Boson stars efficiently nucleate vacuum phase transitions
by Thomas John Brueckner

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics
Montana State University

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Doctor of Philosophy

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MONTANA STATE UNIVERSITY-BOZEMAN
Bozeman, Montana

April 1997

APPROVAL

of a thesis submitted by

Thomas John Brueckner

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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CHAPTER 1

WHY SHOULD ONE STUDY BOSON STARS AS SEEDS FOR VACUUM PHASE TRANSITIONS?

In the beginning, the universe was hot and dense, and a variety of unusual processes and objects existed. One of the more unusual processes is that of a first order vacuum phase transition in a quantum field. Of particular interest is the theory of a scalar field that undergoes a vacuum phase transition, since the scalar Higgs field is a crucial part of larger theories, like electroweak theory, that unify some of the fundamental forces of nature.

In the early universe's menagerie of exotic objects are boson stars, a self-gravitating configuration of a quantum field. Each boson in the star is in the same quantum state. These stars, prevented from collapsing by the Heisenberg uncertainty principle, can range in mass from a few thousand kilograms up to astrophysical size, depending upon the boson's mass. Some scientists view bosonic matter as a possible part of the dark matter content of the universe.

It is logical to ask whether boson stars affect first order phase transitions in the scalar field. The most basic model of a first order vacuum phase transition is one in which a region of empty spacetime spontaneously changes phase,[1] much like drops of rain form spontaneously in a pure water vapor. This region of the new phase is called a vacuum bubble. The scalar field, which I shall call φ, tunnels from its initial state at a local minimum of the potential, through the barrier in the potential $V(\varphi)$, to the true vacuum at the global minimum. Figure 1 shows the tunnelling process for spontaneous bubble formation in empty space.

A second process, induced nucleation, can also generate first order phase transitions. Induced nucleation is like the mundane process of using silver iodide crystals to seed clouds and form precipitation over arid regions of land. Using a boson star to seed a phase transition has a notable advantage over the spontaneous formation process. The boson star is a configuration of the scalar field, $\varphi(r)$, that starts out with a positive central value, $\varphi(0) > 0$. The field in the ground state decreases gradually in size as it extends out from the center of the star, asymptotically approaching zero as radial distance r approaches infinity. In terms of the potential in figure 1, the central value of the star is high up on the "bump" in the potential. According to quantum theory, the barrier is more easily penetrable when the initial value of φ is high up on the potential barrier. It is therefore reasonable to conjecture that, in comparison to spontaneous bubble formation in empty space, a

[1]S. Coleman, "Fate of the false vacuum: Semiclassical theory," Physical Review D, 15, 2929 (1977).

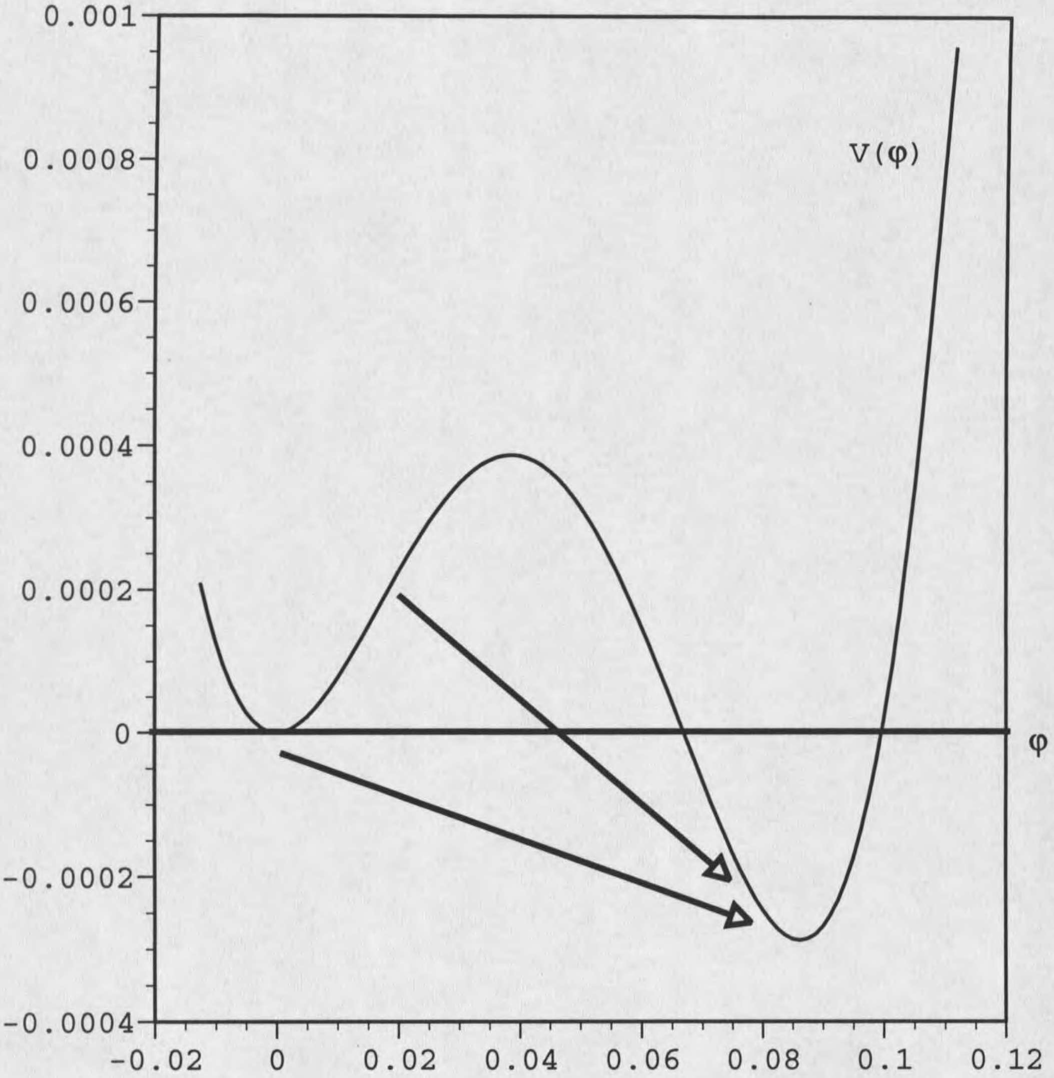

Figure 1. Tunnelling events for first order vacuum phase transitions. The lower arrow shows the tunnelling event corresponding to spontaneous formation of vacuum bubbles in empty space. The upper arrow shows the tunnelling event for a first order phase transition that a boson star has nucleated. The potential barrier is more easily penetrated in the latter case.

boson star might more readily initiate a tunnelling event in the vacuum field. The verification of that conjecture is the subject of this thesis. I compared the two processes, and I found that a boson star has a significantly greater efficiency at nucleating first order phase transitions in the case of a small star nucleating a large bubble of the new phase.

