, II, and III were 1.98, 1.73, and 0.38 (Table 9).
There was a significant difference in mean scores of treatment groups on the post-instruction CRT (Hypothesis 1C). Both the computer-manipulative group and the manipulative group (Treatments I and II) scored higher than the traditional textbook treatment (Treatment III). The least square means for Treatment I, II, and III were 5.85, 5.44 and 3.79 (Table 10).

The significance of Hypothesis 1A is more subtle. The improvement in students' scores was strongly related to their OHPT pretest score. Perhaps not surprisingly, students with higher OHPT pretest scores improved less. There is an effect on posttest OHPT scores due to treatment but it is less for students who
initially scored higher, and the raw dependent variable scores do not show this clearly. The relevant data can be visualized in three graphs which plot the pretest and posttest (OHPT) scores for each of the three treatments. Graph 1 plots the pretest/posttest OHPT scores for all students who underwent treatment I (manipulatives with computer), Graph 2 plots the pretest/posttest OHPT scores for all students who underwent treatment II (manipulatives only), and Graph 3 plots the pretest/posttest OHPT scores for all students who underwent treatment III (textbook only). The line $y = x$ has been drawn to represent no improvement from the pretest to the posttest OHPT scores. Students who improved are above this line. Note that in Graph 1, students are further above this line for small pretest values than students in Graph 3. Students are further above this line for small pretest values in Graph 2 than in Graph 3 as well. The effect due to treatment as measured by the difference of pre-instruction OHPT and post-instruction OHPT scores is not very evident without accounting for the pretest scores. Other graphs found in Appendix V demonstrate a finer breakdown of the data by ethnicity, gender, grade level and school.
GRAPH 1
Plot of Pretest and Posttest OHPT Scores for Treatment I (Manipulatives with computer)

C = ONE OBSERVATION
2 = TWO OBSERVATIONS
3 = THREE OBSERVATIONS
GRAPH 2
Plot of Pretest and Posttest OHPT Scores for Treatment II (Manipulatives only)

$N = \text{ONE OBSERVATION}$
$2 = \text{TWO OBSERVATIONS}$
$5 = \text{FIVE OBSERVATIONS}$

PRETEST

POSTTEST
One observation is out of range (49,60).
Effect Due to Ethnicity

Hypothesis 2A: There was no significant difference in achievement between American Indian students and other students as measured by the difference of pre-instruction and post-instruction OHPT scores.

Decision: Retain the null hypothesis at \( p = 0.4602 \)

Hypothesis 2B: There was no significant difference in achievement between American Indian students and other students as measured by the difference of pretest scores and post-instruction CRT scores.

Decision: Retain the null hypothesis at \( p = 0.5392 \).

Hypothesis 2C: There was no significant difference in achievement between American Indian students and other students as measured by items on the post-instruction CRT.

Decision: Retain the null hypothesis at \( p = 0.5969 \).

| ETHNICITY           | MEAN | Std Err | \( p > |T| \) |
|---------------------|------|---------|------------|
| American Indian     | 1.19 | 0.277   | 0.5392     |
| Non-American Indian | 1.54 | 0.502   |            |
Graphs found in Appendix V show a breakdown of the data of the effects on achievement due to treatment with ethnicity.

**Effect Due to Grade Level**

Hypothesis 3A: There was no significant difference in achievement between sixth and seventh grade students as measured by the difference of pre-instruction and post-instruction OHPT scores.

Decision: Retain the null hypothesis at p = 0.4911.

Hypothesis 3B: There was no significant difference in achievement between sixth and seventh grade students as measured by the difference of pretest scores and post-instruction CRT scores.

Decision: Retain the null hypothesis at p = 0.3258.

Hypothesis 3C: There was no significant difference in achievement between sixth and seventh grade students as measured by items on the post-
instruction CRT.

Decision: Retain the null hypothesis at $p = 0.2634$

### TABLE 13.
Difference of Pre Instruction and Post Instruction CRT Scores (Grade Level)
Means, Standard Errors and P-values: Least Square Means adjusted for
treatment, ethnicity, gender, and pretest OHPT scores.

| GRADE LEVEL | MEAN | Std Err | $p > |T|$ |
|-------------|------|---------|------|
| 6th grade   | 1.02 | 0.294   | 0.3258 |
| 7th grade   | 1.71 | 0.572   |       |

### TABLE 14.
Post Instruction CRT Scores (Grade Level)
Means, Standard Errors and P-values: Least Square Means adjusted for
treatment, ethnicity, gender, and pretest OHPT scores.

| GRADE LEVEL | MEAN | Std Err | $p > |T|$ |
|-------------|------|---------|------|
| 6th grade   | 4.72 | 0.231   | 0.2634 |
| 7th grade   | 5.33 | 0.450   |       |

Graphs found in Appendix V show a breakdown of the data of the effects on achievement due to treatment with grade level.

**Effect Due to Gender**

Hypothesis 4A: There was no significant difference in achievement between gender groups as measured by the difference of pre-instruction and post-instruction OHPT scores.
Decision: Retain the null hypothesis at $p = 0.2605$.

Hypothesis 4B: There was no significant difference in achievement between gender groups as measured by the difference of pretest scores and post-instruction CRT scores.

Decision: Reject the null hypothesis at $p = 0.0059$.

Hypothesis 4C: There was no significant difference in achievement between gender groups as measured by items on the post-instruction CRT.

Decision: Reject the null hypothesis at $p = 0.0456$.

There is a significant difference in mean scores of gender groups on the difference of pre-instruction and post-instruction CRT scores (Hypothesis 4B). Females scored higher than males on the least square means. Mean scores were 2.07 for females and 0.71 for males (Table 15).

| GENDER | MEAN | Std Err | $p > |T|$ |
|--------|------|---------|--------|
| female | 2.07 | 0.35    | 0.0059 |
| male   | 0.71 | 0.39    |        |
There is a significant difference in mean scores of gender groups on the post-instruction CRT (Hypothesis 4C). Females scored higher than males on the least square means. Mean scores were 5.40 for females and 4.66 for males (Table 16).

| GENDER | MEAN | StdErr | p>|T| |
|--------|------|--------|---|
| female | 5.40 | 0.274  | 0.0456 |
| male   | 4.66 | 0.310  | |

Graphs found in Appendix V show a breakdown of the data of the effects on achievement due to treatment with gender.

**Analysis of Two-Way Interactions**

There were no significant interactions of the independent variables. The data relating to hypotheses 5A, 6A, 7A, and 8A stated in Chapter III are reported in Table 17. The data relating to hypotheses 5B, 6B, 7B, and 8B stated in Chapter III are reported in Table 18. The data relating to hypotheses 5C, 6C, 7C, and 8C stated in Chapter III are reported in Table 19.
TABLE 17.
Difference of Pre Instruction and Post Instruction OHPT Scores (Interactions)
ANOVA of Test Scores (N = 196):

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>Sum of squares</th>
<th>F value</th>
<th>p &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treat*Gender</td>
<td>2</td>
<td>9.57</td>
<td>0.13</td>
<td>0.8788</td>
</tr>
<tr>
<td>Treat*Ethnic</td>
<td>2</td>
<td>27.5</td>
<td>0.37</td>
<td>0.6908</td>
</tr>
<tr>
<td>Treat*Grade</td>
<td>2</td>
<td>44.8</td>
<td>0.61</td>
<td>0.5471</td>
</tr>
<tr>
<td>Treat*School</td>
<td>2</td>
<td>45.3</td>
<td>0.61</td>
<td>0.5438</td>
</tr>
<tr>
<td>Gender*Ethnic</td>
<td>1</td>
<td>110.9</td>
<td>3.00</td>
<td>0.0852</td>
</tr>
<tr>
<td>Residual</td>
<td>177</td>
<td>6554.3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 18.
Difference of Pre Instruction and Post Instruction CRT Scores (Interactions)
ANOVA of Test Scores (N = 196):

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>Sum of squares</th>
<th>F value</th>
<th>p &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treat*Gender</td>
<td>2</td>
<td>13.7</td>
<td>0.93</td>
<td>0.3960</td>
</tr>
<tr>
<td>Treat*Ethnic</td>
<td>2</td>
<td>5.94</td>
<td>0.40</td>
<td>0.6684</td>
</tr>
<tr>
<td>Treat*Grade</td>
<td>2</td>
<td>8.98</td>
<td>0.61</td>
<td>0.5443</td>
</tr>
<tr>
<td>Treat*School</td>
<td>2</td>
<td>4.39</td>
<td>0.30</td>
<td>0.7422</td>
</tr>
<tr>
<td>Gender*Ethnic</td>
<td>1</td>
<td>8.89</td>
<td>1.21</td>
<td>0.2733</td>
</tr>
<tr>
<td>Residual</td>
<td>177</td>
<td>1302.4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 19.
Post Instruction CRT Scores (Interactions)
ANOVA of Test Scores (N = 196):

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>Sum of squares</th>
<th>F value</th>
<th>p &gt; F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treat*Gender</td>
<td>2</td>
<td>11.56</td>
<td>1.27</td>
<td>0.2833</td>
</tr>
<tr>
<td>Treat*Ethnic</td>
<td>2</td>
<td>0.72</td>
<td>0.08</td>
<td>0.9237</td>
</tr>
<tr>
<td>Treat*Grade</td>
<td>2</td>
<td>1.97</td>
<td>0.22</td>
<td>0.8055</td>
</tr>
<tr>
<td>Treat*School</td>
<td>2</td>
<td>0.66</td>
<td>0.07</td>
<td>0.9297</td>
</tr>
<tr>
<td>Gender*Ethnic</td>
<td>1</td>
<td>1.16</td>
<td>0.26</td>
<td>0.6134</td>
</tr>
<tr>
<td>RESIDUAL</td>
<td>177</td>
<td>804.8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Hypothesis 5A: There was no interaction between treatment and gender as measured by the difference of pre-instruction and post-instruction OHPT scores.

Decision: Retain the null hypothesis at $p = 0.8788$.

Hypothesis 5B: There was no interaction between treatment and gender as measured by the difference of pretest scores and post-instruction CRT scores.

Decision: Retain the null hypothesis at $p = 0.3960$.

Hypothesis 5C: There was no interaction between treatment and gender as measured by items on the post-instruction CRT.

Decision: Retain the null hypothesis at $p = 0.2833$.

Hypothesis 6A: There was no interaction between treatment and ethnicity as measured by the difference of pre-instruction and post-instruction OHPT scores.

Decision: Retain the null hypothesis at $p = 0.6908$.

Hypothesis 6B: There was no interaction between treatment and ethnicity as measured by the difference of pretest scores and post-instruction CRT scores.
Decision: Retain the null hypothesis at $p = 0.6684$.

Hypothesis 6C: There was no interaction between treatment and ethnicity as measured by items on the post-instruction CRT.

Decision: Retain the null hypothesis at $p = 0.9237$.

Hypothesis 7A: There was no interaction between treatment and grade level as measured by the difference of pre-instruction and post-instruction OHPT scores.

Decision: Retain the null hypothesis at $p = 0.5471$.

Hypothesis 7B: There was no interaction between treatment and grade level as measured by the difference of pretest scores and post-instruction CRT scores.

Decision: Retain the null hypothesis at $p = 0.5443$.

Hypothesis 7C: There was no interaction between treatment and grade levels as measured by items on the post-instruction CRT.

Decision: Retain the null hypothesis at $p = 0.8055$.

Hypothesis 8A: There was no interaction between gender and ethnicity as
measured by the difference of pre-instruction and post-instruction OHPT scores.

Decision: Retain the null hypothesis at $p = 0.0852$.

Hypothesis 8B: There was no interaction between gender and ethnicity as measured by the difference of pretest scores and post-instruction CRT scores.

Decision: Retain the null hypothesis at $p = 0.2733$.

Hypothesis 8C: There was no interaction between gender and ethnicity as measured by items on the post-instruction CRT:

Decision: Retain the null hypothesis at $p = 0.6134$.

Summary

This chapter contained the results of the tests of the hypotheses located in Chapter III. Hypotheses 1A, 1B, and 1C were rejected and the results were significant at the .05 level. There is a significant difference in student achievement due to treatments. The manipulatives with computer treatment and the manipulative only treatment were both significantly better than the traditional textbook only treatment on the overall means of the difference between pre instruction and post instruction OHPT scores, on the overall means of the
difference between pre instruction and post instruction CRT scores, and on the overall means of posttest CRT scores.

Hypotheses 4B and 4C were rejected and the results were significant at the .05 level. In the overall means of the difference between pre instruction and post instruction CRT scores and on the overall means of posttest CRT scores, there was a significant difference in student achievement due to gender. Females did significantly better than males at the .05 level on the CRT which tests the improvement as addressed by the treatments. Hypothesis 4A was retained. There was no significant difference in student achievement on the overall means of the difference between pre instruction and post instruction OHPT scores.

There were no significant differences in student achievement due to grade level or ethnicity and there were no significant interactions between any of the independent variables.
CHAPTER V

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

Summary

The primary purpose of this study was to investigate the effectiveness of using concrete, semi-concrete, and abstract pre-algebra learning models, utilizing multiple embodiments on the achievement of sixth and seventh grade students. The secondary purpose of the study was to analyze whether or not American Indian students did as well as other students, whether or not sixth grade students did as well as seventh grade students, and whether or not female students did as well as male students when using these learning models. Three dependent variables were considered including the difference between the pretest and posttest scores on the OHPT, the difference between the pretest and posttest scores on the CRT, and posttest scores on the CRT (see Appendix II for tests). The independent variables were method of instruction, ethnic background, grade level, and gender. In each case, the pretest OHPT score was a covariate. The general questions underlying the purpose of this analysis were to determine if there was a significant difference in student achievement due to method of instruction, ethnic background, grade level, or gender. Other questions involved interactions between the independent variables and student achievement.
Subjects

The sample for the study included 105 sixth grade students at Browning Middle School in Browning, Montana, 51 seventh grade students at Poplar Middle School in Poplar, Montana, and 40 sixth grade students at Poplar Middle School in Poplar, Montana (n=196). Browning, Montana is on the Blackfeet Indian Reservation in north-west Montana and Poplar, Montana is on the Fort Peck Indian Reservation in north-east Montana.

At Browning Middle School six sections of sixth grade students were randomly selected to receive treatment I, treatment II, and treatment III with two classes per treatment. All six sections were taught by a single instructor. At Poplar Middle School, one instructor from the sixth grade and one instructor from the seventh grade each taught three sections, which were randomly assigned either treatment I, treatment II, or treatment III. A total of three instructors, teaching twelve classes, taught four sections of treatment I (n=56), four sections of treatment II (n=75) and four sections of treatment III (n=64).

Instrumentation

The Orleans-Hanna Algebra Prognosis test was administered prior to instruction. This test was designed to be used as a predictor of student success in first-year algebra and to measure a student’s readiness for algebra. Analysis of covariance was used to reduce the effects of initial group differences. The
student's score on the OHPT was used as the covariate in the analysis of covariance. By adjusting the posttest means, any difference found in treatment would not be due to initial differences in the groups (Borg & Gall, 1989). In addition, a criterion-referenced test (see Appendix II) was administered prior to instruction and after treatment. Since the information in this study addressed students' abilities to solve systems of linear equations and not overall algebraic ability, a test was designed to analyze this skill.

Duration

The instructional sequence consisted of ten consecutive fifty-minute periods. Teachers administered the Orleans-Hanna Algebra Prognosis test and the criterion-referenced test at least a week prior to instruction. There were three treatments, two were experimental treatments and one was a control treatment. The two experimental treatments used physical models to represent a linear equation with one unknown, a linear equation with two unknowns, and two linear equations with two unknowns. In both treatments, students progressed from the concrete level to a semi-concrete or pictorial level, and then to an abstract level using traditional symbolic representations. In one of the experimental treatments, students used a computer modeling program to facilitate their transitions from the concrete level to the semi-concrete or pictorial level, and then from the pictorial level to the abstract level. The control treatment was based on a traditional textbook method of instruction. The teacher demonstrated
a technique for solving a linear equation with one unknown, or a linear equation with two unknowns, or a system of two linear equations with two unknowns. Students were then asked to emulate the instructor’s actions. The Orleans-Hanna Algebra Prognosis test and a criterion referenced test were administered to the students in the week after the ten class periods as posttests to determine the effectiveness of the treatments.

**Research Design**

The data was analyzed using an analysis of covariance with the scores on the Orleans-Hanna Algebra Prognosis test as the covariate. Three dependent variables were considered including the difference between the pretest and posttest scores on the OHPT, the difference between the pretest and posttest scores on the CRT, and posttest scores on the CRT (see appendix II). In order to show educational significance as well as statistical significance, the mean scores are displayed in Tables 9 and 10 in Chapter IV and Tables 20 and 21 in Chapter V. A general linear model was used, since the design was unbalanced. All two-way interactions between the independent variables were tested using each of the three dependent variables.
TABLE 20.
Difference of Pre Instruction and Post Instruction CRT Scores (Treatment)
Means, Standard Errors and P-levels of Pairwise Comparisons: Least Square Means adjusted for ethnicity, grade, gender, and pretest OHPT scores.

<table>
<thead>
<tr>
<th>TREAT</th>
<th>MEAN</th>
<th>Std Err</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>computer</td>
<td>1.98</td>
<td>0.56</td>
<td></td>
<td>0.7107</td>
<td>0.0275</td>
</tr>
<tr>
<td>non-computer</td>
<td>1.73</td>
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TABLE 21.
Post Instruction CRT Scores (Treatment)
Means, Standard Errors and P-levels of Pairwise Comparisons: Least Square Means adjusted for ethnicity, grade, gender, and pretest OHPT scores.

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Findings

The following are the findings of this investigative study, as determined by an analysis of the data.

1. Null Hypothesis 1A was rejected at the 0.05 level, with p = 0.0076. There was a significant difference in mean scores of treatment groups on the difference between pre-instruction and post-instruction OHPT scores.
2. Null Hypothesis IB was rejected at the 0.05 level, with $p = 0.0468$. There was a significant difference in mean scores of treatment groups on the difference between pre-instruction and post-instruction CRT scores. Both the computer-manipulative group (Treatment I) and the manipulative group (Treatment II) scored higher than the traditional textbook group (Treatment III).

3. Null Hypothesis IC was rejected at the 0.05 level, with $p = 0.0005$. There was a significant difference in mean scores of treatment groups on the post-instruction CRT scores. Both the computer-manipulative group (Treatment I) and the non-computer manipulative group (Treatment II) scored higher than the traditional textbook group (Treatment III).

4. Null Hypothesis 4B was rejected at the 0.05 level with $p = 0.0059$. There was a significant difference in mean scores of gender groups on the difference between pre-instruction and post-instruction CRT scores. Females scored higher than males on the least square means. Mean scores were 2.07 for females and 0.71 for males.

5. Null Hypothesis 4C was rejected at the 0.05 level with $p = 0.0456$. There was a significant difference in mean scores of gender groups on the post-instruction CRT scores. Females scored higher than males on the least square means. Mean scores were 5.40 for females and 4.66 for males.

6. The rest of the null hypotheses were not rejected at $p = 0.05$ level.
105

Conclusions

There are several conclusions based on the analyses of the data in this study.

1. Students learn to solve a linear equation or a system of linear equations better using a multiple embodiment sequence. The students who started with a concrete representation of a linear equation or a system of linear equations and proceeded to an abstract representation learned to solve these problems better than students who started in the traditional manner, using abstract manipulations. This agrees with research by Chalouh & Herscovics, 1988, Demana & Leitzel, 1988, and Thompson, 1988. The sequence which used a computer modeling program with the manipulative sequence to facilitate the transfer from the concrete level to the semi-concrete level and finally to the abstract level was significantly better than the traditional textbook technique. This agrees with Kaput, 1989, who concluded that the computer was an important tool in making connections between multiple notations in algebra. The study also supports the theories of mathematical psychologists Piaget, Bruner, and Dienes, who supported the concrete, semi-concrete, and abstract learning model.

2. Although the use of a computer modeling program to facilitate the transfer from the concrete level to the semi-concrete level and finally to the abstract level was not statistically significantly better than the manipulative sequence without the computer program, there was an attributable difference with \( p = .14 \). This suggests that using a computer modeling program as an
interfacing agent was important. Additional studies may conclude that this aspect of the learning sequence would be significant.

3. There was no statistically significant difference in the difference between the pretest and post-instruction OHPT scores between males and females, but females did score significantly better on the means of the difference between the pretest and post-instruction CRT scores and on the overall mean scores on the CRT scores. This suggests that there is not a significant difference between the performances of males and females on overall algebraic ability as evaluated by the OHPT. However, females did perform better on CRT, which suggests that content specific examinations may detect performance differences better.

4. The fact that there was no statistically significant difference in achievement due to ethnicity is an important result. American Indian students will do as well as non-Indian students when using either the manipulative sequence or the manipulative sequence with the computer.

5. The fact that there was no statistically significant difference in achievement is also an important factor in grade level. Sixth and seventh grade students showed no statistically significant difference in achievement. This study suggests that students can learn algebraic concepts prior to ninth grade. Additional studies may be able to compare the success of students in grades six and seven to those in grade nine.

This study agrees with the research of Whitman (1976) who found that students who had been taught to solve equations using just a formal method, were unable to perform the operations with success. The findings of this study
disagree with Filloy and Rojano (1985) who found that concrete models did not
significantly increase students’ abilities to operate at the symbolic level.

Recommendations

Recommendations for Education

The following recommendations are suggested based on the analysis of the
data collected for this study.

