The topology of laminations
by Luther William Johnson

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences
Montana State University
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Abstract:
A lamination of a surface is a one dimensional subset of the surface, generically locally the product of a Cantor set with an arc. Some laminations arise in conjunction with pseudo-Anosov maps of the surface, others are more general. This thesis addresses the question of detecting which laminations are topologically equivalent. By following the construction of Harer and Penner, we realize the lamination as an inverse limit on wedges of circles. Then, taking two such laminations which are topologically equivalent, a “nearly commuting” diagram is obtained, which induces a weak equivalence diagram in the homology maps. This in turn enables us to partially classify such laminations- if such laminations are equivalent, then the systems of matrices which arise are weakly equivalent. Another incomplete invariant which turns out to be equivalent to that of weak equivalence in this setting, is the relationship of the transverse measures the laminations support by a square integer matrix with determinant plus or minus one. Ergodicity of laminations is discussed, and the result is further refined in the case of uniquely ergodic laminations to the relationship of a unique weight vector associated to either lamination by such a matrix.
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Luther William Johnson

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This dissertation has been read by each member of the dissertation committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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ABSTRACT

A lamination of a surface is a one-dimensional subset of the surface, generically locally the product of a Cantor set with an arc. Some laminations arise in conjunction with pseudo-Anosov maps of the surface, others are more general. This thesis addresses the question of detecting which laminations are topologically equivalent. By following the construction of Harer and Penner, we realize the lamination as an inverse limit on wedges of circles. Then, taking two such laminations which are topologically equivalent, a "nearly commuting" diagram is obtained, which induces a weak equivalence diagram in the homology maps. This in turn enables us to partially classify such laminations— if such laminations are equivalent, then the systems of matrices which arise are weakly equivalent. Another incomplete invariant which turns out to be equivalent to that of weak equivalence in this setting, is the relationship of the transverse measures the laminations support by a square integer matrix with determinant plus or minus one. Ergodicity of laminations is discussed, and the result is further refined in the case of uniquely ergodic laminations to the relationship of a unique weight vector associated to either lamination by such a matrix.
CHAPTER 1

INTRODUCTION

The goal of this thesis is to topologically classify certain one dimensional subsets of surfaces called measured laminations by giving algebraic invariants for topological classes of these subsets. Generically, a lamination is locally the product of a zero dimensional set with an arc. As such, they have much in common with the generalized solenoids of Williams [20] and the matchbox manifolds Fokkink [7] studied in his thesis. Our laminations, however, are constrained to live on a surface, while many of the above mentioned objects do not admit an embedding into a surface, although they are the same as laminations locally. We limit our investigation to laminations on compact surfaces without boundary. The theory admits a straightforward extension to other surfaces, but the overhead of extra definitions and cases is awkward and we avoid it.

The study of laminations as a class of objects was begun in the 70’s by Thurston [18] in his investigation of surface automorphisms — certain laminations can be viewed as the closure of the unstable manifold of the fixed points of pseudo-Anosov mappings, split apart. An Anosov map is a hyperbolic map which leaves invariant a pair of transverse foliations, (without singularities) known as the stable and unstable foliations. Invariance is meant in the sense that leaves are mapped to leaves, and
the map contracts or expands the measures on these foliations by a factor known as the dilatation factor, the log of which is the topological entropy of the map. The canonical examples of such maps on surfaces are the maps of the torus given by projecting the action on the plane of integer matrices with determinant 1 and distinct real eigenvalues, where the invariant foliations are lines parallel to the eigenvectors.

Since there are few surfaces supporting foliations without singularities, the class of mappings investigated was broadened to pseudo-Anosov maps, which are hyperbolic except at finitely many singular points. These maps also have associated to them a pair of transverse foliations with singularities which are invariant under the mapping in the same sense as above, and are expanded and contracted in measure by some dilatation factor. Thurston's famous classification theorem states that within each homotopy equivalence class of irreducible diffeomorphisms of a surface there is a map which is either periodic or pseudo-Anosov. Here, irreducible means that the map doesn't fix any homotopy class of simple closed curves. If a map is reducible, then some power of the map will, modulo homotopy, send the complementary regions of the fixed curve to themselves, and the theorem applies to this iterate of the map on the complementary regions. Thus, a reducible map fixes some classes of simple closed curves, and a power of this map is homotopic to a map which is periodic or pseudo-Anosov on each complementary region of the surface with the curves removed [18].

In his work on surface automorphisms, starting with a pseudo-Anosov map of
the surface, Thurston introduced an object called a weighted train track, which is a special one dimensional graph on the surface upon which the action of the mapping gives a Markov transition matrix, and a system of weights satisfying a "switch condition". The (log of the) Perron-Frobenius eigenvalue of the matrix gives the topological entropy of the mapping, which is the minimal entropy of any mapping in its homotopy class, and the associated eigenvector gives a system of weights associated to the edges of the track which will give rise to the transverse measure on the unstable foliation of the map.