I shall provide greater detail on what makes a boson star small in relation to a large bubble, how to construct a model boson star, and other aspects of this nucleation process. Specifically, in chapter 2, I will discuss the time evolution of the temperature of the universe in the Robertson-Walker cosmological model. Also, I will discuss the temperature dependence of effective potentials that allow vacuum phase transitions. In chapter 3, I review some of the foundational concepts of boson stars, and I discuss the calculation of boson star models for later use in the bubble nucleation process. In chapter 4, I review the theory of spontaneous formation of vacuum bubbles, and I introduce the "small star-large bubble" limit. I discuss the details of the calculation of a nucleation rate, and I conclude chapter 4 with results confirming that boson stars are quite efficient at nucleating phase transitions. The thesis ends with a brief chapter 5 in which I discuss the prospects for future work.

CHAPTER 2

WHAT WAS THE EARLY UNIVERSE LIKE?

The Early Universe Was Hot and Dense.

The early universe was hot and dense in comparison to the present.[1] This idea has convincing observational support, principally in the observation of nearly perfect isotropy of the remnant 2.7 K cosmic background radiation. To understand the significance of the cosmic background radiation (CBR), one needs to examine the big bang theory of the early universe. In 1948, George Gamow, Ralph Alpher and Robert Herman constructed a big bang model to study nucleosynthesis. They envisioned a cosmic fireball of neutrons that cooled adiabatically and eventually synthesized the lighter nuclei hydrogen, helium, lithium, beryllium, and boron. This nuclear "soup," consisting of a hot, dense neutron gas,[2] that filled the universe was in thermal equilibrium with all the radiation. Eventually, though, the nuclear soup cooled enough that the individual nuclei began to capture and hold their ration of electrons. When this

[1]The discussions in this section are due mainly to P.J.E. Peebles, Principles of Physical Cosmology (Princeton University Press, Princeton, 1993) and to R.M. Wald, General Relativity (University of Chicago Press, Chicago, 1984.

[2]G. Gamow, "The Evolution of the Universe," Nature, 162, 680 (1948).

happened, the density of free electrons, which had hitherto efficiently scattered radiation, decreased dramatically. Scattering of radiation became less frequent, and thermal equilibrium broke down. The radiation began to stream freely, without further significant interaction with the matter. This decoupling of matter and radiation occurred over a period of time, not all at one instant, but one refers to this period as the "decoupling time." One may also say that, at decoupling, the universe became transparent. The light then was mainly in the visible and near infrared, with a blackbody spectrum at a temperature of a few thousand Kelvins. Alpher and Herman predicted[3] in 1948 that the radiation, like a gas of photons in an enclosed space, should have expanded with the universe and cooled to a temperature today of roughly 5 K. After 1948, other scientists in Britain, the US and the Soviet Union refined the estimate of the present temperature of the CBR. In 1965 Arno Penzias and Robert Wilson discovered an isotropic source of "excess antenna temperature" of approximately 3.5 K.[4] Robert Dicke, P.J.E. Peebles and their collaborators gave the immediate

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[3]R.A. Alpher and R. Herman, "Evolution of the Universe," Nature, 162, 774 (1948).

[4]A.A. Penzias and R.W. Wilson, "A Measurement of Excess Antenna Temperature at 4080 Mc/s," Astrophysical Journal, 142, 419 (1965).

interpretation[5] that this "excess antenna temperature" was actually the red-shifted primordial radiation that Gamow and his collaborators had predicted 17 years earlier. This discovery implied the hot, dense nature of the big bang:

> "A temperature in excess of 10^{10} °K during the highly contracted phase of the universe is strongly implied by a present temperature of 3.5°K for blackbody radiation....If the cosmological solution has a singularity, the temperature would rise much higher than 10^{10} °K in approaching the singularity."[6]

Since 1965, very fine measurements[7] of the cosmic background radiation have yielded a temperature $T = 2.736 \pm 0.017$°K. These observations have fully vindicated the prediction of Gamow and his collaborators.

Gamow built his big bang theory of nucleosynthesis upon the theory of the expanding universe. I shall now review the calculation of the scale, density, and temperature in the early universe based upon the Robertson-Walker model of an expanding universe.

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[5]R.H. Dicke, et al., "Cosmic Black-body Radiation," Astrophysical Journal, 142, 414 (1965).

[6]Ibid.

[7]Peebles, Physical Cosmology, 131.

Most cosmological models assume that the universe is homogeneous and isotropic -- the so-called cosmological principle. This assumption is a reasonable one, for astronomical observations, especially of the CBR, record a homogeneous and isotropic distribution of radiation and matter in the universe.[8]

In the Robertson-Walker model, one envisions the four dimensional spacetime manifold as a foliation of spacelike hypersurfaces. A parameter, t, labels each spacelike hypersurface of the foliation. The 3-geometry of each hypersurface is homogeneous and isotropic.

Each isotropic observer in the spatial leaf moves upon a world line that is orthogonal to each leaf. This allows one to call t the proper time that an isotropic observer would measure. It also allows one to synchronize all clocks on each spatial leaf.

The imposition of isotropy forces the geometry of each leaf to be that of a space of constant curvature. Such homogeneous spaces are of three kinds: spherical, flat, and hyperbolic. It is customary to refer to the former as a "closed" space-time, to the latter as an "open" spacetime.

To get an intuitive grasp of how spherical and hyperbolic geometries compare to flat space, one might examine a circle. In a spherical geometry, the circumference of a

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[8]One must understand this homogeneity and isotropy to be evident on some suitably large scale, certainly larger than galactic.

circle is less than $2\pi r$. One can understand this by considering this circle and its radii to be confined to the surface of an ordinary sphere. In a flat geometry, the circumference is exactly $2\pi r$. In a hyperbolic geometry, the circumference is more than $2\pi r$. One can understand this by considering the circle and its radii to be confined to the surface of a saddle of hyperboloidal shape.