1. When preparing a sixth or seventh grade curriculum, teachers
should consider teaching algebraic concepts, specifically the solving of linear
equations. Students at that level can benefit from exposure to this concept when
taught using a multiple embodiment sequence involving manipulatives. Using a
concrete, semi-concrete, and abstract learning model can allow this topic to be
introduced and understood by students.

2. Although there was not a statistically significant difference between
the use of a computer modeling program and just using the manipulative
sequence, there was a difference that suggests that using a computer modeling
program does benefit students. Teachers should consider using a computer
program that helps students transfer from a concrete level to an abstract level
when solving linear equations.

The recommendations that are a result of this study are in concurrence
with current trends in mathematics as reflected by the NCTM Curriculum and
Evaluation Standards (1989). This document advocates teaching algebraic topics
to students before they are in a high school algebra classroom. It also suggests that methods of instruction be applied that are appropriate for students at various levels of development, and specifically, for students in grades 5-8 to use concrete representations when exposed to new concepts (p. 67). “Formal equation-solving methods can be developed from, and supported by, informal methods. These informal methods, which may include actions on concrete materials that are paralleled by symbolic actions, can lead to more formal procedures” (p. 104).

Recommendations for Research

The following recommendations are suggested for further research.

1. Further research is necessary to determine if the use of a computer modeling program as an interfacing agent in the learning sequence is significant. Since an attributable difference was found, this may be significant given a larger population, or a longer treatment sequence.

2. This study should be replicated using different demographic areas, to determine if the success of the students on American Indian reservations in Montana can be replicated with other students.

3. Studies which investigate the longitudinal effects of this learning sequence on students as they progress into high school algebra and through their mathematics careers would determine if such a learning experience before formal algebra training is effective.

4. Further studies need to explore the ability of students in middle school to learn various algebraic topics when using a concrete, hands-on, multiple embodiment approach. This study did not find a significant difference between
sixth and seventh grade students, when engaging the students physically in the learning process. Additional studies are necessary to verify this significant outcome.

5. Since this study found no statistical difference between sixth and seventh grade students, when using a multiple embodiment learning model, further research should investigate the use of the model with students in even earlier grades.

6. Further research should continue to explore the effect of the learning model used in this study on American Indian students, possibly from other demographic regions.

7. Further studies should include an attitude survey, to determine if attitude is a factor in learning the algebraic material using the different models. Research suggests that students who use hands-on materials and students who are allowed to use the computer during instruction have a better attitude toward mathematics. A study which included an attitude survey could explore this assumption.

8. Additional research should include using a wider age span, to determine if there is a difference in the ability of students to understand the material presented at various age levels.


Collis, B., & Mason, G. (1983). A coordinate graphing microcomputer unit for elementary grades. In G. Shufelt & J. Smart (Eds.), The agenda in action (pp. 131-142). Reston, VA: NCTM.


Meyer, M. R. (1989). Gender differences in mathematics. In M. Lindquist (Ed.), Results from the fourth mathematics assessment of the National Assessment of Educational Progress (pp. 149-159). Reston, VA: NCTM.


APPENDICES
APPENDIX I

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APPENDIX II

TESTS
The Orleans-Hanna Algebra Prognosis test was written by Gerald S. Hanna and Joseph B. Orleans and is published by the Psychological Corporation of Harcourt Brace Jovanovich Publishers. Since the exam is under copyright, a copy of the exam could not be provided. One third of the questions on this exam relate specifically to the study. The rest of the questions relate to more general algebraic ability. Since a copy of the exam can not be provided, a group of questions similar to the exam questions is provided.
Directions: Study this lesson and then do the test that follows.

(1) $4k$ means $4 \times k$. (Note that the multiplication sign is omitted.) If $k = 3$, then $4k$ equals $4 \times 3$, or 12.

(2) $gh$ means $g \times h$. If $g = 2$ and $h = 3$, then $gh$ equals $2 \times 3$, or 6.

(3) $6ab$ means $6 \times a \times b$. If $a = 2$ and $b = 4$, then $6ab$ equals $6 \times 2 \times 4$, or 48.

Note that letters are used to stand for numbers.

Go on to the test below.

<table>
<thead>
<tr>
<th>1. If $c = 2$ and $d = 4$, the $2cd$ equals</th>
<th>4. If $t = \frac{2}{3}$, then $6m$ equals</th>
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<tbody>
<tr>
<td>A 18</td>
<td>A $\frac{24}{32}$</td>
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<tr>
<td>B 23</td>
<td>B $6 \frac{2}{3}$</td>
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<td>C 16</td>
<td>C $\frac{18}{3}$</td>
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<td>D 33</td>
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<th>2. If $m = 10$, then $8r$ equals</th>
<th>5. If $r = 0.5$, then $20r$ equals</th>
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<td>A 2</td>
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<td>B 8</td>
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<td>C 10</td>
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<td>D 18</td>
<td>D 100</td>
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<th>3. If $x = 0$, then $4x$ equals</th>
<th>6. If $y = \frac{3}{4}$ and $z = \frac{2}{5}$, then $10yz$ equals</th>
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CRITERION-REFERENCED TESTS
LESSON 10

Directions: Study this lesson and then do the test that follows:

The piles on either side of the equal sign (=) contain an equal number of counters. Each box contains the same number of counters. Find the number of counters in each box.

(1):

The number of counters in the box is \(2\).

(2):

The number of counters in the box is \(3\).

(3):

If \(3X + 2 = X + 6\)

\[2X = 4\]

\[X = 2\]

Then \(X\) equals 2.
GO ON TO THE TEST BELOW.

61. The number of counter in the box is:
   A  13
   B  7
   C  3
   D  10
   E  none of the above

62. If X + 3 = 4, then X equals
   A  4
   B  3
   C  7
   D  1
   E  none of the above

63. The number of counters in the box is:
   A  1
   B  5
   C  2
   D  7
   E  none of the above

64. If 1 + 2X = X + 1, then X equals
   A  0
   B  2
   C  1
   D  5
   E  none of the above
65. The number of counters is the box is

A  0  
B  15 
C  3  
D  18 
E  none of the above

66. If $3X + 9 = 15 + X$, then $X$ equals

A  2 
B  3  
C  6  
D  0  
E  none of the above
LESSON 11

Directions: Study this lesson and then do the test that follows:

The same number of counters are found on each side of the equal sign (=). Every container marked X contains the same number of counters and every container marked Y contains the same number of counters.

(1):

So all possible answers include:

\[ X = 0, \ Y = 1 \]
and
\[ X = 1, \ Y = 0 \]

(2):

So all possible solutions include:

\[ X = 1, \ Y = 1 \]
and
\[ X = 4, \ Y = 0 \]
If $X + 3Y + 3 = Y + 4$

Then $X + 3Y = Y = 1$

So all possible solutions include:

$X = 1, Y = 1$

GO ON TO THE TEST BELOW.

TEST 11

67.

All possible answers include:

A $X = 1, Y = 2$ and $X = 0, Y = 1$ and $X = 2, Y = 3$

B $X = 1, Y = 0$ and $X = 0, Y = 1$

C $X = 1, Y = 0$

D $X = 1, Y = 1$

E none of the above

68. If $X + 4Y + 8 = 8$

All possible answers include:

A $X = 1, Y = 4$ and $X = 5, Y = 1$ and $X = 2, Y = 2$

B $X = 0, Y = 0$ and $X = 0, Y = 2$

C $X = 1, Y = 3$

D $X = 0, Y = 0$

E none of the above
69. All possible answers include:

A $X = 1, Y = 2$ and $X = 0, Y = 4$ and $X = 1, Y = 1$
B $X = 2, Y = 1$ and $X = 4, Y = 0$
C $X = 2, Y = 1$
D $X = 1, Y = 1$
E none of the above

70. If $4X + 2Y + 5 = 9$

All possible answers include:

A $X = 1, Y = 0$ and $X = 0, Y = 4$ and $X = 1, Y = 1$
B $X = 1, Y = 0$ and $X = 0, Y = 2$
C $X = 1, Y = 0$
D $X = 4, Y = 1$
E none of the above
71.

All possible solutions include

A  $X = 1, Y = 1$ and $X = 0, Y = 2$ and $X = 2, Y = 0$
B  $X = 1, Y = 1$ and $X = 0, Y = 1$
C  $X = 2, Y = 0$
D  $X = 3, Y = 1$
E  none of the above

72. If $2X + 3Y + 3 = X + Y + 6$

All possible solutions include

A  $X = 2, Y = 3$ and $X = 1, Y = 1$ and $X = 3, Y = 4$
B  $X = 1, Y = 1$ and $X = 3, Y = 0$
C  $X = 2, Y = 3$
D  $X = 1, Y = 1$
E  none of the above
LESSON 12

Directions: Study this lesson and then do the test that follows:

The same number of counters are found on each side of the equal sign (=). Every container marked X contains the same number of counters and every container marked Y contains the same number of counters.

Answer: $X = \frac{2}{1}$
(2): 

\[3X + 4Y + 3 = 2X + 2Y + 7\]

Answer: \[X = 2\]
\[Y = 1\]

*************************************************************

(3): 

\[X + 2Y = 4\]

Answer: \[X = 2\]
\[Y = 1\]

*************************************************************
GO ON TO THE TEST BELOW.

TEST 12

73. If \( X + Y = 5 \) and \( X = 2 \), then \( X \) and \( Y \) equal:

A. \( X = 1, Y = 2 \)
B. \( X = 1, Y = 4 \)
C. \( X = 5, Y = 6 \)
D. \( X = 4, Y = 1 \)
E. none of the above

74. If \( X + Y = 5 \) and \( X = 2 \), then \( X \) and \( Y \) equal:

A. \( X = 1, Y = 4 \)
B. \( X = 5, Y = 0 \)
C. \( X = 3, Y = 2 \)
D. \( X = 2, Y = 3 \)
E. none of the above
then X and Y equal:

A  $X = 0, \ Y = 3$
B  $X = 4, \ Y = 2$
C  $X = 1, \ Y = 1$
D  $X = 9, \ Y = 4$
E  none of the above

If $X + 4 = Y + 1$ and $3x = Y + 1$
then X and Y equal:

A  $X = 2, \ Y = 5$
B  $X = 0, \ Y = 3$
C  $X = 1, \ Y = 2$
D  $X = 1, \ Y = 4$
E  none of the above
77. If \( 2Y + 4 = 3X \) and \( Y + X + 1 = 2X \), then \( X \) and \( Y \) equal:

A. \( X = 3, \ Y = 2 \)
B. \( X = 0, \ Y = 3 \)
C. \( X = 2, \ Y = 2 \)
D. \( X = 1, \ Y = 1 \)
E. none of the above

78. If \( 2Y + 4 = 3X \) and \( Y + X + 1 = 2X \), then \( X \) and \( Y \) equal:

A. \( X = 1, \ Y = 0 \)
B. \( X = 3, \ Y = 2 \)
C. \( X = 2, \ Y = 1 \)
D. \( X = 4, \ Y = 4 \)
E. none of the above
APPENDIX III

LESSON PLANS
LESSON PLANS

TREATMENT I (MANIPULATIVES WITH COMPUTER)
DAY I: MANIPULATIVES

TREATMENT I (With computer)
One equation/one unknown

I. **Overhead:** (5 minutes)
Show students film containers and paper clips. Explain that all film containers in a particular problem have the same number of paper clips in them. Open example problem #1 film containers and show students the number of paper clips in each are the same. Then open example problem #2 and show the number of paper clips from each container is the same, but different from the number in problem #1. Explain their goal is to discover the number of paper clips in the containers without opening the container.

Show students the example #1 lab sheet. Explain that the line in the middle means =. This means there are the same number of paper clips on the left side as there are on the right side. Some of the paper clips are visible and some are hidden in the containers. □ on the lab sheet stands for film containers and ⌂ on the lab sheet stands for paper clips.

Demonstrate on the overhead how to set up the lab problem. Place film containers over □ and paper clips over ⌂.

Show students the worksheets they will use for recording their work. Explain that they must record their answers along with the work they did to get the answer. They are to describe in words and pictures the work they did to the containers and paper clips to find the answer for one problem and for the rest of the problems, just describe the problem using pictures and record their answer to the problem.

Hand out **Lab Worksheet I**.

II. **Lab I** (10 minutes)
Organize students into workstations with four students at each station. Each station should have 6 labs available and the students are to work as many as they can in the 10 minutes provided. They are to do their own problems and must record their answers to the problems they have worked but interaction is allowed and encouraged within the workstations. Make sure the students label on the worksheet the number of the lab they are working. Each lab is labeled 1-6.

Stop the students after 10 minutes. Collect the lab I worksheets.

III. **Overhead:** (5 minutes)
Do this example on the overhead:
I have a river. (Place river on the overhead) There are two stables on one side of the river and one stable on the other. (Place film containers on the overhead) There are also 2 horses outside the stables on one side and 5 horses outside the stable on the other. (Place the horses (paper clips) on the overhead) There are the same number of horses in each of the stables and the same number of horses on both sides of the river, some inside the stables and some outside. I want to know how many horses are in one of the stables.

The overhead should look like this:

*

□ □ 〇〇〇〇 * □ 〇〇〇〇〇
Ask the student "How do I find the number of horses in each stable?" Ask for suggestions but if there are no suggestions then begin by removing the same number of horses from each side. Then ask if you can remove one of the stables from each side of the river. If not, ask why not. If you can, ask why.

Hand out Lab Worksheet II.

IV. Lab II (20 minutes)

Remind students that there are the same number of counters in each of the containers for a particular lab, just as in Lab I. Tell the students they will be filling out these lab worksheets the same way they filled out the first ones. Remind them to label the lab problem as they record their answers and they are to describe once again both with pictures and words the work they did to solve one of the problems. For the rest of the problems, just describe the problem with pictures and record the answer. They are to work as many problems as they can in the time provided.

Allow students the rest of the period to work on the lab, but be sure the students have at least 20 minutes. If some students complete the lab early, give them the extra activity provided with lab II. Collect the lab II worksheets.
DAY 2: PICTURES

TREATMENT I (With computer)

One equation/one unknown

I. Overhead I (5 minutes)

Go back to the example used on day I. Remind them on the overhead of the problem of the horses and stables. The overhead should once again look like this:

* 

Ask the students how we could show this problem without the containers and paper clips. Ask for suggestions. If no suggestions, draw on the overhead and replace the containers with simple stables like this and replace the paper clips with stick horses something like this. The overhead should look like this:

Then ask how we solved this problem. Replace the above overhead with the containers and paper clips. Remind the students by removing two paper clips from each side. Ask how we would show what we did with the pictures. Put picture overhead back up, draw lines through the horses that were removed, and then draw a new picture under the old showing the problem without the horses. Put containers and horses back up. What did we do next? Remove a container from each side and then go back to picture overhead. Draw lines through two stables and then draw one more picture without the stables. How do we know what the answer is? The final drawing should tell us that there are 3 horses in the stable. Open up the container to be sure. The series of pictures on the overhead should look like this:
II. Student Worksheet I (5 minutes)

Put the worksheet on the overhead. Explain that the problems to be worked are given in pictures on the worksheet and the object is to discover the number of objects in the containers. If the students wish to use the manipulatives to help solve the problems, have enough film containers and paper clips available for them. Tell the students they may use the containers and paper clips to help them solve the problems on the worksheet if they want, but it is not required.

Hand out Student Worksheet 1.
Collect worksheets.

III. Computer Lab: (15 minutes)

Demonstrate the computer program for the students. Do several examples together as a class. Make sure the students can tell the difference between the containers and the counters. The containers will be like the film containers from their labs and the counters are like the paper clips. Next tell the students they will be doing some problems on their own. Show them Computer Worksheet I. You will have the computer give them all a problem and they will describe the problem on their worksheet using pictures and then solve the problem. Encourage them to use the film containers and paper clips if they want to solve the problem. Make sure they write the problem on their worksheets before starting to solve the problem so that you can have a new problem on the computer before everyone is finished. The students do not have to do all the problems given on the computer, but must do at least 5. That way if some students are quicker, they can keep doing problems, but the slower students will have enough opportunities to do the problems required on the worksheet. Be sure to read the instructions on the worksheet to the students so they all understand what they must do.

Hand out Computer Worksheet 1
Keep the problems changing on the computer fast enough so that the students are not bored, but slow enough so that they have a chance to at least copy the problem on their worksheets. Remind them they do not have to do every problem.
Collect worksheets.

IV. Overhead II: (10 minutes)

Go back once again to the horses and stable problem. Put up an overhead with a picture of just the right hand side (with 5 horses and 1 stable). Explain to the class that instead of drawing a stable each time, I want to find something easier to write. Ask "What would happen if I had 30 stables or 100 stables or 2000 horses? Would you want to draw all those pictures?" Discuss the option of calling the unknown number of horses in the stables X. Tell the students that in algebra X is sometimes used to stand for a number that is not known, like how many horses are in a stable. Refer to the picture of the horses and stables on the overhead. If the X stands for the number of horses in the stable and there are 5 horses outside the stable, how many horses are there altogether on the right side of the river? Suggest we could write that as X + 5. If X stands for the number of horses in one stable, how could I write two stables? Suggest first X + X. Now put up an overhead with the left hand side of the problem (with 2 stables and 2 horses). Then the left side of the river would look like this: X + X + 2. Instead of writing X + X we could write 2 x X, since this means we have 2 stables.
Go back to the containers and paper clips. Instead of setting up the problem with the containers first and counters second on each side, while the students watch you, move the containers and paper clips so that the paper clips are first with the containers second. Set it up so it looks like this:

![Diagram]

Ask the students if you have changed the problem. If students feel that you have, discuss why the problem is the same. Then ask how this problem would change the way we would write the problem using algebraic notation (the X's). Discuss that the expressions would be the same but might look a little different, for example:

\[
\begin{align*}
2 + 2 \times X \\
2 \times X + 2 \\
5 + X \\
X + 5
\end{align*}
\]

Ask how many horses would be on the right hand side (1 stable and 5 horses) if you knew that there were 6 horses in the stable. (11) How did they come up with that number? (6 + 5 = 11) How many horses would be on the left hand side (2 stables and 2 horses) if there were 4 horses in the stables. (10) How did they find that number? (2 \times 4 + 2 = 10)

Show students student worksheet 2. Explain that they are to write the picture problems shown them using the algebraic notation they have been talking about and answer the questions on the worksheet.

**V. Student Worksheet 2:** (5 minutes)

Hand out **Student Worksheet 2**.

Collect worksheets.
TREATMENT I (With computer)

DAY 3: SYMBOLIC

One equation/one unknown

I. Overhead I: (5 minutes)
Remind students once again of the horses and stable problem. Ask the students what the line in the middle of the sheet means. Be sure the concept of "equal" is expressed in some form. Tell the students that in algebra, that is written with the equals sign (=). So how would they write the entire equation? Ask for a volunteer to write the equation:

\[ 2x + 2 = x + 5 \]

Show students student worksheet 3. Explain that they are to write the picture problems shown them using algebraic notation.

II. Student Worksheet 3: (5 minutes)
Hand out Student Worksheet 3.
Collect worksheets.

III. Overhead II: (5 minutes)
Remind students once again about the horses and stables problem, both with the manipulatives and the algebraic notation. What did we do next when solving the problem? If no one suggests anything, remove two paper clips from both sides. How would we write this new problem using X's? It should look like this:

\[ 2x = x + 3 \]

What did we do next? Remove one container from each side. What will this look like? Have a student come up and write the new problem. It should look like:

\[ x = 3 \]

How many horses are in each stable?
Show lab worksheet III on overhead. Explain students will once again be required to record their work and answers on this sheet. They must do as many of the 8 problems as they can in the time provided. They should label the problems as they record them. This time, instead of using pictures to show their work, they must write each problem on the worksheet using algebraic notation, and describe the process used to solve the problem using the notation. Encourage the students to use the manipulatives as much as they need to and to discuss the problems with other members at their workstation. But each person is to do record their own solutions.

Hand out Lab Worksheet III.

IV. Lab III (15 minutes)
Allow the students just 15 minutes to complete the lab. Reemphasize that they are to do as many of the problems as they can in the 15 minutes. If some students complete the lab early, give them the extra activity provided with lab III. Collect the lab III worksheets.
V. Overhead III (5 minutes)

Tell the students that they will be using algebraic notation to represent the picture problems given to them. Do this example on the overhead:

If we had three houses on one side of the street and six houses on the other side of the street, and on the side of the street with the three houses there were three cats in the front yards and on the side of the street with six houses there were no cats in the yard. If there are the same number of cats on both sides of the street and each house has the same number of cats, how many cats are in each house?

Set up the overhead by drawing the houses and cats as you describe the problem. The overhead should look like this:

```
  *  *  *  *
```

Ask someone to come up and write the algebraic expression for the right hand side. Then have someone else write the expression for the left hand side. Keep asking for suggestions until you get two expressions that look like this:

```
3 \times X + 3
```

```
6 \times X
```

How do we show that the two sides have the same number of cats? The expression should now look like:

```
3 \times X + 3 = 6 \times X
```

Explain that the expressions $3 \times X$ and $6 \times X$ may also be written as $3X$ and $6X$. This is just common algebraic notation. The symbol for multiplication ($\times$) is still there, but we don't have to write it because it may be confused with the $X$ which is the unknown (variable). Tell them that from now on, you are going to write the equations without the multiplication symbol $\times$, but they may either write it in or leave it out.

Now, on the overhead rearrange the houses and cats so that one cat and one house are next to each other. How could we write this expression? If there are no suggestions, ask if this problem is different from the first problem? If students think they are different, discuss why they are not. If they think they are the same, suggest the possibility of using the same equation. Then ask them to once again think of other ways to write the equation. Let students come up and write new equations. Discuss the accuracy of each response.

VI. Computer (5 minutes)

Show the students Computer Worksheet II. You will have the computer give them a problem and they are to express that problem using the algebraic notation that you have been discussing. They are not to solve the problems, just write down the algebraic expressions for the equations. The students do not need to do all the problems that you put on the computer screen, but they are to write down the expressions for as many as they can. That way, you can keep changing the problems for the faster students, but the slower students won't feel rushed to do all the problems.

Hand out Computer Worksheet II.
Keep the problems on the computer changing fast enough so that the students are not bored, but slow enough so that they have a chance to at least copy the problem on their worksheets. Tell the students that they can draw a picture of the problem first and then write the algebraic expression if they need more time. Remind them that they do not have to do every problem. Do as many problems as possible in the time available.

Collect worksheets.
TREATMENT I (With computer)

DAY 4: SYMBOLIC

One equation/one unknown

I. Overhead (5 minutes)

Put a container and five paper clips on the overhead. Ask someone to write the algebraic expression for this problem. Then remove the five paper clips and ask how we could show someone what we did using the expression. Suggest,

\[ X + 5 - 5 \]

Then ask what is left on the overhead. The answer should be just one container. How do we write that? That means that the expression above simplifies to just \( X \). Therefore,

\[ X + 5 - 5 = X \]

On the overhead write the following expressions one at a time and let the students respond:

\[
\begin{align*}
2X - X &= \_\_\_\_\_\_; \quad (X) \\
3X - X &= \_\_\_\_\_\_; \quad (2X) \\
5X - 5X &= \_\_\_\_\_\_; \quad (0) \\
4X + 5 - 4X &= \_\_\_\_\_\_; \quad (5) \\
X + 3 + X + 1 &= \_\_\_\_\_\_; \quad (2X+4)
\end{align*}
\]

II. Student Worksheet 4: (5 minutes)

Hand out Student Worksheet 4.

Encourage the students to use the manipulatives or pictures if they need them to do the problems.

Collect worksheets.

III. Overhead (5 minutes)

On the overhead, put two containers on one side of the line and ten paper clips on the other. Write

\[ 2X = 10 \]

Ask the students how they could find out the number of paper clips in the container (without opening it). Talk about putting the paper clips into two equal piles and then counting the number in each pile. Discuss the idea of dividing the one pile into two piles. Show several notations for this including:

\[
\frac{(2X)}{2} = \frac{10}{2}
\]

\[
\frac{2X}{2} = \frac{10}{2}
\]

This simplifies to:

\[ X = 5 \]
On overhead, put 4 containers on one side of the line and 16 paper clips on the other side of the line. How do we write this? \(4x = 16\). How do we solve this problem with paper clips and containers? How do we write what we just did using algebraic notation? (Similar to above)

IV. Student Worksheet 5 (5 minutes)
Hand out Student Worksheet 5.
Encourage the students to use the manipulatives or pictures if they need them to do the problems.
Collect worksheets.

V. Computer Lab: (10 minutes)
Once again, you are going to give the students problems on the computer. The students are to write down the equations using algebraic notation. This time they are also to solve the problems given. They may use the manipulatives to solve the problems, but must use algebraic notation to describe the work they did to solve the problems. They do not have to do every problem on the computer, but do as many as they can.

Hand out Computer Worksheet III.
Keep the problems changing again. Remind students that they do not have to do every problem. They are just to do as many as they can in the 10 minutes provided. Only allow 10 minutes so that the students have at least 10 minutes for the last worksheet.

VI. Worksheet 6: (10 minutes)
In this worksheet, allow students to use the manipulatives to solve the problems, but instruct them that they must use algebraic notation to describe the work they did to solve the problems. Give the students at least 10 minutes to do this worksheet, but no more than 15 minutes.

Hand out Student Worksheet 6.
Collect worksheets.
TREATMENT I (With computer)

DAY 5: MANIPULATIVES One equation/two unknowns

I. Overhead: (5 minutes)

Show the students two different film containers, one marked with an X and one marked with a Y. All the film containers marked with an X (in a box) in a particular problem have the same number of paper clips in them and all film containers marked with a Y (in a triangle) have the same number of paper clips. However, the number of paper clips in the container marked with an X does not have to be the same as the number of paper clips in the container marked with a Y. Open example problem #4's film containers marked with an X and those film containers marked with a Y. The students should see the same number of paper clips in all the X containers and the same number in all the Y containers. Explain that the X and Y containers may have the same number of paper clips and they may not. Their goal is to once again discover the number in both the X container and the Y container.

Show the students the example #4 lab sheet. Remind them of what the line in the middle means (=). Ask them what this means as far as the containers and paper clips are concerned (Same # of paper clips on either side of the line, some are shown and some are hidden in the containers). Ask what differences they see from the last set of labs and problems (containers called X and Y).

II. Overhead (10 minutes)

Do this example on the overhead:

Place a street on the overhead. On one side of the street, there is a truck and three race cars. (Place a film container that has a square on the bottom on the overhead for the truck and three film containers with triangles on the bottom for the race cars.) On the other side of the street is one race car. (Place a film container with a triangle on the bottom on the other side of the street.) On the side of the street with the truck, there are 4 people standing by the cars (use 4 paper clips) and on the other side of the street, there are 6 people (use 6 paper clips). If there are the same number of people on both sides of the street, and the same number of people in each of the race cars, how many people could be in the race car and how many could be in the truck?

The overhead should look like this:

□ △ △ △ 99999 * △ 999999

Ask the students how they could find the number of people in the cars. Ask for suggestions but if there are none, begin by removing one race car from each side of the street. Ask if this is a "legal" move. Then remove 4 people from each side of the street. Now the problem is simplified and from now on we can just try some numbers. Ask if there could be 1 person in the truck (no). Why not? Could there be 2 people? If there are 2 people in the truck, how many are in the race car (0)? Are there any other possibilities? Any others?
Hand out Lab Worksheet IV.

III. Lab IV (20 minutes)
Organize the students into workstations again with four students at each station. Each station should have 8 labs available and the students are to work as many as they can in the 20 minutes provided. They are to do their own problems and must record their answers to the problems they have worked but interaction is allowed and encouraged within the workstations. Make sure the students label on the worksheet the number of the lab they are working. Each lab is labeled 1-8.

Stop the students after 20 minutes. Remind them that they do not have to have all the problems completed, but do as many as they can. If some students complete the lab early, give them the extra activity provided with lab IV.
Collect Lab IV worksheets.

IV. Overhead (5 minutes)
Put another problem on the overhead, using the triangle container, the square container, and the paper clips. Organize the students into their workstations with four students at each station. Allow the students a couple of minutes to work the problem given on the overhead. Give the students enough manipulatives to set up this problem. Set up this problem on the overhead:

```
□ □ □ △ △ * □ △ 0 0 0 0 0 0 0
```

Ask the students "How do I find the number of counters in the square container?" Ask for suggestions but if there are no suggestions begin by removing counters. Remove one square from both sides of the line. Ask the students if this is a "legal" move. Then remove one triangle from each side. Then ask if this gives them any information. If no response, ask if anyone now knows how many counters could be in the square container. (3). Once everyone sees that there could be 3 counters in each of the square containers, ask how that information could be used to determine the number of counters in the triangle container. Once again, if no suggestions are given, remind the students that EACH the square has 3 counters in it. Then remove the square on the overhead and replace it with 3 counters. Ask the students if this is OK. Once this is agreed as an acceptable move, ask how they can discover now the number of counters in the triangle container. Is this the only possible answer? Keep asking questions until the students are convinced that they have found all the possible answers.
TREATMENT I (With computer)

DAY 6: PICTURES

One equation/two unknowns

I. Overhead (5 minutes)

Go back to the first example used yesterday. Remind them on the overhead of the problem of the people and cars. The overhead should once again look like this:

```
□ △ △ △ △ △ △ △ *
```

Ask the students how we could show this problem without the containers and paper clips. Tell them we don't want to use algebraic notation quite yet, but would like a picture of what the problem looks like. Draw on the overhead and replace the cars with pictures. Use two different colored pens to emphasize the difference between the cars and the truck. Replace the paper clips with stick people.

Now ask how we solved this problem. Put the containers and paper clips back on the overhead and ask someone to show the class what we did next. Remove one race car from both sides. Ask how we could show what we did with the pictures. Put the picture overhead back up, and draw lines through one race car on each side of the line. Draw a new picture without the race cars. What conclusion did we come to about the number of people in the truck? What did we do next? Go back to the manipulatives. We replaced the truck with the three people that were in the truck. Draw this on the picture overhead. Now what? Follow the steps that you used yesterday to verify the answer. Ask for other suggestions.

II. Student Worksheet 8 (15 minutes)

Put the worksheet on the overhead. Explain that the problems to be worked are given in pictures on the worksheet and the object is to discover the number of objects in the X containers and the number in the Y containers. Encourage the students to use the manipulatives to help solve the problems. Have enough labeled film containers and paper clips available. Tell the students they may use the manipulatives if they wish, although they do not have to.

Hand out Student Worksheet 8.
Collect worksheets.

III. Computer Lab (20 minutes)

Demonstrate the computer program for the students again. Remind them that this is similar to the program they used with just one unknown, but now there are two. Do several examples together as a class. Make sure the students can tell the difference between the different types of containers, and the difference between the containers and the counters. The containers are like the film containers from their labs and the counters are like the paper clips. Next tell the students they will be doing some problems on their own, just like the last time. Show them Computer Worksheet IV. You will once again have the computer give them all a problem and they will describe the problem on the worksheet using pictures and then solve the problem. Encourage them to use the manipulatives if they want to solve the problems. Make sure they write the problem on their worksheets before
starting to solve the problem so that you can have a new problem on the computer before
everyone is finished. The students do not have to do all the problems given on the
computer, but they must do at least 5. If some students are quicker, they can keep doing
problems. The slower students will have enough opportunities to do the problems required
on the worksheet. Be sure to read the instructions on the worksheet to the students so they
all understand what they must do.

Hand out Computer Worksheet IV.

Keep the problems changing on the computer fast enough so that the students are
not bored, but slowly enough so that they have a chance to at least copy the problem on
their worksheets. Remind them they do not have to do every problem.

Collect worksheets.
DAY 7: MANIPULATIVES Two equations/two unknowns

I. Overhead: (10 minutes)

Show the students two different film containers, one marked with an X and one marked with a Y. All the film containers marked with an X in a particular problem have the same number of paper clips in them and all film containers marked with a Y have the same number of paper clips. However, the number of paper clips in the container marked with an X does not have to be the same as the number of paper clips in the container marked with a Y. Open example problem #4’s film containers marked with an X and those film containers marked with a Y. The students should see the same number of paper clips in all the X containers and the same number in all the Y containers. Explain that the X and Y containers may have the same number of paper clips and they may not. Their goal is to once again discover the number of paper clips in the containers without opening them up. But this time they have to discover the number in both the X container and the Y container.

Show the students the example #4 lab sheet. Remind them of what the line in the middle means (=). Ask them what this means as far as the containers and paper clips are concerned (Same # of paper clips on either side of the line, some are shown and some are hidden in the containers). Ask what differences they see from the last set of labs and problems (containers labeled X and Y, another line).

Explain that the line in the center of the page, separating the two equations from one another means that the two problems are connected. Just as before, the line down the middle of the page indicates the same number of paper clips on both sides. But the line down the center means that the containers marked with an X on both the top line and the bottom line have the same number of paper clips and the containers marked with a Y on both top and bottom have the same number of paper clips. So we can use both the top and the bottom to discover the number of paper clips in the X container and the Y container.

Remind students of what the labs sheets look like. On the lab sheets, the different containers are labeled either with an X or a Y. In the labs, the containers are also labeled with an X or a Y. Show students the example problem #4 containers. They will once again have worksheets to record their work. Explain that they must record their answers along with the work they did to get the answer. They are to describe once again in words and pictures the work they did to the containers and paper clips to find the answer for one problem and the rest of the problems, they can just describe the problem using pictures and record their answer to the problem.

Hand out Lab Worksheet VI.

II. Lab VI (10 minutes)

Organize the students into workstations again with four students at each station. Each station should have 6 labs available and the students are to work as many as they can in the 10 minutes provided. They are to do their own problems and must record their answers to the problems they have worked but interaction is allowed and encouraged within the workstations. Make sure the students label on the worksheet the number of the lab they are working. Each lab is labeled 1-6.

Tell the students before they begin that if they do not understand how to find the answers, that is all right. This lab is designed to let the students play around with the new containers and discuss among themselves possible ways to figure out the answers. They will not have had any further instruction at this point. Don't let them get frustrated. Encourage discussion and interaction within the workstations. If the students can only do
one problem in the time provided, that is fine.

Stop the students after 10 minutes. Remind them that they do not have to have all
the problems completed, but hopefully have at least one problem that they have worked on.
What you want from them now are suggestions and comments about the work they just
did. Were they frustrated? Confused? Now you are going to discuss ways of solving
these types of problems.
Collect Lab VI worksheets.

III. Overhead (10 minutes)

Do this example on the overhead:

I have two streets, Montana Street and Dakota Street. (Place streets on the
overhead) On one side of Montana Street there are 2 white houses (Place
film containers that have triangles on the bottom on the overhead) On the
other side of Montana Street there is one white house and 3 dogs. (Place
another container with a triangle on the bottom and 3 paper clips on the
overhead) On one side of Dakota Street there is one brown house (Place
container with square on the bottom on overhead), one white house (place
container with triangle on bottom), and one dog (place paper clip on
overhead). On the other side of Dakota Street there are 6 dogs (place 6
paper clips on overhead). There are the same number of dogs on both sides
of Montana Street and the same number of dogs on both sides of Dakota
Street. There are the same number of dogs in the white houses. I want to
know how many dogs are in the white houses and how many are in the
brown house.

The overhead should look like this:

\[ \triangle \triangle = \triangle 999 \]

Ask the students "How do I find the number of dogs in the houses?" Ask for
suggestions but if there are no suggestions begin by dealing just with Montana street.
Remove one house from both side of the street. Ask the students if this is a "legal" move.
Then ask if this gives them any information. If no response, ask if anyone now knows
how many dogs are in the white house. (3). Once everyone sees that there are 3 dogs in
each of the white houses, ask how that information could be used to determine the number
of dogs in the brown house. Once again, if no suggestions are given, remind the students
that EACH the white house has 3 dogs in it. Then remove the white house on the overhead
on Dakota Street and replace it with 3 dogs. Ask the students if this is OK. Once this is
agreed as an acceptable move, ask how they can discover now the number of dogs in the
brown house. (Remove four paper clips from either side of Dakota Street) How many dogs are in the brown house? (2)

IV. Overhead (10 minutes)
Put another problem on the overhead, using the triangle container, the square container, and the paper clips. Organize the students into their workstations with four students at each station. Allow the students the rest of the period to work the problem given on the overhead. Give the students enough manipulatives to set up this problem. Set up this problem on the overhead:

There is one blue house and two cats on one side of Main Street and a grey house on the other side. There are two blue houses and a grey house on one side of 5th Street and 8 cats on the other side of the street. If all the blue houses have the same number of cats and all the grey houses have the same number of cats, and there are the same number of cats on either side of Main Street and the same number of cats on either side of 5th Street, how many cats are in the blue houses and how many are in the grey houses.

As you tell the story, place the containers and paper clips on the overhead. The overhead should look like this:

\[ \text{\triangle} \quad \text{\square} \quad \text{\circle} \quad \text{= \square} \]

Allow students up to 5 minutes to figure out how to solve this problem. Then ask how the problem could be solved. Ask for suggestions. Allow as many different ways to solve the problem as you can in the remaining 5 minutes. Tell the students that there may be many ways to solve each of the problems they will be asked to solve, but there is only one answer for each lab.
DAY 8: PICTURES

TREATMENT I (With computer)
Two equations/two unknowns

I. Overhead (5 minutes)

Show Lab Worksheet V on the overhead. Remind students that this lab is similar to the lab they worked yesterday. They will be filling out the lab worksheets in the same way they filled out the first ones. They must remember to label the lab problem as they record their answers and are to describe with both pictures and words the work they did to solve on of the problems. For the rest of the problems, just describe the problem with pictures and record the answer. They are to work as many problems as they can in the time provided.

Hand out Lab Worksheet V.

II. Lab V (20 minutes)

Allow students at least 20 minutes for this lab. If some students complete the lab early, give them the extra activity provided with lab V.

Collect the worksheets.

III. Overhead (5 minutes)

Go back to the first example used yesterday. Remind them on the overhead of the problem of the dogs and houses. The overhead should once again look like this:

\[ \begin{array}{ccc}
\triangle & \triangle & \triangle \\
\end{array} \]

*********

\[ \begin{array}{ccc}
\square & \triangle & \triangle \\
\end{array} \]

Ask the students how we could show this problem without the containers and paper clips. Tell them we don't want to use algebraic notation quite yet, but would like a picture of what the problem looks like. Draw on the overhead and replace the houses with stick houses. Use two different colored pens to show the different colored houses. Replace the paper clips with dogs. Draw one equation on one line and the other equation on the other line. Label the street names so the students see the resemblance to the original problem. The overhead should now look like this:
Now ask how we solved this problem. Put the containers and paper clips back on the overhead and ask someone to show the class what we did next. Remove one house from both sides of Montana Street. Ask how we could show what we did with the pictures. Put the picture overhead back up, and draw lines through one house on each side of Montana Street. Draw a new picture without the houses. What conclusion did we come to about the number of dogs in the white houses? What did we do next? Go back to the manipulatives. We replaced the white house on Dakota Street with the three dogs that lived there. Draw this on the picture overhead. Now what? We removed four paper clips from both sides of Dakota Street, so draw lines through the pictures of the dogs. The conclusion was that there were two dogs in the brown house. Open up the containers once again to verify the answer. The series of pictures on the overhead should look like this:

1. 

2. 

3. 

---------------------------------------------------
IV. Student Worksheet 8 (10 minutes)

Put the worksheet on the overhead. Explain that the problems to be worked are given in pictures on the worksheet and the object is to discover the number of objects in the \( X \) containers and the number in the \( Y \) containers. Encourage the students to use the manipulatives to help solve the problems. Have enough labeled film containers and paper clips available. Tell the students they may use the manipulatives if they wish, although they do not have to.

Hand out Student Worksheet 8.
Collect worksheets.
TREATMENT I (With computer)

DAY 9: SYMBOLIC

Two equations/two unknowns

I. Computer Lab (20 minutes)

Demonstrate the computer program for the students again. Remind them that this is similar to the program they used with just one unknown, but now there are two. Do several examples together as a class. Make sure the students can tell the difference between the different types of containers, and the difference between the containers and the counters. The containers are like the film containers from their labs and the counters are like the paper clips. Next tell the students they will be doing some problems on their own, just like the last time. Show them Computer Worksheet IV. You will once again have the computer give them all a problem and they will describe the problem on the worksheet using pictures and then solve the problem. Encourage them to use the manipulatives if they want to solve the problems. Make sure they write the problem on their worksheets before starting to solve the problem so that you can have a new problem on the computer before everyone is finished. The students do not have to do all the problems given on the computer, but they must do at least 5. If some students are quicker, they can keep doing problems. The slower students will have enough opportunities to do the problems required on the worksheet. Be sure to read the instructions on the worksheet to the students so they all understand what they must do.

Hand out Computer Worksheet IV.

Keep the problems changing on the computer fast enough so that the students are not bored, but slowly enough so that they have a chance to at least copy the problem on their worksheets. Remind them they do not have to do every problem.

Collect worksheets.

II. Overhead (5 minutes)

Go back to the houses and dogs problem. Remind the students what the problem was. Put up an overhead with a picture of just the first equation, with the houses and dogs on Montana Street. Ask how they would write this equation using algebraic notation. Remind the students that we will be using an $X$ to represent the number of dogs in the white house and a $Y$ to represent the number of dogs in the brown house. So the equation for the houses and dogs on Montana Street should look like this:

$$2X = X + 3$$

Now put up an overhead with the picture of the houses and dogs on Dakota Street. Ask for a volunteer to come up and write the algebraic equation for this picture. It should look something like this:

$$X + Y = 6$$

When we write these two equations together so that people know that they are connected, we usually write one on top of the other like this:

$$2X = X + 3$$

$$X + Y = 6$$
III. **Student Worksheet 9** (10 minutes)
Show students student worksheet 9. Explain that they are to write the picture problems shown to them using the algebraic notation they have been talking about.

Hand out **Student Worksheet 9**.

Collect worksheets.

IV. **Computer** (5 minutes)
Show the students Computer Worksheet V. You will have the computer give them a problem and they are to express that problem using algebraic notation. They are **NOT** to solve the problems, just write down the algebraic expressions for the equations. The students do not need to do all the problems that you put on the computer screen, but they are to write down the expressions for as many as they can. That way you can keep changing the problems for the faster students, but the slower students won't feel rushed to do all the problems.

Hand out **Computer Worksheet V**.

Keep the problems on the computer changing fast enough so that the students are not bored, but slow enough so that they have a chance to at least copy the problem on their worksheets. Tell the students that they can draw a picture first and then write the algebraic expression if they need more time. Remind them that they do **not** have to do every problem. Do as many problems as possible in the time available.