Physically, one can say that a closed spacetime contains matter which is dense enough to close the universe back on itself. The open and flat spacetimes have sparse matter density, so that they do not close back in on themselves.

The assumptions of homogeneity and isotropy yield significant simplifications of the metric. The metric on a four dimensional manifold will in general have ten arbitrary functions of the four-dimensional position. The assumption of homogeneity implies the presence of an isometry, spatial translation, at each point on the manifold. The further assumption of isotropy implies another isometry, spatial rotations. In addition, the isometries reduce the number of independent functions from ten down to only one function, $a(t)$. The most convenient form of the Robertson-Walker line element is

$$ds^2 = -dt^2 + d\ell^2 \ ,$$

with

$$d\ell^2 = a(t)^2 \begin{Bmatrix} d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) & \text{(closed)} \\ d\psi^2 + \psi^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) & \text{(flat)} \\ d\psi^2 + \sinh^2\psi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) & \text{(open)} \end{Bmatrix} . \tag{2.1}$$

In this metric,[9] the function a(t) is the cosmic scale parameter; in the case of a closed spacetime, one interprets it as the size of the universe. For example, if two galaxies are a proper distance L_0 apart at present, then at some earlier time t, they were a distance $L_0 \cdot a(t)$ apart, setting the present value of the scale factor to unity for convenience. Therefore, it is imperative to know the behavior of a(t) over time: does it get smaller or larger with time?

In order to determine the behavior of a(t), one must solve the Einstein field equations for these metrics. In general, the Einstein field equations relate the distribution and motion of matter and radiation to the geometry of the spacetime. One uses a stress-energy tensor, T_{ab}, to mathematically represent the matter and radiation. The Einstein tensor, G_{ab}, represents the curvature of the spacetime. These two tensors form the Einstein equations:

$$G_{ab} = 8\pi T_{ab} \ . \tag{2.2}$$

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[9]Throughout most of this thesis, I use natural units: c = 1, G = 1, k_B = 1 and h = 1. I employ a metric signature -+++ and follow the sign conventions of C.W. Misner, K.S. Thorne, and J.A. Wheeler, Gravitation (San Francisco: Freeman, 1973). I use abstract index notation; cf. R.M. Wald, General Relativity, 24.

The matter of the universe, on the cosmic scale, at least, is described by the stress-energy tensor of a perfect fluid.

$$T_{ab} = \rho\, u_a u_b + \left(\rho + P\right) g_{ab} \ . \qquad (2.3)$$

The metric tensor of the spacetime is g_{ab}. In fact, the stress-energy tensor of a perfect fluid is the most general form compatible with the homogeneity and isotropy of the Robertson-Walker model. The quantity ρ de-notes the energy density of the fluid, P denotes its pressure, and u^a is the fluid's four-velocity. Focussing on only the matter, one approximates it as "dust," with P = 0, while for radiation, ρ = P/3.

To understand the evolution of the scale factor, a(t), and the density, ρ, one must solve the Einstein field equat-ions. The two equations relating a(t) with ρ and P are

$$\frac{3\dot{a}^2}{a^2} = 8\pi\rho - \frac{3k}{a^2} \ , \qquad (2.4)$$

$$\frac{3\ddot{a}}{a} = -4\pi\left(\rho + 3p\right) \ . \qquad (2.5)$$

Overdots denote differentiation with respect to time; k = +1 for closed geometry, k = 0 for flat geometry, and k = -1 for open geometry.

The first mathematical implication one can draw from these equations is that a(t) is not constant. When ρ and P have realistic values $\rho > 0$ and $P \geq 0$, the second time derivative of a(t), its "acceleration," will be negative. This in turn shows that the universe *must* be either expanding or contracting. Indeed, astronomers beginning with Edwin Hubble have observed[10] that the universe is everywhere expanding. Hubble's famous law, V = HD, relates the recessional velocity, V, of a galaxy to its distance, D, with a constant of proportionality H. Since speed is distance divided by time, one may state that the H is the inverse of some time interval, H = 1/T. One can interpret T as the total expansion time, which should be an estimate of the age of the universe. By convention, H is called the Hubble constant, and T is called the Hubble time.

It is now important to return to the earlier example of the distance between two galaxies, $L(t) = L_0 \cdot a(t)$. Differentiation with respect to time gives

$$\dot{L} = L_0 \cdot \dot{a} \text{ and } L_0 = \frac{L(t)}{a(t)} \Rightarrow \dot{L} = \left[\frac{\dot{a}(t)}{a(t)}\right] \cdot L(t) \ . \qquad (2.6)$$

Now, if one dubs the quantity in brackets as H, one has the essential form of the Hubble law, V = H·D, except that, in general, H can be a function of time, i.e., H(t). Thus 1/H

[10]E.P. Hubble, "A Relation between Radial Distance and Velocity among Extra-Galactic Nebulae," Proceedings of the National Academy of Sciences, 15, 168 (1929).

is not really the exact age of the universe. Moreover, if one assumes that the universe has been expanding such that its "acceleration" has always been negative, as shown in figure 2, then the Hubble "constant" was larger in the past. Thus, the Hubble time can overestimate the age of an expanding universe.