Collect worksheets.
TREATMENT I (With computer)

DAY 10: SYMBOLIC

Two equations/two unknowns

I. Overhead (5 minutes)

Go back to the houses and dogs, using both the manipulatives and the pictures.
What did we do next when we solved this problem? (Removed a white house from both sides of Montana Street) How would we write this new problem using algebraic notation? It should look like this:

\[ X = 3 \]
\[ X + Y + 1 = 6 \]

Now what did we do? (Replace the \( X \) in second equation with 3) How could we write that? It should look like this:

\[ X = 3 \]
\[ 3 + Y + 1 = 6 \]

Now what happened? (Removed four dogs from both sides of the Dakota Street) Why did we remove 4 dogs? Suggest that the equations really look like this:

\[ X = 3 \]
\[ 4 + Y = 6 \]

Are these two the same? Now we can see why we really removed four dogs from each side of Dakota Street instead of just three. So what did we end up with? The final equations should look like this:

\[ X = 3 \]
\[ Y = 2 \]

So the conclusion was that there were three dogs in each of the white houses and two dogs in each of the brown houses.

Show lab worksheet VI on the overhead. Explain to the students that they will once again be required to record their work and answers on this sheet. They must do as many of the 8 problems as they can in the time provided. They should label the problems as they record them. This time, instead of using pictures to show their work, they must write each problem on the worksheet using algebraic notation, and describe the process used to solve the problem using the notation. Encourage the students to use the manipulatives as much as they need to and to discuss the problems with other members at their workstations, but each person is to record their own solutions.

Hand out Lab Worksheet VI.

II. Lab VI (20 minutes)

Allow the students just 20 minutes to complete the lab. Emphasize that they are to do as many of the problems as they can in the 20 minutes. If some students complete the lab early, give them the extra activity provided with lab VI.

Collect worksheets.

III. Student Worksheet 10 (15 minutes)

On this worksheet, encourage students to use the manipulatives to solve the problems. Instruct them that they must use algebraic notation to describe the work they did to solve the problems.

Hand out Student Worksheet 10.

Collect worksheets.
LESSON PLANS

TREATMENT II (MANIPULATIVES WITHOUT COMPUTER)
DAY 1: MANIPULATIVES

TREATMENT II (Without computer)  One equation/one unknown

I. Overhead: (5 minutes)

Show students film containers and paper clips. Explain that all film containers in a particular problem have the same number of paper clips in them. Open example problem #1 film containers and show students the number of paper clips in each are the same. Then open example problem #2 and show the number of paper clips from each container is the same, but different from the number in problem #1. Explain their goal is to discover the number of paper clips in the containers without opening the container.

Show students the example #1 lab sheet. Explain that the line in the middle means =. This means there are the same number of paper clips on the left side as there are on the right side. Some of the paper clips are visible and some are hidden in the containers. □ on the lab sheet stands for film containers and ◯ on the lab sheet stands for paper clips.

Demonstrate on the overhead how to set up the lab problem. Place film containers over □ and paper clips over ◯.

Show students the worksheets they will use for recording their work. Explain that they must record their answers along with the work they did to get the answer. They are to describe in words and pictures the work they did to the containers and paper clips to find the answer for one problem and for the rest of the problems, just describe the problem using pictures and record their answer to the problem.

Hand out Lab Worksheet I.

II. Lab I (10 minutes)

Organize students into workstations with four students at each station. Each station should have 6 labs available and the students are to work as many as they can in the 10 minutes provided. They are to do their own problems and must record their answers to the problems they have worked but interaction is allowed and encouraged within the workstations. Make sure the students label on the worksheet the number of the lab they are working. Each lab is labeled 1-6.

Stop the students after 10 minutes. Collect the lab I worksheets.

III. Overhead: (5 minutes)

Do this example on the overhead:

I have a river. (Place river on the overhead) There are two stables on one side of the river and one stable on the other. (Place film containers on the overhead) There are also 2 horses outside the stables on one side and 5 horses outside the stable on the other. (Place the horses (paper clips) on the overhead) There are the same number of horses in each of the stables and the same number of horses on both sides of the river, some inside the stables and some outside. I want to know how many horses are in one of the stables.

The overhead should look like this:

*  

□ □ ◯ ◯ ◯ ◯  

*

□ ◯ ◯ ◯ ◯ ◯ ◯ ◯
Ask the student "How do I find the number of horses in each stable?" Ask for suggestions but if there are no suggestions then begin by removing the same number of horses from each side. Then ask if you can remove one of the stables from each side of the river. If not, ask why not. If you can, ask why.

Hand out Lab Worksheet II.

IV. Lab II (20 minutes)

Remind students that there are the same number of counters in each of the containers for a particular lab, just as in Lab I. Tell the students they will be filling out these lab worksheets the same way they filled out the first ones. Remind them to label the lab problem as they record their answers and they are to describe once again both with pictures and words the work they did to solve one of the problems. For the rest of the problems, just describe the problem with pictures and record the answer. They are to work as many problems as they can in the time provided.

Allow students the rest of the period to work on the lab, but be sure the students have at least 20 minutes. If some students complete the lab early, give them the extra activity provided with lab II. Collect the lab II worksheets.
DAY 2: PICTURES

TREATMENT II (Without computer)
One equation/one unknown

I. Overhead: (5 minutes)

Go back to the example used on day I. Remind them on the overhead of the problem of the horses and stables. The overhead should once again look like this:

*  

Then ask how we solved this problem. Replace the above overhead with the containers and paper clips. Remind the students by removing two paper clips from each side. Ask how we would show what we did with the pictures. Put picture overhead back up, draw lines through the horses that were removed, and then draw a new picture under the old showing the problem without the horses. Put containers and horses back up. What did we do next? Remove a container from each side and then go back to picture overhead. Draw lines through two stables and then draw one more picture without the stables. How do we know what the answer is? The final drawing should tell us that there are 3 horses in the stable. Open up the container to be sure. The series of pictures on the overhead should look like this:

*  

*  

*  

*  

*
II. Student Worksheet I (10 minutes)

Put the worksheet on the overhead. Explain that the problems to be worked are given in pictures on the worksheet and the object is to discover the number of objects in the containers. If the students wish to use the manipulatives to help solve the problems, have enough film containers and paper clips available for them. Tell the students they may use the containers and paper clips to help them solve the problems on the worksheet if they want, but it is not required.

Hand out Student Worksheet I.

Collect worksheets.

III. Overhead: (10 minutes)

Go back once again to the horses and stable problem. Put up an overhead with a picture of just the right hand side (with 5 horses and 1 stable). Explain to the class that instead of drawing a stable each time, I want to find something easier to write. Ask "What would happen if I had 30 stables or 100 stables or 2000 horses? Would you want to draw all those pictures?" Discuss the option of calling the unknown number of horses in the stables $X$. Tell the students that in algebra $X$ is sometimes used to stand for a number that is not known, like how many horses are in a stable. Refer to the picture of the horses and stables on the overhead. If the $X$ stands for the number of horses in the stable and there are 5 horses outside the stable, how many horses are there altogether on the right side of the river? Suggest we could write that as $X + 5$. If $X$ stands for the number of horses in one stable, how could I write two stables? Suggest first $X + X$. Now put up an overhead with the left hand side of the problem (with 2 stables and 2 horses). Then the left side of the river would look like this: $X + X + 2$. Instead of writing $X + X$ we could write $2 \times X$, since this means we have 2 stables.

Go back to the containers and paper clips. Instead of setting up the problem with the containers first and counters second on each side, while the students watch you, move the containers and paper clips so that the paper clips are first with the containers second. Set it up so it looks like this:

```
  *  
 0 0 [ ] [ ]  
```

Ask the students if you have changed the problem. If students feel that you have, discuss why the problem is the same. Then ask how this problem would change the way we would write the problem using algebraic notation (the $X$'s). Discuss that the expressions would be the same but might look a little different, for example:

```
2 + 2 \times X
2 \times X + 2
5 + X
X + 5
```

Ask how many horses would be on the right hand side (1 stable and 5 horses) if you knew that there were 6 horses in the stable. (11) How did they come up with that number? ($6 + 5 = 11$) How many horses would be on the left hand side (2 stables and 2 horses) if there were 4 horses in the stables. (10) How did they find that number? ($2 \times 4 + 2 = 10$)

Show students student worksheet 2. Explain that they are to write the picture problems shown them using the algebraic notation they have been talking about and answer
the questions on the worksheet.

IV. Student Worksheet 2: (5 minutes)
Hand out **Student Worksheet 2**.
Collect worksheets.

V. Overhead: (5 minutes)
Go back once again to the horses and stable problem. Ask the students what the line in the middle of the sheet means. Be sure the concept of "equal" is expressed in some form. Tell the students that in algebra, that is written with the equals sign \( = \). So how would they write the entire equation? Ask for a volunteer to write the equation:

\[ 2 \times X + 2 = X + 5 \]

Show students student worksheet 3. Explain that they are to write the picture problems shown them using algebraic notation.

VI. Student Worksheet 3: (5 minutes)
Hand out **Student Worksheet 3**.
Collect worksheets.
TREATMENT II (Without computer)

One equation/one unknown

DAY 3. SYMBOLIC

I. Overhead: (5 minutes)

Remind students once again about the horses and stables problem, both with the manipulatives and the algebraic notation. What did we do next when solving the problem? If no one suggests anything, remove two paper clips from both sides. How would we write this new problem using X's? It should look like this:

\[2 \times X = X + 3\]

What did we do next? Remove one container from each side. What will this look like?

Have a student come up and write the new problem. It should look like:

\[X = 3\]

How many horses are in each stable?

Show lab worksheet III on overhead. Explain to the students that they will once again be required to record their work and answers on this sheet. They must do as many of the 8 problems as they can in the time provided. They should label the problems as they record them. This time, instead of using pictures to show their work, they must write each problem on the worksheet using algebraic notation, and describe the process used to solve the problem using the notation. Encourage the students to use the manipulatives as much as they need to and to discuss the problems with other members at their workstation, but each person is to do record their own solutions.

Hand out Lab Worksheet III.

II. Lab III (15 minutes)

Allow the students just 15 minutes to complete the lab. Reemphasize that they are to do as many of the problems as they can in the 15 minutes. If some students complete the lab early, give them the extra activity provided with lab III.

Collect worksheets.

III. Overhead: (10 minutes)

Tell the students that they will be using algebraic notation to represent the picture problems given to them. Do this example on the overhead:

If we have three houses on one side of the street and six houses on the other side of the street, and on the side of the street with the three houses there are three cats in the front yards and on the side of the street with six houses there are no cats in the yard. If there are the same number of cats on both sides of the street and each house has the same number of cats, how many cats are in each house?

Set up the overhead by drawing the houses and cats as you describe the problem. The overhead should look like this:

Ask someone to come up and write the algebraic expression for the right hand side. Then have someone else write the expression for the left hand side. Keep asking for suggestions until you get two expressions that look like this:

\[3 \times X + 3\]
\[6 \times X\]
How do we show that the two sides have the same number of cats? The expression should now look like:

\[ 3 \times X + 3 = 6 \times X \]

Explain that the expressions \( 3 \times X \) and \( 6 \times X \) may also be written as \( 3X \) and \( 6X \). This is just common algebraic notation. The symbol for multiplication (\( \times \)) is still there, but we don't have to write it because it may be confused with the \( X \) which is the unknown (variable). Tell them that from now on, you are going to write the equations without the multiplication symbol \( \times \), but they may either write it in or leave it out.

Now, on the overhead rearrange the houses and cats so that one cat and one house are next to each other. How could we write this expression? If there are no suggestions, ask if this problem is different from the first problem? If students think they are different, discuss why they are not. If they think they are the same, suggest the possibility of using the same equation. Then ask them to once again think of other ways to write the equation. Let students come up and write new equations. Discuss the accuracy of each response.

IV. Overhead: (5 minutes)

Remove the picture of the cats and houses and put a container and five paper clips on the overhead. Ask someone to write the algebraic expression for this problem. Then remove the five paper clips and ask how we could show someone what we did using the expression. Suggest,

\[ X + 5 - 5 \]

Then ask what is left on the overhead. The answer should be just one container. How do we write that? That means that the expression above simplifies to just \( X \). Therefore,

\[ X + 5 - 5 = X \]

On the overhead write the following expressions one at a time and let the students respond:

\[
\begin{align*}
2X - X & = \text{______;} \quad (X) \\
3X - X & = \text{______;} \quad (2X) \\
5X - 5X & = \text{______;} \quad (0) \\
4X + 5 - 4X & = \text{______;} \quad (5) \\
X + 3 + X + 1 & = \text{______;} \quad (2X+4)
\end{align*}
\]

V. Student Worksheet 4: (5 minutes)

Hand out Student Worksheet 4.

Encourage the students to use the manipulatives or pictures if they need them to do the problems.

Collect worksheets.
TREATMENT II (Without computer)

DAY 4: SYMBOLIC

One equation/one unknown

I. Overhead: (5 minutes)

On the overhead, put two containers on one side of the line and ten paper clips on the other. Write

\[ 2X = 10 \]

Ask the students how they could find out the number of paper clips in the container (without opening it). Talk about putting the paper clips into two equal piles and then counting the number in each pile. Discuss the idea of dividing the one pile into two piles. Show several notations for this including:

\[
\frac{(2X)}{2} = \frac{10}{2}
\]

This simplifies to:

\[
X = 5
\]

On overhead, put 4 containers on one side of the line and 16 paper clips on the other side of the line. How do we write this? \[ 4X = 16 \]. How do we solve this problem with paper clips and containers? How do we write what we just did using algebraic notation? (Similar to above)

II. Student Worksheet 5 (5 minutes)

Hand out Student Worksheet 5.

Encourage the students to use the manipulatives or pictures if they need them to do the problems.

Collect worksheets.

III. Student Worksheet 6: (15 minutes)

In this worksheet, allow students to use the manipulatives to solve the problems, but instruct them that they must use algebraic notation to describe the work they did to solve the problems.

Hand out Student Worksheet 6.

Collect worksheets.

IV. Overhead (5 minutes)

The answer to a problem is \[ X = 2 \]. Make up a problem that involves addition that could have this as its answer. Ask for suggestions but if no one has any, use \[ (X + 3) = 5 \]. Now ask for another suggestion. What if I wanted multiplication to be part of the problem? \[ (3X) = 6 \]. Ask for more suggestions. What if I wanted both multiplication and addition? Give them an example and then ask for other suggestions. \[ (5X + 1) = 11 \]
V. **Student Worksheet 7**: (10 minutes)

Ask students to make up their own problems for the answers provided in worksheet 7. Each problem will give a suggestion on what type of process should be used. For example:

Problem: \[ X = 3 \] (addition)

Answer: \[ X + 1 = 4 \]

Hand out **Student Worksheet 7**.

Collect worksheets.
TREATMENT II (Without computer)

DAY 5: MANIPULATIVES One equation/two unknowns

I. Overhead: (5 minutes)
Show the students two different film containers, one marked with an X and one marked with a Y. All the film containers marked with an X (in a box) in a particular problem have the same number of paper clips in them and all film containers marked with a Y (in a triangle) have the same number of paper clips. However, the number of paper clips in the container marked with an X does not have to be the same as the number of paper clips in the container marked with a Y. Open example problem #4's film containers marked with an X and those film containers marked with a Y. The students should see the same number of paper clips in all the X containers and the same number in all the Y containers. Explain that the X and Y containers may have the same number of paper clips and they may not. Their goal is to once again discover the number of paper clips in the containers without opening them up. But this time they have to discover the number in both the X container and the Y container.

Show the students the example #4 lab sheet. Remind them of what the line in the middle means (=). Ask them what this means as far as the containers and paper clips are concerned (Same # of paper clips on either side of the line, some are shown and some are hidden in the containers). Ask what differences they see from the last set of labs and problems (containers called X and Y).

II. Overhead (10 minutes)
Do this example on the overhead:
Place a street on the overhead. On one side of the street, there is a truck and three race cars. (Place a film container that has a square on the bottom on the overhead for the truck and three film containers with triangles on the bottom for the race cars.) On the other side of the street is one race car. (Place a film container with a triangle on the bottom on the other side of the street.) On the side of the street with the truck, there are 4 people standing by the cars (use 4 paper clips) and on the other side of the street, there are 6 people (use 6 paper clips). If there are the same number of people on both sides of the street, and the same number of people in each of the race cars, how many people could be in the race car and how many could be in the truck?

The overhead should look like this:

□ △ △ △ 0000 * △ 000000

Ask the students how they could find the number of people in the cars. Ask for suggestions but if there are none, begin by removing one race car from each side of the street. Ask if this is a "legal" move. Then remove 4 people from each side of the street. Now the problem is simplified and from now on we can just try some numbers. Ask if there could be 1 person in the truck (no). Why not? Could there be 2 people? If there are 2 people in the truck, how many are in the race car (0)? Are there any other possibilities? Any others?
Hand out Lab Worksheet IV.

III. Lab IV (20 minutes)
Organize the students into workstations again with four students at each station. Each station should have 8 labs available and the students are to work as many as they can in the 20 minutes provided. They are to do their own problems and must record their answers to the problems they have worked but interaction is allowed and encouraged within the workstations. Make sure the students label on the worksheet the number of the lab they are working. Each lab is labeled 1-8.

Stop the students after 20 minutes. Remind them that they do not have to have all the problems completed, but do as many as they can. If some students complete the lab early, give them the extra activity provided with lab IV.
Collect Lab IV worksheets.

IV. Overhead (5 minutes)
Put another problem on the overhead, using the triangle container, the square container, and the paper clips. Organize the students into their workstations with four students at each station. Allow the students a couple of minutes to work the problem given on the overhead. Give the students enough manipulatives to set up this problem. Set up this problem on the overhead:

Ask the students "How do I find the number of counters in the square container?"
Ask for suggestions but if there are no suggestions begin by removing counters. Remove one square from both sides of the line. Ask the students if this is a "legal" move. Then remove one triangle from each side. Then ask if this gives them any information. If no response, ask if anyone now knows how many counters could be in the square container. Once everyone sees that there could be 3 counters in each of the square containers, ask how that information could be used to determine the number of counters in the triangle container. Once again, if no suggestions are given, remind the students that EACH the square has 3 counters in it. Then remove the square on the overhead and replace it with 3 counters. Ask the students if this is OK. Once this is agreed as an acceptable move, ask how they can discover now the number of counters in the triangle container. Is this the only possible answer? Keep asking questions until the students are convinced that they have found all the possible answers.
TREATMENT II (Without computer)

DAY 6: PICTURES One equation/two unknowns

I. Overhead (5 minutes)

Go back to the first example used yesterday. Remind them on the overhead of the problem of the people and cars. The overhead should once again look like this:

\[ \square \triangle \triangle \bullet \bullet \bullet \bullet \bullet \bullet = \triangle \bullet \bullet \bullet \bullet \bullet \]

Ask the students how we could show this problem without the containers and paper clips. Tell them we don't want to use algebraic notation quite yet, but would like a picture of what the problem looks like. Draw on the overhead and replace the cars with pictures. Use two different colored pens to emphasize the difference between the cars and the truck. Replace the paper clips with stick people.

Now ask how we solved this problem. Put the containers and paper clips back on the overhead and ask someone to show the class what we did next. Remove one race car from both sides. Ask how we could show what we did with the pictures. Put the picture overhead back up, and draw lines through one race car on each side of the line. Draw a new picture without the race cars. What conclusion did we come to about the number of people in the truck? What did we do next? Go back to the manipulatives. We replaced the truck with the three people that were in the truck. Draw this on the picture overhead. Now what? Follow the steps that you used yesterday to verify the answer. Ask for other suggestions.

II. Student Worksheet 8 (15 minutes)

Put the worksheet on the overhead. Explain that the problems to be worked are given in pictures on the worksheet and the object is to discover the number of objects in the \( X \) containers and the number in the \( Y \) containers. Encourage the students to use the manipulatives to help solve the problems. Have enough labeled film containers and paper clips available. Tell the students they may use the manipulatives if they wish, although they do not have to.

Hand out Student Worksheet 8.

Collect worksheets.

III. Overhead (5 minutes)

Show Lab Worksheet V on the overhead. Remind students that this lab is similar to the lab they worked yesterday. They will be filling out the lab worksheets in the same way they filled out the first ones. They must remember to label the lab problem as they record their answers and are to describe with both pictures and words the work they did to solve on of the problems. For the rest of the problems, just describe the problem with pictures and record the answer. They are to work as many problems as they can in the time
provided.

Hand out Lab Worksheet V.

IV. Lab V (15 minutes)
Allow students at least 15 minutes for this lab. If some students complete the lab early, give them the extra activity provided with lab VII.

Collect the worksheets.
DAY 7: MANIPULATIVES  Two equations/two unknowns

I. Overhead: (10 minutes)

Show the students two different film containers, one marked with an X and one marked with a Y. All the film containers marked with an X in a particular problem have the same number of paper clips in them and all film containers marked with a Y have the same number of paper clips. However, the number of paper clips in the container marked with an X does not have to be the same as the number of paper clips in the container marked with a Y. Open example problem #4's film containers marked with an X and those film containers marked with a Y. The students should see the same number of paper clips in all the X containers and the same number in all the Y containers. Explain that the X and Y containers may have the same number of paper clips and they may not. Their goal is to once again discover the number of paper clips in the containers without opening them up. But this time they have to discover the number in both the X container and the Y container.

Show the students the example #4 lab sheet. Remind them of what the line in the middle means (=). Ask them what this means as far as the containers and paper clips are concerned (Same # of paper clips on either side of the line, some are shown and some are hidden in the containers). Ask what differences they see from the last set of labs and problems (containers called X and Y, another line).

Explain that the line in the center of the page, separating the two equations from one another means that the two problems are connected. Just as before, the line down the middle of the page indicates the same number of paper clips on both sides. But the line down the center means that the containers marked with an X on both the top line and the bottom line have the same number of paper clips and the containers marked with a Y on both top and bottom have the same number of paper clips. So we can use both the top and the bottom to discover the number of paper clips in the X container and the Y container.

Remind students of what the labs sheets look like. On the lab sheets, the different containers are labeled either with an X or a Y. In the labs, the containers are also labeled with an X or a Y. Show students the example problem #4 containers. They will once again have worksheets to record their work. Explain that they must record their answers along with the work they did to get the answer. They are to describe once again in words and pictures the work they did to the containers and paper clips to find the answer for one problem and the rest of the problems, they can just describe the problem using pictures and record their answer to the problem.

Hand out Lab Worksheet IV.

II. Lab IV (10 minutes)

Organize the students into workstations again with four students at each station. Each station should have 6 labs available and the students are to work as many as they can in the 10 minutes provided. They are to do their own problems and must record their answers to the problems they have worked but interaction is allowed and encouraged within the workstations. Make sure the students label on the worksheet the number of the lab they are working. Each lab is labeled 1-6.

Tell the students before they begin that if they do not understand how to find the answers, that is all right. This lab is designed to let the students play around with the new containers and discuss among themselves possible ways to figure out the answers. They will not have had any further instruction at this point. Don't let them get frustrated. Encourage discussion and interaction within the workstations. If the students can only do
one problem in the time provided, that is fine.

Stop the students after 10 minutes. Remind them that they do not have to have all the problems completed, but hopefully have at least one problem that they have worked on. What you want from them now are suggestions and comments about the work they just did. Were they frustrated? Confused? Now you are going to discuss ways of solving these types of problems.

Collect Lab IV worksheets.

III. Overhead (10 minutes)

Do this example on the overhead:

I have two streets, Montana Street and Dakota Street. (Place streets on the overhead) On one side of Montana Street there are 2 white houses (Place film containers that have triangles on the bottom on the overhead) On the other side of Montana Street there is one white house and 3 dogs. (Place another container with a triangle on the bottom and 3 paper clips on the overhead) On one side of Dakota Street there is one brown house (Place container with square on the bottom on overhead), one white house (place container with on bottom), and one dog (place paper clip on overhead). On the other side of Dakota Street there are 6 dogs (place 6 paper clips on overhead). There are the same number of dogs on both sides of Montana Street and the same number of dogs on both sides of Dakota Street. There are the same number of dogs in the white houses. I want to know how many dogs are in the white houses and how many are in the brown house.

The overhead should look like this:

\[
\begin{array}{c}
\triangle \triangle \\
\hline
\square \triangle \emptyset
\end{array}
\quad =
\begin{array}{c}
\triangle \emptyset \emptyset \emptyset \\
\hline
\emptyset \emptyset \emptyset \emptyset \emptyset \emptyset \emptyset
\end{array}
\]

Ask the students "How do I find the number of dogs in the houses?" Ask for suggestions but if there are no suggestions begin by dealing just with Montana street. Remove one house from both side of the street. Ask the students if this is a "legal" move. Then ask if this gives them any information. If no response, ask if anyone now knows how many dogs are in the white house. (3). Once everyone sees that there are 3 dogs in each of the white houses, ask how that information could be used to determine the number of dogs in the brown house. Once again, if no suggestions are given, remind the students that EACH the white house has 3 dogs in it. Then remove the white house on the overhead on Dakota Street and replace it with 3 dogs. Ask the students if this is OK. Once this is agreed as an acceptable move, ask how they can discover now the number of dogs in the brown house. (Remove four paper clips from either side of Dakota Street) How many dogs are in the brown house? (2)

IV. Overhead (10 minutes)
Put another problem on the overhead, using the triangle container, the square container, and the paper clips. Organize the students into their workstations with four students at each station. Allow the students the rest of the period to work the problem given on the overhead. Give the students enough manipulatives to set up this problem. Set up this problem on the overhead:

There is one blue house and two cats on one side of Main Street and a grey house on the other side. There are two blue houses and a grey house on one side of 5th Street and 8 cats on the other side of the street. If all the blue houses have the same number of cats and all the grey houses have the same number of cats, and there are the same number of cats on either side of Main Street and the same number of cats on either side of 5th Street, how many cats are in the blue houses and how many are in the grey houses.

As you tell the story, place the containers and paper clips on the overhead. The overhead should look like this:

\[ \triangle \quad \bullet \bullet \quad = \quad \square \]

\[ \triangle \quad \triangle \quad \square \quad = \quad \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \]

Allow students up to 5 minutes to figure out how to solve this problem. Then ask how the problem could be solved. Ask for suggestions. Allow as many different ways to solve the problem as you can in the remaining 5 minutes. Tell the students that there may be many ways to solve each of the problems they will be asked to solve, but there is only one answer for each lab.
TREATMENT II (Without computer)
Two equations/two unknowns

DAY 8: PICTURES

I. Overhead (5 minutes)
   Show Lab Worksheet V on the overhead. Remind students that this lab is similar to
   the lab they worked yesterday. They will be filling out the lab worksheets in the same way
   they filled out the first ones. They must remember to label the lab problem as they record
   their answers and are to describe with both pictures and words the work they did to solve
   on of the problems. For the rest of the problems, just describe the problem with pictures
   and record the answer. They are to work as many problems as they can in the time
   provided.
   Hand out Lab Worksheet V.

II. Lab V (20 minutes)
   Allow students at least 20 minutes for this lab. If some students complete the lab
   early, give them the extra activity provided with lab V.
   Collect the worksheets.

III. Overhead (5 minutes)
   Go back to the first example used yesterday. Remind them on the overhead of the
   problem of the dogs and houses. The overhead should once again look like this:

   \[
   \begin{array}{ccc}
   \triangle & \triangle & = \\
   \hline
   \hline
   \hline
   \hline
   \hline
   \hline
   \hline
   \hline
   \hline
   \hline
   \hline
   \end{array}
   \]

   Ask the students how we could show this problem without the containers and paper
   clips. Tell them we don't want to use algebraic notation quite yet, but would like a picture
   of what the problem looks like. Draw on the overhead and replace the houses with stick
   houses. Use two different colored pens to show the different colored houses. Replace the
   paper clips with dogs. Draw one equation on one line and the other equation on the other
   line. Label the street names so the students see the resemblance to the original problem.
   The overhead should now look like this:

   \[
   \begin{array}{ccc}
   \square & \triangle & \square & = \\
   \hline
   \hline
   \hline
   \hline
   \hline
   \hline
   \hline
   \hline
   \hline
   \hline
   \hline
   \hline
   \hline
   \end{array}
   \]

   Now ask how we solved this problem. Put the containers and paper clips back on
   the overhead and ask someone to show the class what we did next. Remove one house
   from both sides of Montana Street. Ask how we could show what we did with the
   pictures. Put the picture overhead back up, and draw lines through one house on each side
   of Montana Street. Draw a new picture without the houses. What conclusion did we come
   to about the number of dogs in the white houses? What did we do next? Go back to the
   manipulatives. We replaced the white house on Dakota Street with the three dogs that lived
   there. Draw this on the picture overhead. Now what? We removed four paper clips from
   both sides of Dakota Street, so draw lines through the pictures of the dogs. The conclusion
was that there were two dogs in the brown house. Open up the containers once again to verify the answer. The series of pictures on the overhead should look like this:

1.  

2.  

3.  

4.  

5.  

IV. **Student Worksheet 8** (10 minutes)  
Put the worksheet on the overhead. Explain that the problems to be worked are given in pictures on the worksheet and the object is to discover the number of objects in the X containers and the number in the Y containers. Encourage the students to use the manipulatives to help solve the problems. Have enough labeled film containers and paper clips available. Tell the students they may use the manipulatives if they wish, although they do not have to.  
Hand out **Student Worksheet 8**.  
Collect worksheets.
TREATMENT II (Without computer)

DAY 9: SYMBOLIC

Two equations/two unknowns

I. Overhead (5 minutes)

Go back to the houses and dogs problem. Remind the students what the problem was. Put up an overhead with a picture of just the first equation, with the houses and dogs on Montana Street. Ask how they would write this equation using algebraic notation. Remind the students that we will be using an \( X \) to represent the number of dogs in the white house and a \( Y \) to represent the number of dogs in the brown house. So the equation for the houses and dogs on Montana Street should look like this:

\[
2X = X + 3
\]

Now put up an overhead with the picture of the houses and dogs on Dakota Street. Ask for a volunteer to come up and write the algebraic equation for this picture. It should look something like this:

\[
X + Y = 6
\]

When we write these two equations together so that people know that they are connected, we usually write one on top of the other like this:

\[
\begin{align*}
2X &= X + 3 \\
X + Y &= 6
\end{align*}
\]

II. Student Worksheet 9 (10 minutes)

Show students student worksheet 9. Explain that they are to write the picture problems shown them using the algebraic notation they have been talking about. Hand out Student Worksheet 9. Collect worksheets.

III. Overhead (5 minutes)

Go back to the houses and dogs, using both the manipulatives and the pictures. What did we do next when we solved this problem? (Removed a white house from both sides of Montana Street) How would we write this new problem using algebraic notation? It should look like this:

\[
X = 3  \\
X + Y + 1 = 6
\]

Now what did we do? (Replace the \( X \) in second equation with 3) How could we write that? It should look like this:

\[
\begin{align*}
X &= 3 \\
3 + Y + 1 &= 6
\end{align*}
\]

Now what happened? (Removed four dogs from both sides of the Dakota Street) Why did we remove 4 dogs? Suggest that the equations really look like this:

\[
\begin{align*}
X &= 3 \\
4 + Y &= 6
\end{align*}
\]

or

\[
\begin{align*}
X &= 3 \\
Y + 4 &= 6
\end{align*}
\]

Are these two the same? Now we can see why we really removed four dogs from each side of Dakota Street instead of just three. So what did we end up with? The final equations should look like this:

\[
\begin{align*}
X &= 3 \\
Y &= 2
\end{align*}
\]

So the conclusion was that there were three dogs in each of the white houses and two dogs.
in each of the brown houses.

Show lab worksheet VI on the overhead. Explain to the students that they will once again be required to record their work and answers on this sheet. They must do as many of the 8 problems as they can in the time provided. They should label the problems as they record them. This time, instead of using pictures to show their work, they must write each problem on the worksheet using algebraic notation, and describe the process used to solve the problem using the notation. Encourage the students to use the manipulatives as much as they need to and to discuss the problems with other members at their workstations, but each person is to record their own solutions.

Hand out Lab Worksheet VI.

IV. Lab VI (20 minutes)
Allow the students just 20 minutes to complete the lab. Emphasize that they are to do as many of the problems as they can in the 20 minutes. If some students complete the lab early, give them the extra activity provided with lab VI.
Collect worksheets.
TREATMENT II (Without computer)

DAY 10: SYMBOLIC

Two equations/two unknowns

I. Student Worksheet 10 (15 minutes)
   On this worksheet, encourage students to use the manipulatives to solve the problems. Instruct them that they must use algebraic notation to describe the work they did to solve the problems.
   Hand out Student Worksheet 10.
   Collect worksheets.

II. Overhead (5 minutes)
   The answers to a problem are $X = 2$ and $Y = 5$. Make up a problem that involves two equations that could have this solution. Ask for suggestions. If no one makes a suggestion try this one:

   \[
   \begin{align*}
   3 &= X + 1 \\
   3Y + X &= X + 15
   \end{align*}
   \]
   How can we decide if this is a problem with the answers we wanted? Is this the only problem with the answers of 2 and 5? Could there be others? Put this problem on the overhead and ask if this could also be a solution? How could you be sure?

   \[
   \begin{align*}
   X + 4 &= Y + 1 \\
   3X &= Y + 1
   \end{align*}
   \]
   Discuss ways of solving this problem. Suggest that since we know that $Y + 1$ is equal to both $X + 4$ and $3X$ that we could replace the $Y + 1$ in the first equation with the $3X$.
   Then the problem looks like:

   \[
   \begin{align*}
   X + 4 &= 3X \\
   3X &= Y + 1
   \end{align*}
   \]
   So then can solve the first equation to find out what $X$ is equal to, and then use that information to find out the value of $Y$. It turns out that this problem does have the same solution of $X = 2$ and $Y = 5$.

III. Overhead (20 minutes)
   Organize the students into their workstations. There should be four people in each group. Ask each group to come up with a different problem with the same answer $X = 2$ and $Y = 5$. After 5 minutes, ask each group to share their problem. One member of the group can write it on the overhead and another member of the group can describe how their group came up with that problem. Another member can describe on a piece of paper a method that could be used to solve the problem, and the last member can show on the same paper that if the 2 and 5 replace the $X$ and $Y$ the problem is correct.
LESSON PLANS

TREATMENT III CONTROL TREATMENT
Day 1: Subtraction Property and Addition Property for Equations

I. Warm-up Exercises (2 minutes)
On the overhead, do the following exercises. Ask the students to find the value of each expression.
1. \(5 + 3 - 5\)  
2. \(10 + 2 - 2\)  
3. \(15 + 10 - 10\)  
4. \(X + 8 - 8\)

II. Overhead (8 minutes)
The following table shows that subtraction is the opposite or inverse operation of addition.

<table>
<thead>
<tr>
<th>Start with a number</th>
<th>Add any number</th>
<th>Subtract the same number</th>
<th>Get back the original number</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10 + 3</td>
<td>10 + 3 - 3</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>5 + 6</td>
<td>5 + 6 - 6</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>4 + 12</td>
<td>4 + 12 - 12</td>
<td>4</td>
</tr>
<tr>
<td>(n)</td>
<td>(n + 7)</td>
<td>(n + 7 - 7)</td>
<td>(n)</td>
</tr>
</tbody>
</table>

Thus, subtraction reverses the addition step. This idea can be applied to solving equations. An equation is a sentence that contains the equality symbol "=".

To solve an equation means to find the number or numbers that will make the equation true.

If \(n\) is replaced by 7, which of these equations are true?

a. \(n + 5 = 12\)  
b. \(n + 5 - 5 = 12 - 5\)  
c. \(n = 7\)

The number 7 is the solution to all three equations. Equations that have the same solution are equivalent equations. Thus, equations a, b, and c are equivalent. Note that the only difference between equations a and b is that 5 is subtracted from each side of the equation.

So we have the following property:

**Subtraction Property for Equations**

Subtracting the same number from each side of an equation forms an equivalent equation.

\[n + 5 = 12\]
\[n + 5 - 5 = 12 - 5\]
\[n = 7\]

The goal in solving an equation is to get the variable alone on one side of the equation. The Subtraction Property for Equations is a means of doing this, as is demonstrated in the following examples.
Example 1  Solve and check: \( n + 8 = 20 \) Show all steps to students.

Solution: To get \( n \) alone, subtract 8 from each side.

\[
\begin{align*}
  n + 8 &= 20 \\
  n + 8 - 8 &= 20 - 8 \\
  n &= 12
\end{align*}
\]

Check: \( n + 8 = 20 \)
\[
\begin{align*}
  12 + 8 &= 20 \\
  20 &= 20
\end{align*}
\]

Example 2  Solve and check: \( 14 = X + 5 \)

Solution: \[
\begin{align*}
  14 &= X + 5 \\
  14 - 5 &= X + 5 - 5 \\
  9 &= X
\end{align*}
\]

Check: \( 14 = X + 5 \)
\[
\begin{align*}
  9 + 5 &= 14 \\
  14 &= 14
\end{align*}
\]

III. Classroom Exercises (5 minutes)

Do the following exercises on the overhead, asking for suggestions on the next step that should be taken.

1. \( X + 7 - 7 \)
2. \( r + 30 - 30 \)
3. \( n + 19 - 19 \)
4. \( 12 + n - n \)
5. \( 112 + p - p \)
6. \( X + n - n \)
7. \( X + 5 = 19 \)
8. \( a + 12 = 14 \)
9. \( 105 = m + 92 \)
10. \( 87 = r + 36 \)
11. \( z + 56 = 91 \)
12. \( 184 = c + 102 \)

IV. Warm-up Exercises (2 minutes)

On the overhead, ask the students the solutions to the following questions. The goal is to fill in the number which will make each equation true. Write the equation on the overhead without the numbers in bold. Ask students to suggest the number that should be placed in each blank.

1. \( 19 + 0 = 19 \)  
2. \( 4 + 15 = 19 \)  
3. \( 0 + 8 = 8 \)

V. Overhead (8 minutes)

The following table shows that addition is the opposite or inverse operation of subtraction.

<table>
<thead>
<tr>
<th>Start with a number.</th>
<th>Subtract any number.</th>
<th>Add the same number.</th>
<th>Get back the original number.</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>13 - 8</td>
<td>13 - 8 + 8</td>
<td>13</td>
</tr>
<tr>
<td>9</td>
<td>9 - 3</td>
<td>9 - 3 + 3</td>
<td>9</td>
</tr>
<tr>
<td>( n )</td>
<td>( n - 20 )</td>
<td>( n - 20 + 20 )</td>
<td>( n )</td>
</tr>
</tbody>
</table>

What is the value of \( 5 - 18 + 18 \)?

You can find the answer to that example even though you have not yet learned how to solve problems such as \( 5 - 18 \). Just write \( 5 - 18 + 18 \) as \( 5 + 18 - 18 \). You solved this type of problem in the last section.

If \( X \) is replaced by 8, which of these equations are true?

a. \( X - 3 = 5 \)  
b. \( X - 3 + 3 = 5 + 3 \)  
c. \( X = 8 \)
Equations a, b, and c are equivalent. Note the 3 was added to each side of equation a to get equation b and equation c.

This suggests the following property.

**Addition Property for Equations**

Adding the same number to each side of an equation forms as equivalent equation.

**Example 1**  Solve and check: $X - 12 = 9$. Show the students all the steps.

Solution: To get $X$ alone, add 12 to each side.

\[
X - 12 = 9 \\
X - 12 + 12 = 9 + 12 \\
X = 21
\]

Check: $X - 12 = 9$

\[
21 - 12 = 9
\]

**Example 2**  Solve and check: $13 = t - 5$

Solution: $13 + 5 = t - 5 + 5$

\[
18 = t
\]

Check: $13 = t - 5$

\[
18 - 5 = 13
\]

VI. Classroom Exercises (5 minutes)

Do the following exercises on the overhead, asking for suggestions on the next step that should be taken.

1. $r - 12 + 12$
2. $X - 37 + 37$
3. $g - 8 + 8$
4. $3 - X + X$
5. $t - 7 = 19$
6. $a - 12 = 5$
7. $23 = s - 27$
8. $47 = t - 83$
9. $w - 3 = 21$

Decide whether the value shown is the solution for the given equation. Answer either YES or NO.

10. $m - 23 = 56$
   \[
   m = 33
   \]
11. $16 = p - 4$
   \[
   p = 20
   \]
12. $X - 1 = 8$
   \[
   X = 9
   \]

VII. Written Exercises **Worksheet I** (10 minutes)

Have the students work as many of these problems as they can in the 10 minutes remaining in class.

Hand out **Worksheet I**

Allow the students whatever time remains in the class period. Tell the students that they do not have to have all the problems finished but should finish as many as they can.

Collect worksheets.
Treatment III (Control)
One equation with one unknown

Day 2: Division Property and Multiplication Property for Equation

I. Warm-up Exercises (2 minutes)
   On the overhead, ask the students what is the first step in the solution of each equation.

1. \( x + 5 = 12 \)  \( x + 5 - 5 = 12 - 5 \)
2. \( a - 7 = 2 \)  \( a - 7 + 7 = 2 + 7 \)
3. \( y + 3 = 4 \)  \( y + 3 - 3 = 4 - 3 \)
4. \( s - 1 = 9 \)  \( s - 1 + 1 = 9 + 1 \)

II. Overhead (8 minutes)
   The following table shows that division is the inverse operation of multiplication.

<table>
<thead>
<tr>
<th>Start with a number</th>
<th>Multiply by any number</th>
<th>Divide by the same number</th>
<th>Get back the original number</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>8 ( \times 3 )</td>
<td>( \frac{8 \times 3}{3} )</td>
<td>8</td>
</tr>
<tr>
<td>27</td>
<td>27 ( \times 5 )</td>
<td>( \frac{27 \times 5}{5} )</td>
<td>27</td>
</tr>
<tr>
<td>9</td>
<td>( 9n )</td>
<td>( \frac{9n}{n} )</td>
<td>9</td>
</tr>
</tbody>
</table>

If \( n \) is replaced by 3, which of these equations are true?

a. \( 4n = 12 \)
   b. \( \frac{4n}{4} = \frac{12}{4} \)
   c. \( n = 3 \)

Equations a, b, and c are equivalent. Note that each side of equation a was divided by 4 to get equation b and equation c.