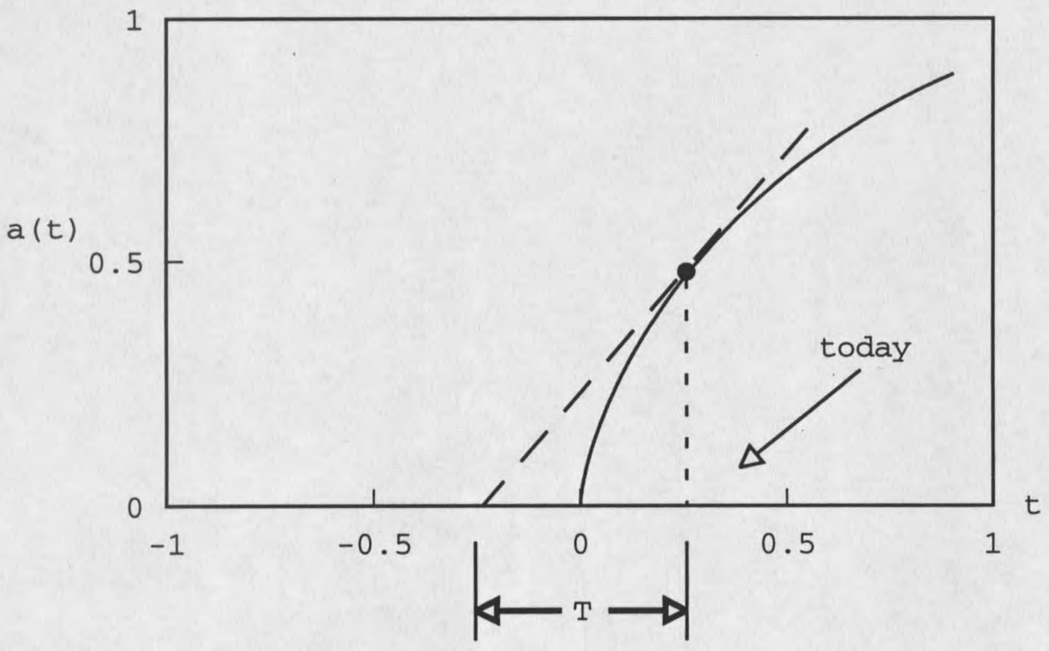

Figure 2. Sample solution for a(t) showing overestimation of the present age of the universe. The diagonal dotted line shows the overestimating effect of extrapolating Hubble's law back in time.

Was the scale factor zero at $t = 0$? To get an answer, one must combine the two Einstein field equations. With an algebraic combination of the two equations --

$$\dot{\rho} + 3\left(\rho + P\right)\frac{\dot{a}}{a} = 0 \qquad (2.7)$$

-- one can form a pair of conserved quantities. For the galactic "dust" of matter, P = 0, hence

$$\partial_t\left(\rho_{matter}\, a^3\right) = 0\ . \qquad (2.8)$$

The quantity ρa^3 is a constant. Thus the density of matter varies as the inverse cube of the scale factor:

$$\rho_{matter} \sim \frac{1}{a^3}\ . \qquad (2.9)$$

For radiation, $P = \rho/3$, so

$$\partial_t\left(\rho_{rad}\, a^4\right) = 0\ . \qquad (2.10)$$

The quantity $\rho \cdot a^4$ is a constant over time. Therefore the density of radiation energy varies with the inverse fourth power of the scale factor.

$$\rho_{rad} \sim \frac{1}{a^4}\ . \qquad (2.11)$$

In the Robertson-Walker model containing both matter and radiation, for sufficiently small values of the scale factor, the energy density of radiation will be larger than the energy density of matter. Later, as a(t) increases, the energy density of matter will approach and eventually exceed the energy density of radiation.

During the radiation-dominated period one can determine the behavior of a(t). Multiplying (2.4) by a^4 and replacing the quantity $\rho \cdot a^4$ with a constant of integration C, one obtains a nonlinear differential equation for a(t):

$$\dot{a}^2 - \frac{\left(\frac{8\pi C}{3}\right)}{a^2} + 3k = 0 \ . \qquad (2.12)$$

It is easy to determine the behavior of a(t) when a(t) becomes relatively small. Specifically, when $a^2 << 8\pi C/9|k|$, the differential equation becomes

$$\dot{a}^2 - \frac{\left(\frac{8\pi C}{3}\right)}{a^2} \approx 0 \ . \qquad (2.13)$$

The approximate solution for small a(t) is $a(t) \doteq A_0 t^{1/2}$, with A_0 being a constant that incorporates the previous constant C. This thime dependence shows that at some finite time in the past, the scale factor a(t) was tending toward zero as the square root of the time.[11] The energy density of the radiation, proportional to the inverse fourth power of the scale factor, must have been correspondingly large.

To show that the early universe was very hot, it is sufficient to state that for a radiation gas, ρ is proportional to the fourth power of the temperature, T.

$$\rho \sim T^4 \quad (2.14)$$

Since the energy density varies with the inverse fourth power of the scale factor, the temperature varies inversely with the scale factor.

$$T \sim \frac{1}{a(t)} \quad . \quad (2.15)$$

As a(t) tends toward zero, the temperature T increases, and we know indeed that the scale factor was smaller in the past. Therefore, we can say that, during the radiation-dominated period, the temperature was much higher than it is today.

In summary, the implications of homogeneity and isotropy, and the focus on the radiation-dominated period, lead to the prediction of a hot, dense, compact early

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[11]Whether one may extend this cosmological model all the way back to a singularity at t = 0 is a matter of current debate. When the energy density of the universe approaches the Planck density, approximately 10^{93} grams/cm^3, one must apply a quantum theory for the gravitational field. This theory is not currently complete. Thus, one may not presume to describe a time when a(t) = 0. For the purposes of this thesis, however, we will not need to resolve this question beyond saying the universe was very compact and dense in its early stage.

universe. This is the exact picture for which Penzias and Wilson found observational evidence in 1965 when they discovered the cosmic background radiation.

Matter Fields Are Effectively Massless at High Temperature.

Quantum theory has been spectacularly successful in describing the behavior of matter in many physical situations, from very low temperatures in a superconducting quantum interference device (SQUID) to very high energies in a large particle collider. When the temperature of the universe is significantly larger than the rest mass of a species of particle,

$$T \gg mc^2 \,, \tag{2.16}$$

then the relativistic form of the energy, ε, for a single particle,

$$\varepsilon = \sqrt{(pc)^2 + (mc^2)^2} \,, \tag{2.17}$$

can be approximated with its ultrarelativistic form, $\varepsilon = pc$. (For this section, I briefly reintroduce explicit constants c and $\hbar$.)