This leads to the following property:

**Division Property for Equations**

Dividing each side of an equation by the same nonzero number forms an equivalent equation.

\[ 4n = 12 \]
\[ \frac{4n}{4} = \frac{12}{4} \]
\[ n = 3 \]

**Example 1** Solve and check: \( 3n = 18 \). Show the students all steps.

Solution: To get \( n \) alone, divide each side by 3.

\[ 3n = 18 \]
\[ \frac{3n}{3} = \frac{18}{3} \]
\[ n = 6 \]

**Check:** \( 3n = 18 \)
\[ 3 \times 6 = 18 \]
Example 2  Solve and check: 56 = 8w

Solution: 56 = 8w

\[
\frac{56}{8} = \frac{8w}{8}
\]

7 = w

Check: 56 = 8w

III. Classroom Exercises (5 minutes)
Do the following exercises on the overhead, asking the students to help you simplify each expression.

1. \(\frac{13x}{13}\) 2. \(\frac{27r}{27}\) 3. \(\frac{9t}{9}\) 4. \(\frac{4q}{4}\)

IV. Warm-up Exercises (2 minutes)
On the overhead do the following exercises by asking the students to help you solve each equation.

1. 6x = 54 2. x + 3 = 93 3. x - 5 = 60 4. 7x = 77
5. 26 = 2x 6. 18 = x + 2 7. 19 = x - 1

V. Overhead (8 minutes)
The following table shows that multiplication is the inverse operation of division.

<table>
<thead>
<tr>
<th>Start with a number</th>
<th>Divide by any number</th>
<th>Multiply by the same number</th>
<th>Get back the original number</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>12/4</td>
<td>12/4 x 4</td>
<td>12</td>
</tr>
<tr>
<td>45</td>
<td>45/9</td>
<td>45 x 9</td>
<td>45</td>
</tr>
<tr>
<td>n</td>
<td>n/8</td>
<td>n x 8</td>
<td>n</td>
</tr>
</tbody>
</table>

If X is replaced by 12, which of these equations are true?

a. \(\frac{X}{4} = 3\)  b. \(\frac{X}{4} x 4 = 3 x 4\)  c. \(X = 12\)

Equations a, b, and c are equivalent. Note that each side of equation a was multiplied by 4 to get equation b and equation c.
This suggests the following property:

**Multiplication Property for Equations**

Multiplying each side of an equation by the same nonzero number forms an equivalent equation.

\[
\frac{x}{4} = 3
\]

\[
\frac{x \times 4}{4} = 3 \times 4
\]

\[
x = 12
\]

**Example 1** Solve and check: \( t = 9 \). Show the students all the steps on the overhead.

Solution: To get \( t \) alone, multiply each side by 5.

\[
\frac{t}{5} = 9
\]

\[
\frac{t \times 5}{5} = 9 \times 5
\]

\[
t = 45
\]

Check: \( \frac{t}{5} = 9 \)

\[
\frac{45}{5}
\]

\[
t = 9
\]

**Example 2** Solve and check: \( 12 = \frac{y}{8} \)

Solution: \( 12 = \frac{y}{8} \)

Check: \( 12 = \frac{y}{8} \)

\[
12 \times 8 = \frac{y \times 8}{8}
\]

\[
96 = y
\]

\[
96 = \frac{y}{8} \times 8
\]

\[
y = 12
\]

**Example 3** Solve and check: \( \frac{m}{6} = 4 \)

Solution: \( \frac{m}{6} = 4 \)

Check: \( \frac{m}{6} = 4 \)

\[
\frac{m \times 6}{6} = 4 \times 6
\]

\[
m = 24
\]

\[
\frac{24}{6}
\]

\[
m = 4
\]
VI. Classroom Exercises (5 minutes)

Do the following exercises on the overhead, asking the students to help you simplify each expression.

1. \( \frac{a}{19} \times 19 \)
2. \( \frac{s}{9} \times 9 \)
3. \( \frac{20p}{11} \times 11 \)
4. \( \frac{12h}{12} \)

Write the equivalent equation formed as the first step in solving each equation.

5. \( \frac{x}{4} = 13 \)
6. \( 7 = \frac{s}{6} \)
7. \( w = \frac{9}{11} \)
8. \( 12 = \frac{r}{8} \)

VII. Written Exercises Worksheet II (10 minutes)

Have the students work as many of these problems as they can in the 10 minutes remaining in class.

Hand out Worksheet II

Allow the students whatever time remains in the class period. Tell the students that they do not have to have all the problems finished but should finish as many as they can.

Collect worksheets.
Day 3: Equations with More Than One Operation

I. Warm-up Exercises (2 minutes)
   On the overhead, do the following problems. Ask a student to write the name of the Property for Equations that you would use as the first step in solving each equation.

   1. \(X + 5 = 17\)  **Subtraction**  2. \(X - 3 = 6\)  **Addition**
   3. \(3X = 18\)  **Division**  4. \(\frac{X}{4} = 5\)  **Multiplication**
   5. \(12X = 24\)  **Division**  6. \(\frac{X}{7} = 2\)  **Multiplication**

II. Overhead (15 minutes)
   To solve equations such as the following
   \[3n + 5 = 26\]  \[14 = 4X - 10\]  \[\frac{t}{5} + 7 = 18\]
   the first step is to get the term with the variable alone on one side of the equation.

   **To solve an equation with more than one operation, addition and subtraction are performed before multiplication and division.**

**Example 1** Solve and check: \(3n + 5 = 26\) Show the students all the steps.

   Solution:
   Subtract 5 from each side.
   \[3n + 5 = 26\]
   \[3n + 5 - 5 = 26 - 5\]
   \[3n = 21\]

   Divide each side by 3.
   \[\frac{3n}{3} = \frac{21}{3}\]
   \[n = 7\]

   **Example 2** Solve and check: \(14 = 4X - 10\)

   Solution:
   \[14 = 4X - 10\]
   \[14 + 10 = 4X - 10 + 10\]
   \[24 = 4X\]
   \[\frac{24}{4} = \frac{4X}{4}\]
   \[6 = X\]

   **Check:**
   \[14 = 4X - 10\]
   \[4\times6 - 10\]
   \[24 - 10\]
   \[14\]
Example 3  Solve and check: $8s + 7 = 39$

Solution: 

\[
\begin{align*}
8s + 7 &= 39 \\
8s + 7 - 7 &= 39 - 7 \\
8s &= 32 \\
\frac{8s}{8} &= \frac{32}{8} \\
s &= 4 \\
\end{align*}
\]

Check: 

\[
\begin{align*}
8s + 7 &= 39 \\
8 \times 4 + 7 &= 39 \\
32 + 7 &= 39 \\
\end{align*}
\]

Example 4  Solve and check: $\frac{t + 2}{5} = 6$

Solution: 

\[
\begin{align*}
\frac{t}{5} + 2 &= 6 \\
\frac{t}{5} + 2 - 2 &= 6 - 2 \\
\frac{t}{5} &= 4 \\
\frac{t \times 5}{5} &= 4 \times 5 \\
t &= 20 \\
\end{align*}
\]

Check: 

\[
\begin{align*}
\frac{t}{5} + 2 &= 6 \\
\frac{20}{5} + 2 &= 6 \\
4 + 2 &= 6 \\
\end{align*}
\]

Example 5  Solve and check: $9 = \frac{w + 4}{3}$

Solution: 

\[
\begin{align*}
9 &= \frac{w}{3} + 4 \\
9 - 4 &= \frac{w}{3} + 4 - 4 \\
5 &= \frac{w}{3} \\
5 \times 3 &= \frac{w \times 3}{3} \\
15 &= w \\
\end{align*}
\]

Check: 

\[
\begin{align*}
9 &= \frac{w}{3} + 4 \\
\frac{15}{3} + 4 &= 9 \\
5 + 4 &= 9 \\
\end{align*}
\]

III. Classroom Exercises (8 minutes)

Ask the students what operation to perform first in solving each equation.

1. $2y - 5 = 13$ addition 
2. $7 = \frac{X}{3} + 2$ subtraction
3. $5 = 3t + 2$ subtraction 
4. $2a - 3 = 3$ addition
5. $\frac{r}{3} - 1 = 4$ addition 
6. $2 = 3m + 2$ subtraction
Write the equivalent equation formed as the first step in solving each equation above.

7. $2y - 5 + 5 = 13 - 5$
8. $7 - 2 = \frac{X}{3} + 2 - 2$
9. $5 - 2 = 3t + 2 - 2$
10. $2a - 3 + 3 = 3 + 3$
11. $\frac{r}{3} - 1 + 1 = 4$
12. $2 - 2 = 3m + 2 - 2$

Decide whether the value of X shown is the solution for the given equation. Answer YES or NO.

13. $3X + 4 = 22$: $X = 6$ (yes) 14. $2X + 5 = 21$: $X = 4$ (no)
15. $12 = 5X - 2$: $X = 2$ (no) 16. $16 = 7X - 5$: $X = 3$ (yes)
17. $\frac{X}{3} - 5 = 8$: $X = 3$ (no) 18. $26 = \frac{X}{2} + 14$: $X = 24$ (yes)

IV. Written Exercises Worksheet III (15 minutes)

Have the students work as many of these problems as they can in the 15 minutes remaining in class.

Hand out Worksheet III

Allow the students whatever time remains in the class period. Tell the students that they do not have to have all the problems finished but should finish as many as they can.

Collect worksheets.
Day 4: More Equations: Like Terms

I. Warm-up Exercises (2 minutes)
On the overhead, ask the students to solve each equation.

1. \(X - 7 = 13\) \((X = 20)\)
2. \(3X = 18\) \((X = 6)\)
3. \(2X - 5 = 19\) \((X = 12)\)
4. \(\frac{X}{3} = 5\) \((X = 15)\)
5. \(\frac{X}{7} + 11 = 14\) \((X = 21)\)
6. \(\frac{X}{32} - 5 = 3\) \((X = 32)\)

II. Overhead (15 minutes)
When you solve an equation such as \(2X + 4X - 5 = 13\)
you must first combine (add or subtract) the like terms.

Example 1 Solve and check: \(2X + 4X - 5 = 13\). Show the students all the steps.

Solution: \(2X + 4X - 5 = 13\)

Check: \(2X + 4X - 5 = 13\)

Combine like terms: \(6X - 5 = 13\)
Add 5 to each side: \(6X - 5 + 5 = 13 + 5\)
\(6X = 18\)
Divide each side by 6: \(\frac{6X}{6} = \frac{18}{6}\)
\(X = 3\)

Example 2 Solve and check: \(23 = 10t + 3 - 5t\)

Solution: \(23 = 10t + 3 - 5t\)

Check: \(23 = 10t + 3 - 5t\)

Combine like terms: \(23 = 5t + 3\)
Subtract 3 from each side: \(23 - 3 = 5t + 3 - 3\)
\(20 = 5t\)
Divide each side by 5: \(\frac{20}{5} = \frac{5t}{5}\)
\(4 = t\)
Example 3  Solve and check: 24 = 12p - p - 9

Solution:  
\[24 = 12p - p - 9\]

Combine like terms:  
\[24 = 11p - 9\]

Add 9 to each side:  
\[24 + 9 = 11p - 9 + 9\]
\[33 = 11p\]

Divide each side by 11  
\[\frac{33}{11} = \frac{11p}{11}\]
\[3 = p\]

Check:  
\[24 = 12p - p - 9\]

Example 4  Solve and check: 5w - 3w + 4 = 14

Solution:  
\[5w - 3w + 4 = 14\]

Combine like terms:  
\[2w + 4 = 14\]

Subtract 4 from each side:  
\[2w + 4 - 4 = 14 - 4\]
\[2w = 10\]

Divide each side by 2:  
\[\frac{2w}{2} = \frac{10}{2}\]
\[w = 5\]

Check:  
\[5w - 3w + 4 = 14\]

Example 5  Solve and check: 18 = 3y + 5y + 2

Solution:  
\[18 = 3y + 5y + 2\]

Combine like terms:  
\[18 = 8y + 2\]

Subtract 2 from each side:  
\[18 - 2 = 8y + 2 - 2\]
\[16 = 8y\]

Divide each side by 8:  
\[\frac{16}{8} = \frac{8y}{8}\]
\[2 = y\]

Check:  
\[18 = 3y + 5y + 2\]

To review:

To Solve an Equation with Like Terms

1. Combine like terms.
2. Use the Addition or Subtraction Property for Equations.
3. Use the Multiplication or Division Property for Equations.
4. Check the solution.
III. Classroom Exercises (8 minutes)

Ask the students what equivalent equation is formed as the first step in solving each equation.

1. 12r + 8r + 3 = 10
2. 22 = 6w + 8 + w
3. 3a + 9 + 4a + 2 = 27
4. 9t + 6 + t = 46
5. 56 = 7w + 5w - 4
6. 6p + 2p + 3 = 27
7. n - 6 + n = 28
8. 44 = 5 + 7s + 3 + 2s
9. 10b - 7b + 6 = 19
10. 28 = 5s + 12 - s
11. 9y + 12 - 3y - 4 = 50

IV. Written Exercises Worksheet IV (15 minutes)

Have the students work as many of these problems as they can in the 15 minutes remaining in class.

Hand out Worksheet IV

Allow the students whatever time remains in the class period. Tell the students that they do not have to have all the problems finished but should finish as many as they can.

Collect worksheets.
Day 5: Distributive Property in Equations

Treatment III (Control)
One equation with one unknown

I. Warm-up Exercises (2 minutes)
Write the following equations on the overhead, and ask the students what the equivalent equation formed as the first step in solving each equation.

1. \(2r + 5r + 4 = 24\) \(7r + 4 = 24\)
2. \(3x + x - 5 = 7\) \(4x - 5 = 7\)
3. \(6y - 2y - 3 = 17\) \(4y - 3 = 17\)
4. \(8 = 5t - 2t + 5\) \(8 = 3t + 5\)
5. \(13 = 3s + 5 + 6s\) \(13 = 9s + 5\)
6. \(12n - 6 - 7n = 24\) \(5n - 6 = 24\)

II. Overhead (15 minutes)
In some equations you must apply the Distributive Property as the first step. Then you add or subtract.

Example 1 Solve and check: \(54 = 3(4t - 2)\).

Solution:
Use the Distributive Property: \(54 = 3x4t - 3x2\)
Simplify: \(54 = 12t - 6\)
Add 6 to each side: \(54 + 6 = 12t - 6 + 6\)
Divide each side by 12: \(60 = 12t\)
\(5 = t\)

Check: \(54 = 3(4t - 2)\)
\(3(4x8 - 2)\)
\(3(20 - 2)\)
\(3(18)\)
\(54\)

Example 2 Solve and check: \(2(3a + 7) + a = 70\)

Solution:
Use Distributive Property: \(2x3a + 2x7 + a = 70\)
Add 14 from each side: \(7a + 14 = 70 - 14\)
Divide each side by 7: \(\frac{7a}{7} = \frac{56}{7}\)
\(a = 8\)
Example 3  Solve and check: \(7(2b - 8) + b = 4\)

Solution:

Use the Distributive Property

Add 56 to each side

Divide each side by 15

Check:

\[\begin{align*}
7(2b - 8) + b &= 4 \\
14b - 56 + b &= 4 \\
15b - 56 &= 4 \\
15b &= 60 \\
\frac{15b}{15} &= \frac{60}{15} \\
b &= 4
\]

Using the Distributive Property to Solve an Equation

1. Use the Distributive Property.
2. Combine like terms.
3. Use the Addition or Subtraction Property for Equations.
4. Use the Multiplication or Division for Equations.
5. Check the solution.

III. Classroom Exercises (8 minutes)

Write the following equations on the overhead and have the students give the equivalent equation formed as the first step in solving each equation.

1. \(7(n + 2) = 56\)  \(7n + 14 = 56\)
2. \(12(y - 3) = 108\)  \(12y - 36 = 108\)
3. \((4n - 9)5 = 144\)  \(20n - 45 = 144\)
4. \(117 = (12w + 5)3\)  \(117 = 36w + 15\)
5. \(2(x - 3) + x = 9\)  \(2x - 6 + x = 9\)
6. \(3(2x + 7) + 3x = 30\)  \(6x + 21 + 3x = 30\)
7. \(3x + (3x - 5)2 + 4x = 126\)  \(3x + 6x - 10 + 4x = 126\)

IV. Written Exercises Worksheet V (15 minutes)

Have the students work as many of these problems as they can in the 15 minutes remaining in class.

Hand out Worksheet V

Allow the students whatever time remains in the class period. Tell the students that they do not have to have all the problems finished but should finish as many as they can.

Collect worksheets.
Day 6: Equations with Two Variables

I. Warm-up Exercises (2 minutes)
On the overhead, ask the students to evaluate each expression:

1. \( y + 5 \) when \( y = 6 \)
2. \( 2y + 5 \) when \( y = 6 \)
3. \( x + 6 \) when \( x = 2 \)
4. \( 3x + 6 \) when \( x = 2 \)
5. \( 3(x + 2) \) when \( x = 1 \)
6. \( 4y + 1 \) when \( y = 0 \)
7. \( x + y \) when \( x = 1 \) and \( y = 3 \)
8. \( 2x + y + 3 \) when \( x = 0 \) and \( y = 1 \)

II. Overhead (15 minutes)
The equation \( x + y = 2 \) is an example of a linear equation in two variables.

**Definition:** The standard form of a linear equation is \( Ax + By = C \) in which
\( A, B, \) and \( C \) are just any number other than 0.

What are the values of \( A, B, \) and \( C \) in the equation \( x + y = 2 \)?

**Definition:** The solution to a linear equation is an ordered pair that makes the equation true.

A solution to the linear equation \( x + y = 5 \) is \( x = 2, y = 3 \). If you replace the \( x \) in the equation with 2 and the \( y \) with 3, the equation is \( 2 + 3 = 5 \) which is a true statement. This solution can be written as an ordered pair \((2,3)\) where \( x \) is the first number and \( y \) is the second number.

**Example 1** Which ordered pairs are solutions of \( y = x + 3 \)?

a) (1,4)  
b) (2, 7)  
c) (0, 3)  
d) (4, 1)

Substitute the given values for \( x \) and \( y \) in the equation.

**Example 2** Which ordered pairs are solutions of \( y + 1 = 2x \)?

a) (1, 1)  
b) (0, 1)  
c) (3, 1)  
d) (2, 3)

Substitute the given values for \( x \) and \( y \) in the equation.
To find a solution of a linear equation, choose a positive, integer value for \( x \). Then substitute the chosen value in the equation and solve for \( y \). If \( y \) is a positive integer, then this is a solution.

*Note* The solutions that we are interested in finding are solutions of positive integer values. These are not the only solutions to an equation, but it is the only type of solution that we are interested in finding. Both \( x \) and \( y \) must be positive and an integer. If students don’t know what a positive integer is, take a couple of minutes to explain.

**Example 3:** Make a table showing two solutions of \( 2x + y = 3 \)

**Solution:** Choose 2 different values for \( x \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

III. Classroom Exercises (8 minutes)
Determine whether the given ordered pair is a solution of the given equation.

1. \( y = 3x + 1 \): (2, 5)  
2. \( y + 1 = 2x \): (0, 5)  
3. \( x + y = 1 \): (1, 1)  
4. \( 2x + y = 4 \): (2, 0)  
5. \( y + 1 = 3x \): (1, 2)  
6. \( x + 2y = 0 \): (0, 0)

Find three solutions for each equation using a table.

1. \( x + y = 5 \)  
2. \( x + 2y = 17 \)  
3. \( 3x + y = 9 \)  
4. \( 3x + 2y = 11 \)

IV. Written Exercises Worksheet VI (15 minutes)
Have the students work as many of these problems as they can in the 15 minutes remaining in class.

Hand out Worksheet VI
Allow the students whatever time remains in the class period. Tell the students that they do not have to have all the problems finished but should finish as many as they can.
Collect worksheets.
Day 7: Solution Set to Equations with Two Variables

I. Warm-up Exercises (2 minutes):
On the overhead, ask the students to complete the tables.

1. \[ y + 1 = x \]
   \[ \begin{array}{cccc}
   x & 0 & 1 & 2 \\
   y & ? & ? & ? \\
   \end{array} \]

2. \[ x + y = 3 \]
   \[ \begin{array}{cccc}
   x & 0 & 1 & 2 \\
   y & ? & ? & ? \\
   \end{array} \]

3. \[ x + y = 2 \]
   \[ \begin{array}{cccc}
   x & 0 & 1 & 2 \\
   y & ? & ? & ? \\
   \end{array} \]

4. \[ 2x + y = 5 \]
   \[ \begin{array}{cccc}
   x & 0 & 1 & 2 \\
   y & ? & ? & ? \\
   \end{array} \]

II. Overhead (10 minutes)

**Definition:** The solution set of a linear equation is all the sets of ordered pairs of positive integers that make the equation true.

*Reminder* The solutions that we are interested in finding are solutions of positive integer values. These are not the only solutions to an equation, but it is the only type of solution that we are interested in finding. Both x and y must be positive and an integer. If students still don't know what a positive integer is, take a couple of minutes to explain once again.

A solution set of the linear equation \( x + y = 5 \) can be found by finding all the positive integers that make the equation true. We can do this by making a table to show the solutions.
Example 1: Make a table to show all the solutions of the equation $2x + y = 4$.

<table>
<thead>
<tr>
<th>Select values for x.</th>
<th>Evaluate $2x + y = 4$</th>
<th>For each value of x, y equals</th>
<th>Resulting ordered pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2(0) + y = 4$</td>
<td>4</td>
<td>(0, 4)</td>
</tr>
<tr>
<td>1</td>
<td>$2(1) + y = 4$</td>
<td>2</td>
<td>(1, 2)</td>
</tr>
<tr>
<td>2</td>
<td>$2(2) + y = 4$</td>
<td>0</td>
<td>(2, 0)</td>
</tr>
<tr>
<td>3</td>
<td>$2(3) + y = 4$</td>
<td>No solution</td>
<td></td>
</tr>
</tbody>
</table>

Solution set \{ (0, 4), (1, 2), (2, 0) \}

To find all the positive integer solutions to a linear equation, make a table with $x = 0, 1, 2, 3, \ldots$ until you have found all the solutions that give a $y$ value that is a positive integer value.

Example 2: Make a table to show all the solutions of the equation $x + 2y = 5$.

<table>
<thead>
<tr>
<th>Select values for x.</th>
<th>Evaluate $x + 2y = 5$</th>
<th>For each value of x, y equals</th>
<th>Resulting ordered pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(0) + 2y = 4</td>
<td>2</td>
<td>(0, 2)</td>
</tr>
<tr>
<td>1</td>
<td>(1) + 2y = 4</td>
<td>No solution</td>
<td>No pair</td>
</tr>
<tr>
<td>2</td>
<td>(2) + 2y = 4</td>
<td>1</td>
<td>(2, 1)</td>
</tr>
<tr>
<td>3</td>
<td>(3) + 2y = 4</td>
<td>No solution</td>
<td>No pair</td>
</tr>
<tr>
<td>4</td>
<td>(4) + 2y = 4</td>
<td>0</td>
<td>(4, 0)</td>
</tr>
<tr>
<td>5</td>
<td>(5) + 2y = 4</td>
<td>No solution</td>
<td>No pair</td>
</tr>
</tbody>
</table>

Solution set \{ (0, 2), (2, 1), (4, 0) \}

III. Classroom Exercises (13 minutes)
Find the solution set for the following equations using a table. Be sure to write the solution set using ordered pairs.

1. $x + 2y = 3$
2. $2x + 2y = 4$
3. $x + y = 1$
4. $2x + y = 4$
5. $3x + y = 1$
6. $x + 2y = 0$
IV. Written Exercises *Worksheet VII* (15 minutes)

Have the students work as many of these problems as they can in the 15 minutes remaining in class.

Hand out *Worksheet VII*

Allow the students whatever time remains in the class period. Tell the students that they do not have to have all the problems finished but should finish as many as they can.

Collect worksheets.
Day 8: Addition Method

I. Warm-up Exercises (2 minutes)
On the overhead, ask the students to simplify each expression.

1. $3x - 3x$
2. $2x - x$
3. $7x - 3x$
4. $7x - 3x + x$
5. $4x - 7 - 3x + 7$
6. $9x - 3 + 7x + 3$
7. $13x - 10x$
8. $8x + 3x$

II. Overhead (15 minutes)
The equation $x + y = 2$ is an example of a linear equation in two variables.

Definition: The standard form of a linear equation is $Ax + By = C$ in which $A$, $B$, and $C$ are just any number other than 0.

What are the values of $A$, $B$, and $C$ in the equation $x + y = 2$?

Definition: Two equations in the same two variables form a system of equations.

\[
\begin{align*}
x - 2y &= 3 \\
3x + 4y &= 19
\end{align*}
\]

Definition: The solution set of a system of equations is the set of ordered pairs that makes both equations true.

The solution set of the system below can be read like: $x = 2$
\[
y = 3 \quad \text{Solution set: } \{(2,3)\}
\]

When the corresponding sides of the two equations of a system are added, a sum equation is formed.

For the system
\[
\begin{align*}
x &= 2, \\
y &= 3 \\
x + y &= 5
\end{align*}
\]
the sum equation is

For the system
\[
\begin{align*}
x &= 2, \\
y &= 1 \\
x + y &= 3
\end{align*}
\]
the sum equation is

For the system
\[
\begin{align*}
x &= 0, \\
y &= 12 \\
x + y &= 12
\end{align*}
\]
the sum equation is
To solve a system of equations algebraically, you can add the corresponding sides of the equations. This method is useful when the coefficients of either the $x$ terms or of the $y$ terms are opposites.

For example: $x + y = 5$ and $x - y = 1$ the coefficients of the $y$ terms are opposites.

**Example 1**

Solve:

\[
\begin{align*}
x + y &= 5 \\
x - y &= 1
\end{align*}
\]

Solution:

Add the corresponding sides of both equations to eliminate $y$.

\[
\begin{align*}
x + y &= 5 \\
x - y &= 1
\end{align*}
\]

Solve for $x$.

\[
\begin{align*}
2x &= 6 \\
x &= 3
\end{align*}
\]

Check: Substitute $x = 3$ and $y = 2$ in $x + y = 5$.

\[
\begin{align*}
x + y &= 5 \\
3 + 2 &= 5
\end{align*}
\]

Substitute 3 for $x$ in one of the equations.

\[
x + y = 5 \\
3 + y = 5
\]

Solve for $y$.

\[
\begin{align*}
3 - 3 + y &= 5 - 3 \\
y &= 2
\end{align*}
\]

**Example 2**

Solve:

\[
\begin{align*}
2x + y &= 3 \\
-2x + 3y &= 1
\end{align*}
\]

Solution:

Add the corresponding sides of the equations. This eliminates the $x$ terms.

\[
\begin{align*}
2x + y &= 3 \\
-2x + 3y &= 1
\end{align*}
\]

Substitute 1 for $y$ in one of the equations.

\[
\begin{align*}
2x + y &= 3 \\
2x + 1 &= 3 \\
2x &= 2 \\
x &= 1
\end{align*}
\]

Check: Substitute $x = 1$ and $y = 1$.

\[
\begin{align*}
2x + y &= 3 \\
2(1) + 1 &= 3
\end{align*}
\]

\[
\begin{align*}
-2x + 3y &= 1 \\
-2(1) + 3(1) &= 1
\end{align*}
\]
Steps for the Addition Method

1. Add to eliminate one of the variables. Solve for the remaining variable.

2. Substitute the known value of one variable in one of the original equations of the system. Solve for the other variable.

3. Check the solution in both equations of the system.

III. Classroom Exercises (8 minutes)

Have the students tell you the equation that results if you add the corresponding sides of two equations in each system.

1. \( x + 2y = 3 \)
2. \( 2x - 3y = 2 \)
3. \( 4x - 5y = 2 \)
4. \( -x + y = 6 \)
5. \( x + 3y = 4 \)
6. \( x + 5y = 8 \)

Add the corresponding sides of the above equations and then solve for either \( x \) or \( y \).

1. \( 3y = 9 \) \( y = 3 \)
2. \( 3x = 6 \) \( x = 2 \)
3. \( 5x = 10 \) \( x = 2 \)
4. \( 7x = 14 \) \( x = 2 \)
5. \( 8y = 8 \) \( y = 1 \)
6. \( 6x = 24 \) \( x = 4 \)

IV. Written Exercises Worksheet VIII (15 minutes)

Have the students work as many of these problems as they can in the 15 minutes remaining in class.

Hand out Worksheet VIII

Allow the students whatever time remains in the class period. Tell the students that they do not have to have all the problems finished but should finish as many as they can. Collect worksheets.
Day 9: Multiplication/Addition Method

I. Warm-up Exercises (2 minutes)
Write the following expressions on the overhead, and ask the students to simplify each expression.

1. \(-2x + 4y + (3x - 4y)\)  \(x\)  
2. \(6x - 2y + (2x + 2y)\)  \(8x\)  
3. \(3x - 3y + (-3x + 2y)\)  \(-y\)  
4. \(4x - 2y + (5x + 2y)\)  \(9x\)  
5. \(-6x + 15y + (6x - 4y)\)  \(11y\)  
6. \(3x + 2y + (-10x - 2y)\)  \(-7x\)

II. Overhead (15 minutes)
When you add the corresponding sides of the equations of a system, you may get an equation that has two variables rather than only one. This occurs because the coefficients of the variables are not opposite. So, the multiplication/addition method is used to solve such a system of equations.

Example 1

1. \(X - 5Y = 1\)

Solve:
2. \(2X - 4Y = 14\)

Solution:
Multiply each side of equation 1 by -2
\(-2(X - 5Y) = 1(-2)\)
\(-2X + 10Y = -2\)

Add equation 2.
\(2X - 4Y = 14\)
\(6Y = 12\)
\(Y = 2\)

Substitute 2 for \(Y\) in either equation 1 or 2. Solve for \(X\).
\(X - 5Y = 1\)
\(X - 5 \times 2 = 1\)
\(X - 10 = 1\)
\(X - 10 + 10 = 1 + 10\)
\(X = 11\)

Check: Substitute \(X = 11\) and \(Y = 2\) in both equations 1 and 2.
\(X - 5Y = 1\)
\(11 - 5 \times 2\)
\(11 - 10\)
\(1\)
\(2X - 4Y = 14\)
\(2 \times 11 - 4 \times 2\)
\(22 - 8\)
\(14\)
Ask the students what number to multiply equation 1 by that will form coefficients of \( x \) which are opposites.

\[
\begin{align*}
1 \quad &x - 2y = 4 \\
a. \quad &2 \quad x + 3y = 14 \\
1 \quad &-x + 2y = 5 \\
b. \quad &2 \quad 3x + 5y = 29 \\
x + 5y = 31 \\
c. \quad &2 \quad 5x - 4y = 19
\end{align*}
\]

Example 2

Solve:

\[
\begin{align*}
1. \quad &3x - 2y = -1 \\
2. \quad &-2x + 3y = 4
\end{align*}
\]

Solution:

Multiply each side of 1 by 3

\[3(3x - 2y) = -1(3)\]

Multiply each side of 2 by 2

\[2(-2x + 3y) = 4(2)\]

Add equations

\[
\begin{align*}
9x - 6y &= -3 \\
-4x + 6y &= 8 \\
\hline
5x &= 5 \\
x &= 1
\end{align*}
\]

Substitute 1 for \( x \) in either equation 1 or 2. Solve for \( y \)

\[
\begin{align*}
3x - 2y &= -1 \\
3x1 - 2y &= -1 \\
3 - 2y &= -1 \\
-2y &= -4 \\
y &= 2
\end{align*}
\]

Check: Substitute \( x = 1 \) and \( y = 2 \) in both equations 1 and 2.

\[
\begin{align*}
3x - 2y &= -1 \\
3x1 - 2x2 &= -2 + 6 \\
3 - 4 &= 4
\end{align*}
\]

Steps for the Multiplication/Addition Method

1. Check for opposite coefficients of one variable. If necessary, multiply each side of either or both equations by numbers that will make opposite coefficients for one variable.

2. Add to eliminate one of the variables. Solve the resulting equation.

3. Substitute the known value of one variable in one of the original equations of the system. Solve for the other variable.

4. Check the solution in both equations of the system.
III. Classroom Exercises (8 minutes)

Ask the students what number they would multiply each side of each equation in order to eliminate x by addition.

1. \( x + y = -2 \)
   \(-2x + 3y = -1\)
   **Mult. each side of first equation by 2.**

2. \( 3x - y = -5 \)
   \( x + 2y = 1 \)
   **Mult. each side of second equation by -3.**

3. \(-3x + 2y = 5 \)
   \( x - y = -3 \)
   **Mult. each side of second equation by 3.**

4. \( 2x - y = -2 \)
   \(-3x + y = 1 \)
   **Mult each side of first equation by 3 and second equation by 2.**

5. \( 2x - y = -3 \)
   \( 5x + 2y = 1 \)
   **Mult each side of first equation by 5 and second equation by -2.**

6. \( 4x - y = 2 \)
   \(-3x + 2y = 1 \)
   **Mult each side of first equation by 3 and second equation by 4.**

IV. Written Exercises **Worksheet IX** (15 minutes)

Have the students work as many of these problems as they can in the 15 minutes remaining in class.

**Hand out Worksheet IX**

Allow the students whatever time remains in the class period. Tell the students that they do not have to have all the problems finished but should finish as many as they can.

**Collect worksheets.**
Day 10: Substitution Method

I. Warm-up Exercises (2 minutes)
Write the following expressions on the overhead, and ask the students what they
would do to each side of the equation to get x alone on one side of the equation.

1. $x + 5 = 12$ \textbf{Subtract} 5
2. $x - 8 = 7$ \textbf{Add} 8
3. $x + y = 3$ \textbf{Subtract} $y$
4. $x - 2y = 7$ \textbf{Add} 2$y$
5. $2x + 3y = 4$ \textbf{Subtract} 3$y$, \textbf{divide by} 2
6. $3x - 2y = 1$ \textbf{Add} 2$y$, \textbf{divide by} 3

II. Overhead (15 minutes)
Another method for solving linear equations is the substitution method.

\textbf{Example 1}

\textbf{Solve:}

1. $x - 5y = 1$
2. $x - 4y = 13$

\textbf{Solution: Solve one equation for one of the variables.}

1. $x - 5y = 1$ \hspace{1cm} \text{Choose to solve for $x$ because its}
\hspace{1cm} \text{coefficient is 1.}
\hspace{1cm} x = 1 + 5y

Substitute $(1 + 5y)$ for $x$ in equation 2. Solve for $y$.

2. \hspace{1cm} $x - 4y = 13$
\hspace{1cm} $(1 + 5y) - 4y = 13$
\hspace{1cm} $1 + 5y - 4y = 14$
\hspace{1cm} $1 + y = 13$
\hspace{1cm} $y = 12$

Substitute 12 in for $y$ in one of the original equations. Solve for $x$.

1. \hspace{1cm} $x - 5y = 1$
\hspace{1cm} $x - 5(12) = 1$
\hspace{1cm} $x - 60 = 1$
\hspace{1cm} $x = 61$

\textbf{Check: Substitute 61 for $x$ and 12 for $y$ in both equations.}

1. $x - 5y = 1$
\hspace{1cm} $61 - 5(12)$
\hspace{1cm} $61 - 60$
\hspace{1cm} $1$

2. $x - 4y = 13$
\hspace{1cm} $61 - 4(12)$
\hspace{1cm} $61 - 48$
\hspace{1cm} $13$
In Example 1, the number 2 in \(2x + y = 1\) is the coefficient of \(x\). The coefficient of \(y\) is 1 since \(2x + y = 1\) can be written as \(2x + 1y = 1\). In the substitution method, try to solve first for a variable that has 1 or -1 as its coefficient.

**Example 2**

Solve:

1. \(2x - 3y = -4\)
2. \(-x + 2y = 3\)

Solution: Solve one equation for one of the variables.

2. \[-x + 2y = 3\]
   \[-x + 2y + x = 3 + x\]
   \[2y = 3 + x\]
   \[2y - 3 = 3 + x - 3\]
   \[2y - 3 = x\]

Substitute \(2y - 3\) for \(x\) in equation 1. Solve for \(y\).

1. \[2x - 3y = -4\]
   \[2(2y - 3) - 3y = -4\]
   \[4y - 6 - 3y = -4\]
   \[y - 6 = -4\]
   \[y - 6 + 6 = -4 + 6\]
   \[y = 2\]

Substitute 2 for \(y\) in either equation. Solve for \(x\).

2. \[2x - 3y = -4\]
   \[2x - 3(2) = -4\]
   \[2x - 6 = -4\]
   \[2x - 6 + 6 = -4 + 6\]
   \[2x = 2\]
   \[x = 1\]

**Check:** Substitute 1 for \(x\) and 2 for \(y\) in both equations.

1. \[2x - 3y = -4\]
   \[2(1) - 3(2)\]
   \[2 - 6\]
   \[-4\]
2. \[-x + 2y = 3\]
   \[-(1) + 2(2)\]
   \[-1 + 4\]
   \[3\]
Steps for the Substitution Method

1. Solve one equation for one of the variables.

2. Substitute the resulting expression in the other equation. Solve the equation.

3. Substitute the value of the variable from Step 2 in either equation. Solve the resulting equation.

4. Check by substituting both values in both equations.

III. Classroom Exercises (8 minutes)
Write the following equations on the overhead and ask the students to solve for one of the variables.

1. \( x + y = 10 \)  
2. \( x - 3y = -4 \)  
3. \( -3x - y = -1 \)  
4. \( 2y - x = 7 \)  
5. \( x + y = -9 \)  
6. \( 4x + y = 5 \)

What would be the next step in solving each of the following equations?

7. \( 3x - 5(x + 2) = 3 \)  
8. \( -2(y - 3) + 3y = 1 \)  
9. \( 3(2y - 2) - 4y = -5 \)  
10. \( -x + 2(-2x - 3) = -4 \)  
11. \( -3x - (4x - 2) = 6 \)  
12. \( 2(y - 1) + y = 5 \)

IV. Written Exercises Worksheet X (15 minutes)

Have the students work as many of these problems as they can in the 15 minutes remaining in class.

Hand out Worksheet X

Allow the students whatever time remains in the class period. Tell the students that they do not have to have all the problems finished but should finish as many as they can.

Collect worksheets.
LAB WORKSHEETS
Lab Worksheet I, II, and V

Instructions: Record your answers to Lab *. For all the problems in Lab *, describe the problem using pictures and then demonstrate how you solved the problem. For the first problem in the lab that you work, give a written description of how you solved the problem. Record your answer to every problem in the space provided. Be sure to label each problem.

LAB *. The number of counters in the container is ___.

LAB *. The number of counters in the container is ___.

LAB *. The number of counters in the container is ___.

LAB *. The number of counters in the container is ___.
Lab Worksheet III, IV and VI

Instructions: Record your answers to Lab *. For all the problems in Lab *, describe the problem using algebraic notation and then demonstrate how you solved the problem using the same notation. For the first problem in the lab that you work, give a written description of how you solved the problem. Record your answer to every problem in the space provided. Be sure to label each problem.

LAB * ____. The number of counters in the container is ____.

LAB * ____. The number of counters in the container is ____.

LAB * ____. The number of counters in the container is ____.

LAB * ____. The number of counters in the container is ____.
**Student Worksheet 1**

Instructions: The piles on either side of the equal sign (=) contain an equal number of counters. Each box contains the same number of counters. Find the number of counters in each box.

Example 1:

![Image of counters]

The number of counters in the box is **2**

Example 2:

![Image of counters]

The number of counters in the box is **3**

Problem 1:

![Image of counters]

1. The number of counters in the box is ______

Problem 2:

![Image of counters]

2. The number of counters in the box is ______

Problem 3:

![Image of counters]

3. The number of counters in the box is ______
Problem 4: \[ \square \square \square \square \square = \boxed{\square \square \square \square \square \square \square \square \square \square \square \square} \]

4. The number of counters in the box is ______

Problem 5: \[ \square \square = \boxed{\square \square \square \square \square \square \square \square \square \square \square \square} \]

5. The number of counters in the box is ______

Problem 6: \[ \square \square \square \square \square = \boxed{\square \square \square \square \square \square \square \square \square \square \square \square} \]

6. The number of counters in the box is ______

Problem 7: \[ \square \square \square = \boxed{\square \square \square \square \square \square \square \square \square \square \square \square} \]

7. The number of counters in the box is ______

Problem 8: \[ \square \square \square \square \square = \boxed{\square \square \square \square \square \square \square \square \square \square \square \square} \]

8. The number of counters in the box is ______

Problem 9: \[ \square \square \square \square \square = \boxed{\square \square \square \square \square \square \square \square \square \square \square \square} \]

9. The number of counters in the box is ______

Problem 10: \[ \square \square \square \square \square = \boxed{\square \square \square \square \square \square \square \square \square \square \square \square} \]

10. The number of counters in the box is ______
Student Worksheet 2

Instructions: The number of counters in each box is given by "X". Write an algebraic expression for the total number of counters in the pile.

Example:

\[ 2X + 3 \]

An expression for the number is \[ 2X + 3 \] or \[ 3 + 2X \]

When \( x \) represents 12, the expression represents 27.

\[ (2\times12 + 3 = 27) \]

Problem 1:

\[ \square \square \bigcirc \bigcirc \bigcirc \]

1. An expression for the number is ______________

Problem 2:

\[ \square \bigcirc \bigcirc \bigcirc \]

2. An expression for the number is ______________

Problem 3:

\[ \square \square \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \]

3. An expression for the number is ______________

Problem 4:

\[ \square \bigcirc \bigcirc \square \square \square \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \]

4. An expression for the number is ______________

Problem 5:

\[ \square \square \bigcirc \bigcirc \square \square \bigcirc \bigcirc \bigcirc \bigcirc \]

5. An expression for the number is ______________

When \( X \) represents 3, the expression represents ___
Student Worksheet 3

Instructions: The two piles have equal total numbers of counters. Write an algebraic equation expressing that fact.

Example:

\[ \square \square \sigma = \sigma \sigma \sigma \]

The equation is \( 2X + 1 = 3 \)

Problem 1:

\[ \square = \sigma \sigma \sigma \]

1. The equation is ____________________

Problem 2:

\[ \square \square = \square \sigma \sigma \sigma \]

2. The equation is ____________________

Problem 3:

\[ \square \sigma \sigma \sigma = \square \square \]

3. The equation is ____________________

Problem 4:

\[ \square \square \square = \square \sigma \sigma \sigma \sigma \]

4. The equation is ____________________

Problem 5:

\[ \square \square \sigma \sigma = \square \sigma \sigma \sigma \sigma \]

5. The equation is ____________________

Problem 6:

\[ \square \square \sigma = \square \sigma \sigma \sigma \]

6. The equation is ____________________
Instructions: Simplify the algebraic expressions.

Examples:
The problem: \( X + 5 - 5 \)  
The simplified expression is \( X \)
The problem: \( 3X + 14 - 2X -8 \)  
The simplified expression is \( X + 6 \)

Problem 1: \( X + 7 - 7 \)  
The simplified expression is ______

Problem 2: \( X + 10 - 3 \)  
The simplified expression is ______

Problem 3: \( 2X - X \)  
The simplified expression is ______

Problem 4: \( 9X - 4X \)  
The simplified expression is ______

Problem 5: \( 3X - X \)  
The simplified expression is ______

Problem 6: \( 2X + 7 - 4 \)  
The simplified expression is ______

Problem 7: \( X + 7 - X \)  
The simplified expression is ______

Problem 8: \( X + 7 - X + 4 \)  
The simplified expression is ______
Student Worksheet 5

Instructions: Simplify the algebraic expressions.