Consider a degenerate relativistic electron gas.[12] The number density of electrons in a region of phase space Γ is

$$dn_p = \frac{V\,d^3p}{h^3}\cdot 2 = \frac{4\pi V p^2\,dp}{h^3}\cdot 2\ , \qquad (2.18)$$

where V is the volume, and Planck's constant h represents the "cell-size" in phase space. The final factor of 2 accounts for the multiplicity of states for spin = 1/2 electrons. To find the energy of the gas, one simply integrates the product of $\varepsilon(p)$ and dn_p over all momenta up to the Fermi limit, p_F. The limiting momentum, p_F, depends on the number, N, of electrons one can pack in the given volume, V, in the following way:

$$N = \frac{V\cdot\frac{4\pi}{3}p_F{}^3}{h^3} = \frac{V}{\pi^2\hbar^3}\cdot\tfrac{1}{3}p_F{}^3\ . \qquad (2.19)$$

The total energy, E, is

$$E = \int_\Gamma \varepsilon\,dn$$

$$= \int_0^{p_F}(pc)\,dn_p = \frac{cV}{\pi^2\hbar^3}\int_0^{p_F}p^3\,dp = \frac{c}{4\pi^2\hbar^3}p_F{}^4\,V = \tfrac{3}{4}\hbar c\,N\left[\frac{3\pi^2 N}{V}\right]^{1/3}\ . \qquad (2.20)$$

Now, making use of standard thermodynamic relationships

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[12] L.D. Landau and E.M. Lifshitz, Statistical Physics (Part I). trans. J.B. Sykes and M.J. Kearsley, 3rd ed., (Oxford: Pergamon Press, 1980), 178.

relating energy, pressure and volume,

$$-\left.\frac{\partial E}{\partial V}\right)_S = P = \frac{1}{4}\left(3\pi^2\right)^{1/3}\hbar c\left[\frac{N}{V}\right]^{4/3} = \frac{1}{3}\frac{E}{V} \ , \tag{2.21}$$

one sees that, for high temperature, the electron pressure, P, and energy density, ρ, form a particularly simple equation of state, $P = \rho/3$. This is exactly the same equation of state as a photon gas, as mentioned above. In addition, one can derive the same equation of state for systems of other relativistic particles.

At some early time, when the temperature was significantly greater than the rest mass of the heaviest elementary particle (assuming one exists), the equation of state for all species of particles was that of radiation. As the temperature decreased, the radiation-like description for different species of particles began to fail. The first to lose their radiation-like behavior were the quarks; their equation of state changed gradually to that of "dust," $P = 0$. Then successive species of elementary particle lost their photon-like behavior until finally the electrons left the relativistic regime. The time of dominance for radiation was at an end and the time of matter dominance began.

* * * * * * *

Cooling of the Universe Leads to the Possibility of Phase Transitions in the Vacuum.

In finite temperature quantum field theory, phase transitions in the vacuum becomes possible. What is "the vacuum?" This refers to the ground state of the quantized fields in the absence of sources. Consider a simple φ^4 field theory in flat space. Its Lagrangian density is

$$\mathcal{L} = -\left(\nabla\varphi\right)^2 - \tfrac{1}{2}m^2\varphi^2 - \tfrac{\lambda}{4}\varphi^4 \ . \qquad (2.22)$$

The potential is $V(\varphi) = (1/2)m^2\varphi^2 + (\lambda/4)\varphi^4$. If $m^2 > 0$, there is a minimum at $\varphi = 0$. In terms of quantum field theory, one must instead work with the effective potential, which one derives from a Legendre transformation of the classical action.[13]

To understand the kinds of phase transitions that are possible, one must contrast a quantum field theory at zero temperature with one at non-zero or "finite" temperature, both with quantum corrections to leading order in $\hbar$, the so-called one-loop order. Incorporating the corrections for one-loop quantum fluctuations results in an effective

[13] In this section, I follow R.J. Rivers, Path Integral Methods in Quantum Field Theory (Cambridge: Cambridge University Press, 1987), 37, 86.

potential that may have a different number and location of the extrema, compared to the zero-temperature effective potential. For instance, one might consider the φ^4 theory of (2.22). If $\lambda > 0$ and $m^2 < 0$, there will be a second extremum; I discuss this situation below.

Although exact details about the location, etc. of the extrema depend upon which field theory one is studying, there are a few ideas that are relatively simple to express: the temperature dependent mass term, spontaneous breaking of an internal "hidden" symmetry, and a critical temperature. These are the areas in which the differences between zero-temperature and finite-temperature field theories become most important.

The effective mass of the quantum field can acquire a temperature dependence in a finite temperature field theory. Normally, one interprets the mass of the field via the coefficient of the term in the Lagrangian which is quadratic in the field: the square of the mass is twice that coefficient. For the Lagrangian of (2.22), the φ field has mass m. With quantum corrections at finite temperature, however, the mass becomes a function of the temperature, $m^2(T)$. If $m^2(T)$ becomes negative, two real-valued extrema occur at $\varphi = 0$ and the global minimum, $\varphi = (-m^2/\lambda)^{1/2}$, as figure 3 shows.

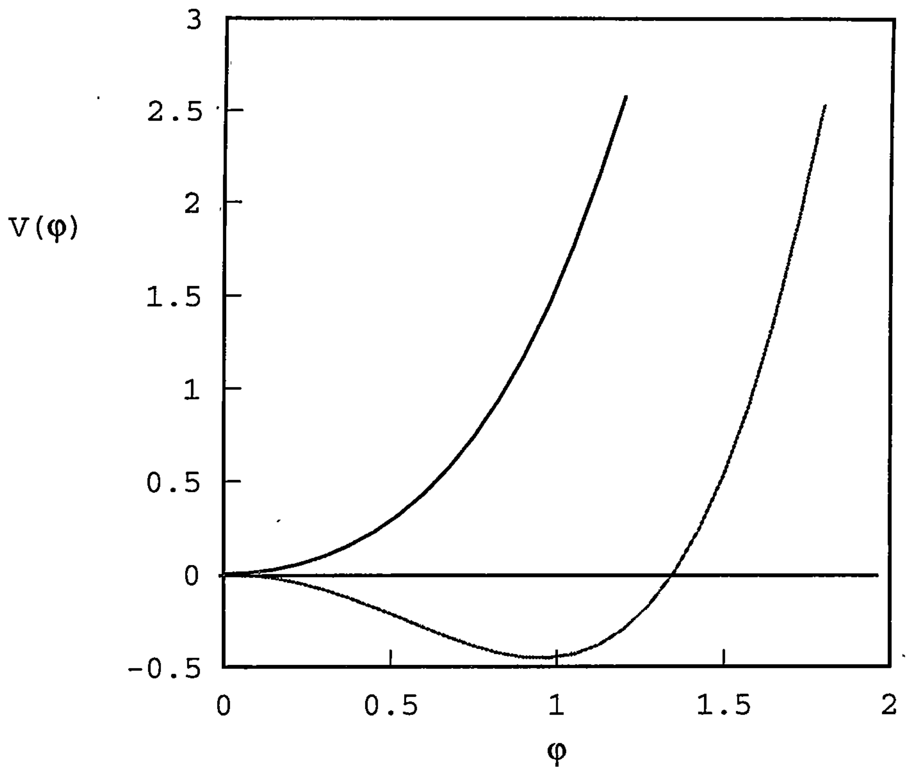

Figure 3. Shape of an effective potential for the two cases $m^2 > 0$ (upper curve) and $m^2 < 0$ (lower curve).