Example:
The problem: \( \frac{6X}{2} \)  
The simplified expression is \( 3X \)

Problem 1: \( \frac{2X}{2} \)  
The simplified expression is 

Problem 2: \( \frac{9X}{3} \)  
The simplified expression is 

Problem 3: \( 2X + \frac{8}{2} \)  
The simplified expression is 

Problem 4: \( \frac{2X}{2} + \frac{8}{2} \)  
The simplified expression is 

Problem 5: \( \frac{2(X + 4)}{2} \)  
The simplified expression is 

Problem 6: \( \frac{12X}{2} + 16 - 6 \)  
The simplified expression is 

Problem 7: \( 2X + 5 - \frac{4X}{4} \)  
The simplified expression is 

Problem 8: \( \frac{5X + 16}{4} - 3X - 2 \)  
The simplified expression is
Student Worksheet 6

Instructions: Solve the following problems. For all the problems, demonstrate how you solved the problem. You may use the film containers and paper clips to help you solve the problems. For the first problem that you work, give a written description of how you solved the problem. Record your answer to each problem in the space provided.

Problem 1: \[3X + 5 = 17\]  
\[X = \phantom{0}\]

Problem 2: \[X + 3 = 2X\]  
\[X = \phantom{0}\]

Problem 3: \[9X + 30 = 48\]  
\[X = \phantom{0}\]

Problem 4: \[15 + 3X = 5 + 8X\]  
\[X = \phantom{0}\]

Problem 5: \[3 + 6X + 30 = 105\]  
\[X = \phantom{0}\]
Student Worksheet 7

Instructions: For each problem, you are given the answer. You are also told whether to use addition or multiplication. Use this information to create your own problem.

Example: \( X = 3 \) (addition)  
My problem is \( X + 7 = 10 \)

Problem 1: \( X = 4 \) (addition)  
My problem is ________________

Problem 2: \( X = 2 \) (multiplication)  
My problem is ________________

Problem 3: \( X = 14 \) (addition twice)  
My problem is ________________

Problem 4: \( X = 23 \) (addition)  
My problem is ________________

Problem 5: \( X = 18 \) (multiplication)  
My problem is ________________

Problem 6: \( X = 3 \) (multiplication twice)  
My problem is ________________

Problem 7: \( X = 1 \) (addition and multiplication)  
My problem is ________________

Problem 8: \( X = 0 \) (addition and multiplication)  
My problem is ________________
Student Worksheet 8

Instructions: The same number of counters are found on each side of the equal sign (=). Every container marked X contains the same number of counters and every container marked Y contains the same number of counters. Use counters and containers to solve the following problems.

Example 1:

\[
\begin{array}{c}
X \\
Y
\end{array}
\]

Possible solutions include:

\[
\begin{array}{c}
X = 0, Y = 1 \\
X = 1, Y = 0
\end{array}
\]

Example 2:

\[
\begin{array}{c}
X \\
Y
\end{array}
\]

Possible solutions include:

\[
\begin{array}{c}
X = 0, Y = 3 \\
X = 1, Y = 2 \\
X = 2, Y = 1 \\
X = 3, Y = 0
\end{array}
\]

Problem 1:

\[
\begin{array}{c}
X X \\
Y Y \\
X
\end{array}
\]

Possible solutions include:

Problem 2:

\[
\begin{array}{c}
X X \\
Y Y Y \\
Y
\end{array}
\]

Possible solutions include:

Problem 3:

\[
\begin{array}{c}
X X X \\
Y Y Y \\
X
\end{array}
\]

Possible solutions include:

Problem 4:

\[
\begin{array}{c}
X X X \\
Y Y \\
X
\end{array}
\]

Possible solutions include:

Problem 5:

\[
\begin{array}{c}
\ldots \\
X X X X \\
Y Y
\end{array}
\]

Possible solutions include:
Student Worksheet 9

Instructions: The same number of counters are found on each side of the equal sign (=). Every container marked X contains the same number of counters and every container marked Y contains the same number of counters. Use counters and containers to solve the following problems.

Example:

Each X container contains 2
Each Y container contains 3

Problem 1.

1. Each X container contains ______
   Each Y container contains ______

Problem 2.

2. Each X container contains ______
   Each Y container contains ______

Problem 3.

3. Each X container contains ______
   Each Y container contains ______

Problem 4.

4. Each X container contains ______
   Each Y container contains ______
Instructions: Write an algebraic equation that describes the following pictures.

Example: The equations are \( X + 2 = Y + 1 \) and \( 2X + 2 = Y + 3 \)

Problem 1.

1. The equations are ___________

Problem 2.

2. The equations are ___________

Problem 3.

3. The equations are ___________

Problem 4.

4. The equations are ___________
Student Worksheet 11

Instructions: Solve the following problems. For all the problems, demonstrate how you solved the problem. You may use the film containers and paper clips to help you solve the problems. For the first problem that you work, give a written description of how you solved the problem. Record your answers to each problem in the space provided.

Problem 1. \( X = Y \)
\[ 2X + Y = 9 \]

\( \begin{align*}
X &= \\
Y &=
\end{align*} \]

Problem 2. \( 2X + 3Y = 22 \)
\[ 5X = 25 \]

\( \begin{align*}
X &= \\
Y &=
\end{align*} \]

Problem 3. \( Y + 1 = X + 4 \)
\[ Y + 1 = 3X \]

\( \begin{align*}
X &= \\
Y &=
\end{align*} \]

Problem 4. \( Y + X + 2 = 5X \)
\[ 2x + 2 = Y \]

\( \begin{align*}
X &= \\
Y &=
\end{align*} \]

Problem 5. \( Y + 2 = X \)
\[ 10 + 4X = 7Y \]

\( \begin{align*}
X &= \\
Y &=
\end{align*} \)
COMPUTER WORKSHEETS
Computer Worksheets I and IV

Instructions: Describe, using pictures, the problem given on the computer. Then solve the problem. You may use the film containers and paper clips to help you solve the problem. Remember, you do not have to do every problem that appears on the computer screen. Do as many as you can in the time provided. Be sure you describe the problem before you start to solve it.

Problem 1. The number of counters in the container is ______

Problem 2. The number of counters in the container is ______

Problem 3. The number of counters in the container is ______

Problem 4. The number of counters in the container is ______
Computer Worksheets II and V

Instructions: Describe, using algebraic notation, the problem given on the computer. Do not solve the problem. Remember, you do not have to do every problem that appears on the computer screen. Do as many as you can in the time provided. Be sure you describe the problem using algebraic notation.

Problem 1. The number of counters in the container is ______

Problem 2. The number of counters in the container is ______

Problem 3. The number of counters in the container is ______

Problem 4. The number of counters in the container is ______
Instructions: Describe, using algebraic notation, the problem given on the computer. Then solve the problem. You may use the film containers and paper clips to help you solve the problem. Remember, you do not have to do every problem that appears on the computer screen. Do as many as you can in the time provided. Be sure you describe the problem before you start to solve it.

Problem 1. The number of counters in the container is _______

Problem 2. The number of counters in the container is _______

Problem 3. The number of counters in the container is _______

Problem 4. The number of counters in the container is _______
APPENDIX IV

COMPUTER MODELING PROGRAMS
Program 1

One equation with one unknown

to startup
begin
end

to begin
ct cg cc
show [To begin type START and strike the return key.]
end

to vars
recycle
make "r 1
make "x (random 7 + 1)
make "a1 (random 7 + 1)
make "a2 (random 7)
if :a1 - :a2 <= 0 [vars]
make "c1 random 4
make "c2 ((:a1 - :a2)*:x + :c1)
if :c2 > 36 [vars]
end

to sqr
ht setc 1
pd repeat 4 [fd 15 rt 90]
seth 0 rt 45 pu fd 2 pd fd 19 pu bk 21 rt 45 fd 15 lt 135 pu fd 2 pd fd 19 pu bk 21 lt 45 fd 15 rt 90
end

to bar
setc 1
pu setpos [-245 10] pd seth 90
repeat 2 [fd 490 rt 90 fd 10 rt 90]
setpos [-10 10] seth 0 pd repeat 2 [fd 30 rt 90 fd 10 rt 90] pu fd 10 rt 90 fd 5 pd fill pu
setpos [-10 75] seth 0 pd repeat 2 [fd 35 rt 90 fd 10 rt 90] pu fd 10 rt 90 fd 5 pd fill pu
setpos [-20 55] seth 90 pd repeat 2 [fd 30 rt 90 fd 5 rt 90] lt 90 pu bk 3 rt 90 fd 10 pd
pu bk 10 rt 90 fd 3
pu setpos [-20 65] seth 90 pd repeat 2 [fd 30 rt 90 fd 5 rt 90] lt 90 pu bk 3 rt 90 fd 10 pd
pu bk 10 rt 90 fd 3
end

to d1
ht pu setpos [-240 70] pd
repeat (:a1) [sqr pu rt 90 fd 20 lt 90 pd]
pu setpos [-240 50] pd
end
to d2
ht pu setpos [20 70] pd
repeat (:a2) [sqr pu rt 90 fd 20 lt 90 pd]
pu setpos [20 50] pd
end
to small
end
to counterl
recycle
make "r 1 setc 1
pu seth 0 setpos [-240 30] pd
make "k 1
repeat :c1 [small rt 90 fd 10 lt 90 pd make "k :k+1 if :k = 4 [pu rt 90 fd 4 lt 90 pd] if :k =4 [make "k 1]]
pu setpos [15 30] pd
make "r 1 setc 1 make "k 1 make "m 1
repeat :c2 [small rt 90 fd 10 lt 90 pd make "k :k+1 if :k = 4 [pu rt 90 fd 4 lt 90 pd] if :k =4 [make "k 1] make "m :m + 1 if :m > 18 [pu setpos [15 20] make "m 1]]] pd
pu setpos [15 20] pd setsh 0 pu fd 20 rt 90 fd 5 ht
end
to start
cr cc pu type [Please wait, I am thinking of a challenging problem.]
vars ht cr cc pu
tell 1 ht pu home pd bar tell 0 d1 d2
setc (1 +random 3) counterl
ht
show [Strike any key to see a correct answer]
if readchar = " [cc]
answer
type [.) Strike any key to run the program again or hold down the"apple key" while striking the "period key" to stop the program.]
if readchar = " [cc] start
end
to answer
cc
(type [(x is) :x
end
Program 1 Examples

[Diagrams of various patterns and structures are shown on the page.]
Program 2

One equation with two unknowns

to startup
begin
end

to begin
ct cg cc
show [To begin type START and strike the return key.]
end

to vars
recycle
make "r 1
make "x (random 2 + 1)
make "y (random 2)
make "a1 (random 4 + 1)
make "a2 (random 2)
if :a1 - :a2 <= 0 [vars]
make "a :a1 - :a2

make "b1 (random 4 + 1)
make "b2 (random 2)
if :b1 - :b2 <= 0 [vars]
make "b :b1 - :b2
make "d1 random 2
make "c :a*:x + :b*:y
make "c2 random 10
make "c1 :c2 - :c
if :c > :c2 [vars]
end

to sqr
ht setc 1
pd repeat 4 [fd 15 rt 90]
seth 0 rt 45 pu fd 2 pd fd 19 pu bk 21 rt 45 fd 15 lt 135 pu fd 2 pd fd 19 pu bk 21 lt 45 fd 15 rt 90
end

to tri
pd setc 1 rt 30
repeat 3 [ fd 20 rt 120] lt 30
pu rt 90 fd 10 lt 90 pd fd 5 rt 45 fd 4 bk 4 lt 90 fd 4 pu bk 4 rt 45 bk 5 lt 90 fd 10 seth 0 lt 30
end

to bar
(Strike a key to see the correct answers where x and y are whole numbers.)

if readchar = " [cc]
cc
make "x 0
more.answers

type [Strike any key to run the program again or hold down the "apple key" while striking the "period key" to stop the program.]
if readchar = " [cc] start
end

to more.answers
make "y (:c -a*:x)/:b
if :y < 0 [negative]
make "q int :y
if :q = :y [(type [(x is] :x [, y is] :y [.))]]
make "x :x + 1
if :x < 5 [more.answers]
end

to negative
make "y .5
end

to answer
cc
(type [(x is] :x [, y is] :y [])
end
Program 2 Examples
Program 3

Two equations with two unknowns

to startup
begin
end

to begin
ct cg cc
show [To begin type START and strike the return key.]
end

to vars
recycle
make "r 1
make "x (random 12)
make "y (random 12)
make "a (1 + random 4)
make "a2 random 5
make "a1 :a + :a2
make "b (1 + random 4)
make "b2 random 5
make "b1 :b + :b2
make "c :a* :x + :b* :y
make "c1 random 4
make "c2 :c + :c1
if :c2> 36 [vars]
make "d (1 + random 4)
make "d2 random 5
make "e (1 + random 4)
make "e2 random 5
make "d1 :d + :d2
make "e1 :e + :e2
make "f :d* :x + :e* :y
make "f1 random 5
make "f2 :f + :f1
if :f2> 42 [vars]
if :a* :e = :b* :d [vars]
end
to sq
ht setc 1
pd repeat 4 [fd 15 rt 90]
seth 0 rt 45 pu fd 2 pd fd 19 pu bk 21 rt 45 fd 15 lt 135 pu fd 2 pd fd 19 pu bk 21 lt 45 fd 15 rt 90
end
to tri
pd setc 1 rt 30
repeat 3 [fd 20 rt 120] lt 30
pu rt 90 fd 10 lt 90 pd fd 5 rt 45 fd 4 bk 4 lt 90 fd 4 pu bk 4 rt 45 bk 5 lt 90 fd 10 seth 0 lt 30
end
to bar
setc 1
pu setpos [-245 10] pd seth 90
repeat 2 [fd 490 rt 90 fd 10 rt 90]
setpos [-10 10] seth 0 pd repeat 2 [fd 30 rt 90 fd 10 rt 90] pu fd 10 rt 90 fd 5 pd fill pu
setpos [-10 75] seth 0 pd repeat 2 [fd 35 rt 90 fd 10 rt 90] pu fd 10 rt 90 fd 5 pd fill pu
setpos [-20 55] seth 90 pd repeat 2 [fd 30 rt 90 fd 5 rt 90] lt 90 pu bk 3 rt 90 fd 10 pd
pu bk 10 rt 90 fd 3 pu
pu setpos [-20 65] seth 90 pd repeat 2 [fd 30 rt 90 fd 5 rt 90] lt 90 pu bk 3 rt 90 fd 10 pd
pu bk 10 rt 90 fd 3
setpos [-10 -110] seth 0 pd repeat 2 [fd 40 rt 90 fd 10 rt 90] pu fd 10 rt 90 fd 5 pd fill pu
setpos [-10 -35] seth 0 pd repeat 2 [fd 35 rt 90 fd 10 rt 90] pu fd 10 rt 90 fd 5 pd fill pu
setpos [-20 -45] seth 90 pd repeat 2 [fd 30 rt 90 fd 5 rt 90] lt 90 pu bk 3 rt 90 fd 10 pd
pu bk 10 rt 90 fd 3
pu setpos [-20 -55] seth 90 pd repeat 2 [fd 30 rt 90 fd 5 rt 90] lt 90 pu bk 3 rt 90 fd 10 pd
pu bk 10 rt 90 fd 3
end
to d1
ht pu setpos [-240 70] pd
repeat (:a1) [sqr pu rt 90 fd 20 lt 90 pd]
pu setpos [-240 50] pd
repeat (:b1) [tri seth 0 pu rt 90 fd 20 lt 90 pd]
end
to d2
ht pu setpos [20 70] pd
repeat (:a2) [sqr pu rt 90 fd 20 lt 90 pd]
pu setpos [20 50] pd
repeat (:b2) [tri seth 0 pu rt 90 fd 20 lt 90 pd]
end
to small
end
to d1b
ht pu setpos [-240 -30] pd
repeat (:d1) [sqr pu rt 90 fd 20 lt 90 pd]
pu setpos [-240 -50] pd
repeat (:e1) [tri seth 0 pu rt 90 fd 20 lt 90 pd]
end
to d2b
ht pu setpos [20 -30] pd
repeat (:d2) [sqr pu rt 90 fd 20 lt 90 pd]
pu setpos [20 -50] pd
repeat (:e2) [tri.seth 0 pu rt 90 fd 20 lt 90 pd]
end
to counter1
recycle
make "r1 setc 1
pu seth 0 setpos [-240 30] pd
make "k 1
repeat :c1 [small rt 90 fd 10 lt 90 pd make "k :k+1 if :k = 4 [pu rt 90 fd 4 lt 90 pd] if :k =4 [make "k 1]]
pu setpos [15 30] pd
make "r1 setc 1 make "k 1 make "m 1
repeat :c2 [small rt 90 fd 10 lt 90 pd make "k :k+1 if :k = 4 [pu rt 90 fd 4 lt 90 pd] if :k =4 [make "k 1] make "m :m + 1 if :m > 15 [pu setpos [15 20] make "m 1]] pd
pu setpos [15 20] pd setsh 0 pu fd 20 rt 90 fd 5 ht
end.
to counter1b
recycle
make "r2 setc 1
pu seth 0 setpos [-240 -60] pd make "k 1
repeat :f1 [small rt 90 fd 10 lt 90 pd make "k :k+1 if :k = 4 [pu rt 90 fd 4 lt 90 pd] if :k =4 [make "k 1]]
pu setpos [-240 -70] seth 0 pd
make "r2 setc 1 make "k 1 make "m 1
repeat :f2 [small rt 90 fd 10 lt 90 pd make "k :k+1 if :k = 4 [pu rt 90 fd 4 lt 90 pd] if :k =4 [make "k 1] make "m :m + 1 if :m > 15 [pu setpos [20 -70] make "m 1]] pd
seth 0 pu fd 20 rt 90 fd 5 ht
end
to start
cg cc pu type [Please wait, I am thinking of a challenging problem.]
vars ht cg cc pu

tell 1 ht pu home pd bar tell 0 d1 d2 d1b d2b
setc (1 +random 3) counter1 counter1b
ht
show [Strike any key to see a correct answer]
if readchar = " [cc]
answer
type [. Strike any key to run the program again or hold down the "apple key" while striking the "period key" to stop the program.]
if readchar = " [cc] start
end
to answer
cg
(type [(x is) x [ y is ] y]). Be careful, additional answers may also be true. ]
end
Program 3 Examples
APPENDIX V
ADDITIONAL GRAPHS
Graph 4
Plot of Pretest and Posttest OHPT Scores
Treatment = I (Manipulatives with Computer)
Ethnicity = N (American Indian)

Graph 5
Plot of Pretest and Posttest OHPT Scores
Treatment = I (Manipulatives with Computer)
Ethnicity = W (Non-American Indian)

Graph 6
Plot of Pretest and Posttest OHPT Scores
Treatment = II (Manipulatives without Computer)
Ethnicity = N (American Indian)

Graph 7
Plot of Pretest and Posttest OHPT Scores
Treatment = II (Manipulatives without Computer)
Ethnicity = W (Non-American Indian)
Graph 8
Plot of Pretest and Posttest OHPT Scores
Treatment = III (Textbook Only)
Ethnicity = N (American Indian)

Graph 9
Plot of Pretest and Posttest OHPT Scores
Treatment = III (Textbook Only)
Ethnicity = W (Non-American Indian)

Graph 10
Plot of Pretest and Posttest OHPT Scores
Treatment = I (Manipulatives with Computer)
Grade Level = 6th grade

Graph 11
Plot of Pretest and Posttest OHPT Scores
Treatment = I (Manipulatives with Computer)
Grade Level = 7th grade
Graph 12
Plot of Pretest and Posttest OHPT Scores
Treatment = II (Manipulatives without Computer)
Grade Level = 6th grade

Graph 13
Plot of Pretest and Posttest OHPT Scores
Treatment = II (Manipulatives without Computer)
Grade Level = 7th grade

Graph 14
Plot of Pretest and Posttest OHPT Scores
Treatment = III (Textbook Only)
Grade Level = 6th grade

Graph 15
Plot of Pretest and Posttest OHPT Scores
Treatment = III (Textbook Only)
Grade Level = 7th grade
Graph 16
Plot of Pretest and Posttest OHPT Scores
Treatment = I (Manipulatives with Computer)
Gender = F (Female)

Graph 17
Plot of Pretest and Posttest OHPT Scores
Treatment = I (Manipulatives with Computer)
Gender = M (Male)

Graph 18
Plot of Pretest and Posttest OHPT Scores
Treatment = II (Manipulatives without Computer)
Gender = F (Female)

Graph 19
Plot of Pretest and Posttest OHPT Scores
Treatment = II (Manipulatives without Computer)
Gender = M (Male)
Graph 20
Plot of Pretest and Posttest OHPT Scores
Treatment = III (Textbook Only)
Gender = F (Female)

Graph 21
Plot of Pretest and Posttest OHPT Scores
Treatment = III (Textbook Only)
Gender = M (Male)