The vacuum expectation value of the scalar field is precisely that value of the field which extremizes the potential. In the first case, that value is $\varphi = 0$. In the second case, however, there are two values for the vacuum , one at $\varphi = 0$ and the other at $\varphi_0 = (-m^2/\lambda)^{1/2}$.

In the theory of spontaneous symmetry breaking, one must first expand the potential about the minimum at φ_0. Using the transformation

$$\varphi \rightarrow u = \varphi - \varphi_0 , \tag{2.23}$$

the potential acquires terms cubic in the field u.

$$V(\varphi) \rightarrow V(u) = -m^2u^2 + \lambda\varphi_0 u^3 + \tfrac{\lambda}{4}u^4 + \tfrac{1}{4}m^2\varphi_0{}^2 . \tag{2.24}$$

The u field also acquires a new mass: $(-2m^2)^{1/2}$. The discrete symmetry,

$$\varphi \rightarrow -\varphi , \tag{2.25}$$

which was manifest in the original Lagrangian does not appear in V(u); it is a hidden symmetry. However, the new ground state, u = 0, is asymmetric with respect to the original form of the field theory; the original symmetry is broken.

It is possible to express the temperature dependence of

m^2 with the concept of a critical temperature, T_c. Above T_c, m^2 is positive and the potential has but one extremum, $\varphi = 0$. Below T_c, m^2 is negative and the potential has two extrema. The typical form of the temperature dependence is

$$m^2(T) = m^2\left(1 - \frac{T^2}{T_c^2}\right) , \qquad (2.26)$$

with the exact form of T_c depending on the details of the field theory in question.

In the simple φ^4 theory above, the field φ can "roll" down into the well at $\varphi = \varphi_0$ from an initial state at $\varphi = 0$, a continuous change in the vacuum expectation value. This continuous change from one value to the other is a second-order phase transition. In electroweak theory, for example, it is possible to have a configuration of coupling constants for the gauge and scalar fields such that a second-order phase transition in the scalar field occurs.

Another kind of phase transition is possible, one in which there is a <u>discontinuous</u> change in the expectation value of the scalar field. This is a first-order phase transition. Figure 4 shows an effective potential in which a first order phase transition may occur. In the electroweak example, a first-order phase transition is possible under another configuration of the coupling constants, a relatively strong gauge coupling constant. As the temperature decreases

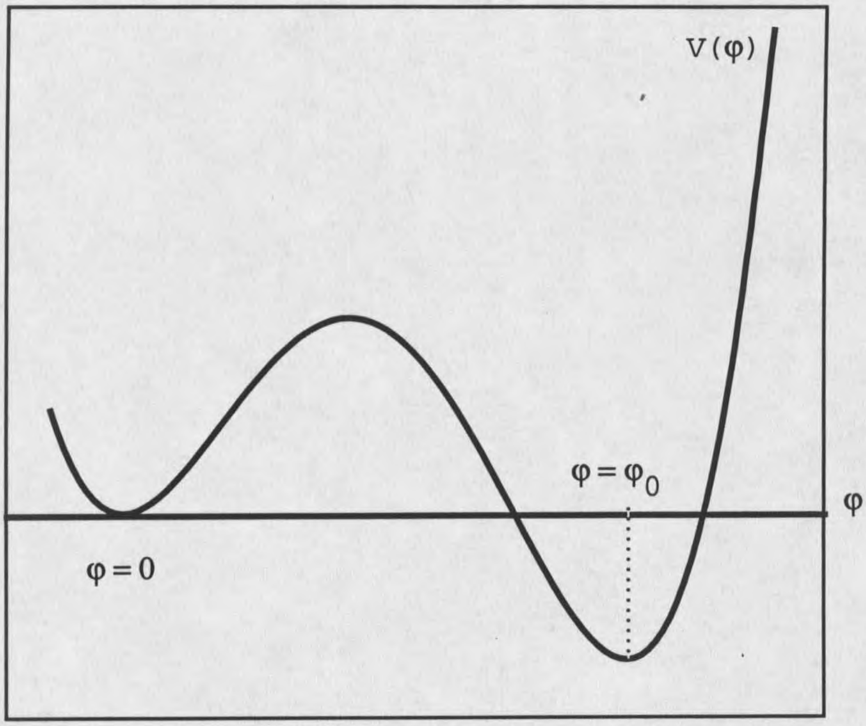

Figure 4. An effective potential in which a first order phase transition may occur.

toward the critical temperature, two wells may appear in the effective potential, as in figure 4. Although the field might originally be located at the quasi-stable minimum at $\varphi = 0$, that state is unstable in quantum field theory. The field may tunnel through the potential barrier to φ_0, a much different process from a second-order phase transition.

First-order phase transitions in quantized scalar fields are the main topic of this thesis. One kind of bubble formation process is of special interest: nucleation by a non-vacuum field configuration, a boson star. This exotic astrophysical object is the topic in chapter 3.

CHAPTER 3

WHAT ARE THE PROPERTIES AND CHARACTERISTICS OF BOSON STARS?

Background Concepts for Calculating Boson Star Models

The idea of a boson star can trace its ancestry back forty years or more to the geometric-electromagnetic entity that John A. Wheeler dubbed a "geon." In German, its name is *Kugelblitz*, the sphere of light. Wheeler suggested the geon as a "self-consistent solution to the problem of coupled electromagnetic and gravitational fields."[1] His most basic geon model was just a stable standing wave beam of light bent into a toroidal shape. Wheeler saw this as a generalization for the concept of material body that was possible within the framework of general relativity.

The idea of a self-gravitating field configuration is a fruitful one. For example, one can model neutron stars in this way, as a self-gravitating spin-1/2 fermionic field configuration. However, one must remember the current idea that neutron stars might not be simply a huge concentration only of neutrons; they might have a shell-like interior structure of exotic meson condensates and a crust of regular baryonic matter. Nonetheless, a neutron star is a well-known

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[1] J.A. Wheeler, "Geons," Physical Review, 97, 511 (1955).

example of a self-gravitating configuration of a quantum field.

Boson stars also fall into this class of exotic objects, and they have been the object of extensive study recently.[2] The most common model is that of a complex scalar field in gravitational equilibrium, with only the uncertainty principle supporting it against gravitational collapse.

Before going on to review some of the previous research on different kinds of boson stars, it is important to distinguish the self-gravitating field method for constructing a star model, from the "traditional" Oppenheimer-Volkoff method. The distinguishing feature is the use of an implicit equation of state in the former method versus the use of an explicit equation of state in the latter.

In the Oppenheimer-Volkoff approach[3], one assumes that the star has spherical symmetry and that there is no time dependence in the solution. Oppenheimer and Volkoff used the Schwarzschild coordinate system. Their method also assumes that the matter has the stress tensor of a perfect fluid, symbolized in the equations below as $T_{\alpha\beta}$. One must also adopt some equation of state $\rho=\rho(P)$ in order to obtain a solution for the system. Thus, one solves the system (3.1): the

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[2]See recent reviews in P. Jetzer, "Boson stars," Physics Report, 220, 163 (1992), and T.D. Lee and Y. Pang, "Nontopological solitons," Physics Reports, 221, 251 (1992).

[3]J.R. Oppenheimer and G.M. Volkoff, "On Massive Neutron Cores," Physical Review, 55, 374 (1939).

Einstein field equations, the equation of motion for the fluid, and the equation of state, viz.

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} ,$$

$$\nabla^{\alpha} T_{\alpha\beta} = 0 , \tag{3.1}$$

$$\rho = \rho(P) ,$$

for $\rho(r)$, $P(r)$, and for the metric functions $g_{tt}(r)$ and $g_{rr}(r)$, where t denotes the time and r the Schwarzschild radial coordinate. The physical dimensions of the object are constructed from these functions. For example, the radius of the object, R, is the radius at which the pressure vanishes, $P(R) = 0$.

The self-gravitating field method, which I shall use, is one which employs the scalar wave equation as the implicit substitute for the concept of an equation of state.[4] Beginning with the same assumptions about symmetry, and using the same coordinate system, one considers a scalar field φ with an Euler-Lagrange equation of motion given by $\Box\varphi + dV/d\varphi = 0$. One still uses the Einstein field equations, except that now, $T_{\alpha\beta}$ is the stress-energy tensor for the scalar field φ. The stress-energy tensor is a construction not of pressure and

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[4]R. Ruffini and S. Bonazzola, "Systems of Self-Gravitating Particles in General Relativity and the Concept of an Equation of State," Physical Review, 187, 1767 (1969).

density (as with the perfect fluid) but of the Lagrangian of the scalar field. Therefore, for the self-gravitating field method, the system of equations one must solve is just a bit simpler, one less equation than for the Oppenheimer-Volkoff method. For the self-gravitating field method, the system of equations is

$$\Box\varphi = -\frac{dV}{d\varphi} \ ,$$
$$G_{\alpha\beta} = 8\pi\, T_{\alpha\beta} \ . \qquad (3.2)$$

One solves (3.2) for $\varphi(t,r)$, $g_{tt}(t,r)$ and $g_{rr}(t,r)$. From those functions, one calculates various quantitative properties of the star. For example, one might find the size of the object by looking not for vanishing pressure but for the radial distance R at which the field φ has decreased to 1% of the central field value.

In the self-gravitating field method, the solutions will depend significantly upon the type of scalar potential one studies. Using different potentials in the self-gravitating field method is akin to working with different equations of state in the Oppenheimer-Volkoff method.

The first boson star models were calculated by D. J. Kaup in 1968 using the free particle potential for a classical complex scalar field,

$$V(\varphi) = \tfrac{1}{2} m^2 |\varphi|^2 \ . \tag{3.3}$$

The corresponding Euler-Lagrange equation is the familiar Klein-Gordon equation:

$$\Box\varphi \equiv \nabla^{\alpha}\nabla_{\alpha}\varphi = -\, m^2\varphi \ . \tag{3.4}$$

Kaup calculated size, mass and several thermodynamic quantities from his solutions for the Klein-Gordon geon.[5]

In 1969, Remo Ruffini and Silvano Bonazzola extended the analysis to a quantized scalar field of self-gravitating free bosons, as well as to a quantized spinor field of free fermions. In their examination of the concept of an equation of state,[6] they constructed star models and were able to calculate the size, mass and thermodynamic quantities of boson stars and neutron stars.

The case of the complex scalar field with a quartic self-interaction potential, came under the investigative eye of Monica Colpi, Stuart Shapiro and Ira Wasserman in 1986. In their paper on the gravitational equilibria of self-interacting scalar fields (hereafter, CSW),[7] they used a

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[5]D.J. Kaup, "Klein-Gordon Geon," Physical Review, 172, 1331 (1968).

[6]Ruffini and Bonazzola, "Self-Gravitating Systems," 1768.

[7]M. Colpi, S.L. Shapiro, I. Wasserman, "Boson Stars: Gravitational Equilibria of Self-Interacting Scalar Fields," Physical Review Letters, 57, 2485 (1986). Hereafter, I shall refer to this paper as CSW.

potential of the form

$$V(\varphi) = \tfrac{1}{2}m^2|\varphi|^2 + \tfrac{\lambda}{4}|\varphi|^4 . \quad (3.5)$$

They calculated the size and masses of boson star models with this potential. Depending on the mass of the scalar boson in question and on the relative strength of the interaction, boson stars of mass comparable to main sequence stars are possible. They even put together an effective equation of state for the case of relatively strong self-interaction.

What other kind of potentials might one use to model new varieties of boson stars? In chapter 2, I discussed some early universe models that call for a scalar potential in which phase transitions are possible. An example from electroweak theory is the Coleman-Weinberg mechanism.[8] The effective potential is of the following form:

$$V_{CW}(\varphi) = A|\varphi|^2 + |\varphi|^4 \left[\ln\left(\frac{|\varphi|^2}{B^2}\right) - \tfrac{1}{2}\right] . \quad (3.6)$$

One sets its parameters A and B using the masses of the W and Z particles and with the value of the field at the global minimum of the potential. In this potential are a pair of potential wells between which a first-order phase transition may proceed. Figure 5 shows an example of this potential.

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[8]R.J. Rivers, Path Integral Methods, 243.

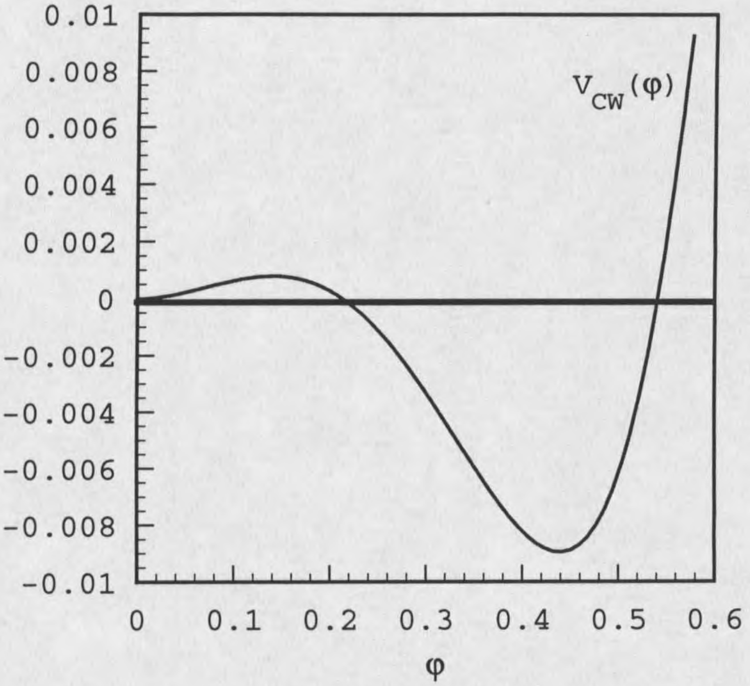

Figure 5. The effective potential in the Coleman-Weinberg process. A = 0.1, B = 0.5.

Another possibility is to use a potential that has terms quadratic, cubic and quartic in the scalar field, the so-called 2-3-4 potential.[9] This potential,

$$V(\varphi) = \tfrac{1}{2}m^2|\varphi|^2 - \eta|\varphi|^3 + \tfrac{\lambda}{4}|\varphi|^4 , \qquad (3.7)$$

is easy to work with and one can force it to have two minima by setting the three parameters, boson mass, m, coupling constant for self-interaction, λ, and a temperature-related coefficient η. In fact, the 2-3-4 potential will be my tool in studying the boson stars that nucleate early universe vacuum phase transitions, which is the topic of the next chapter.

Details of Calculating Boson Star Models with the Self-Gravitating Field Method.

To construct boson star models, I will use the self-gravitating field method, closely following the procedures in CSW. Once I have constructed the models, I will use them to calculate a nucleation rate for vacuum phase transitions.

To begin, I take the 2-3-4 potential as the model potential for the boson field φ. I ignore momentarily any

[9]D. Samuel and W.A. Hiscock, "'Thin-wall' approximations to vacuum decay rates," Physics Letters B, 261, 251 (1991); A. Linde, Particle Physics and Inflationary Cosmology (New York: Harwood Academic Publishers, 1990), 120.

coupling of φ to other fields, even possible gauge fields coupled to φ in the spontaneous symmetry breaking process. The Lagrangian density for the complex scalar field φ is

$$\mathcal{L} = -\tfrac{1}{2} g^{ab} \nabla_a \varphi^* \nabla_b \varphi - V(\varphi) \ . \tag{3.8}$$

The Euler-Lagrange equation for φ is

$$\nabla_a \left[\frac{\partial \mathcal{L}}{\partial \left(\nabla_a \varphi^* \right)} \right] - \frac{\partial \mathcal{L}}{\partial \varphi^*} = 0 \ . \tag{3.9}$$

From the Lagrangian one also constructs the stress energy tensor of the scalar field, which is

$$T^a_b = \tfrac{1}{2} g^{ac} \left[\nabla_c \varphi^* \nabla_b \varphi + \nabla_b \varphi^* \nabla_c \varphi \right]$$
$$- \tfrac{1}{2} \delta^a_b \left[g^{cd} \nabla_c \varphi^* \nabla_d \varphi + m^2 |\varphi|^2 - 2\eta |\varphi|^3 + \tfrac{1}{2} \lambda |\varphi|^4 \right] \ . \tag{3.10}$$

For solution of the scalar wave equation and the Einstein field equations, it is necessary to make several decisions about the nature of the solutions for the field and about the coordinate system which has so far remained unspecified.

The first decision about the type of boson star solution concerns its functional dependence upon time. For a star with fixed number of bosons, N, the time dependence for the ground state must be of the form $\exp[-i\omega t]$.[10] At this point in the discussion, ω is the unspecified Lagrange multiplier with which one minimizes the energy of the system under the constraint of fixed N.

A requirement specifically for the ground state is that there must be no nodes in the ground state solution. Later I will make use of this requirement to determine a specific value of ω. I also require that the solution be "localized" in a finite volume of space. In practice, this will happen because the ground state solution will have a decaying exponential dependence, so that it asymptotically approaches zero far from the center of the star.

Another specification of solutions is that they be spherically symmetric. This further simplifies the system of equations, in that the φ solutions depend only on time and the radial coordinate r. Overall, then, the solutions φ will be of the form $\varphi(t,r) = \varphi(r)\exp[i\omega t]$. This time dependence will simplify the system of equations in that all time derivatives are replaced with factors of $\pm i\omega$.

The requirement of spherical symmetry applies to the metric. In addition, since I am looking for equilibrium solutions for φ, the metric must be static, with no time dependence. In the (t, r, θ, ϕ) coordinate system, the form for the line element is

[10]T.D. Lee and Y. Pang, "Nontopological solitons," 255.

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2\,d\Omega^2 \ , \tag{3.11}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2 \ .$$

In this equation, ϕ stands for the azimuthal angle; θ stands for the co-latitudinal angle. Far from the star, where gravitational effects have diminished, the spacetime should become flat. In the limit of large r, both metric functions B(r) and A(r) must approach unity. Later, I will use this asymptotic condition on the metric to scale the metric function B(r).

Now that I have chosen a specific coordinate system, it is possible to write down the explicit form of the scalar wave equation and the Einstein field equations. Before doing that, however, I will rescale the field, the potential and the coordinate system so that they are all dimensionless, for convenience in calculation.

In the first rescaling, I replace the field $\varphi(r)$ with a dimensionless scalar field $\sigma(r)$ such that

$$\sigma = \frac{\sqrt{4\pi}}{M_{Planck}}\,\varphi \ . \tag{3.12}$$

Then, I change to a dimensionless potential such that