Los [13] discusses an algorithm for producing the Markov matrix for a map of a surface based upon fitting an invariant train track to the action of the map. The Perron eigenvector of the incidence matrix he produces in a pseudo-Anosov case yields a weight vector satisfying the switch conditions on the track, and he uses this to characterize when a given mapping class has a pseudo-Anosov element in it, as opposed to a reducible or periodic element, in which case his algorithm does not produce a track.

We should not become too attached to the historical and dynamical origins of the train tracks however, as the study of laminations has broadened to consideration of sets which are not connected in any way to surface automorphisms. Starting with a train track on some surface, a weight vector which satisfies the "switch condition" (which we will discuss at length later) gives rise to a measured foliation on the surface. We develop a method for "splitting open" the singularities of such a foliation
in such a fashion that we arrive at a measured lamination. Roughly speaking, the foliation is a surface of fabric, with leaves ending at singularities corresponding to infinite zippers, and our splitting procedure will amount to unzipping the zippers completely. The subset of laminations arising in this fashion which correspond to pseudo-Anosov mappings is important as a special class because much is known about pseudo-Anosov foliations and laminations, and examples are easy to come by in this class, but the set of vectors which corresponds to these laminations is of measure zero in the appropriate setting, which we will also discuss.

A means of obtaining the results of this paper is discussed later on which avoids the machinery of train tracks, but we begin with train tracks in spite of this, because of the ease of presentation train tracks afford, and because the concrete, constructive nature of the development offers a nice geometric intuition for the laminations arising from them. Therefore, the first section of the paper is devoted to developing the language of train tracks, measured foliations, and measured laminations.

Following this, we develop a splitting procedure which opens up a measured foliation by splitting apart the singular leaves of the foliation, the result of which is a measured lamination. By carefully controlling this splitting procedure, we realize the lamination as a nested intersection of a sequence of foliated subsets arising from the weighted track. From this we are afforded a description of the lamination as an inverse limit in a common procedure when a set can be written as an infinite intersection in a nice way.
The description of a lamination as an inverse limit allows us a good handle to grasp the topology of the lamination by, and we quickly arrive at the main result of the thesis: If two laminations gotten in such a manner are homeomorphic, their weight vectors are related by a square integer matrix with determinant 1. This result is arrived at by first getting the result that in case two such laminations are homeomorphic, the two sequences of “splitting matrices” are weakly equivalent. The splitting matrices arise during the splitting process, and essentially capture the order in which singular leaves wind around in the foliation, which determines the manner in which each of the successive foliated subsets injects into the preceding one. Weak equivalence of sequences of matrices has been studied fruitfully as a topological invariant for a variety of similar systems by a number of authors of late; see Barge and Diamond [2] for the weak equivalence of matrices associated to the inverse limit of unimodal interval maps, or Anderson and Putnam [1] for an invariant of substitution tilings, for example.

In the course of the development of the invariants in this paper, we also discuss the ergodicity of laminations, and the transverse measures they can possibly support. It turns out to be the case that almost every lamination is uniquely ergodic, that is, the transverse measure which descends from the weight vector on the track is the only invariant measure for the lamination under holonomy. After stating the main result and proving it, we examine some examples, and discuss further the special case of pseudo-Anosov laminations. Finally, we will talk about work other authors
have been pursuing, and the relationship of this thesis to some of their results.
CHAPTER 2

DEFINITIONS

Train Tracks

DEFINITION 2.1. A branched one manifold is a one dimensional manifold except at finitely many points, where it has the topological type of a neighborhood of a point with finitely many rays emanating from it.

DEFINITION 2.2. A train track $\tau$ on a smooth surface $M$ is a branched one manifold with continuously varying tangent directions and no relative boundary, smoothly immersed in $M$, whose complementary regions have negative Euler characteristic (see below for an explanation of this). We call the finitely many points at which $\tau$ is not locally an arc the switches, and the components of $\tau$ less the switches are called branches.

That the complementary domains have negative Euler characteristic is a technical requirement which will guarantee the construction of the foliation given in the next section is well defined. Put more concretely, this requirement is that no complementary domain is a disk with zero, one or two cusps, or an annulus with one.

The restriction that the tangent direction vary continuously gives us a local picture at a switch like the following, where by way of an example, a 5-prong switch with branches $b_1, b_2, \ldots, b_5$ is depicted.
At each switch point we arbitrarily pick a positive direction in the tangent space, and refer to branches ending at the switch as either incoming or outgoing, depending upon the direction they approach the switch from.

**Definition 2.3.** A *weight* on a train track is a function from the branches of the track to the non-negative reals which satisfies the *switch condition*, that the sum of the function values over the incoming branches at each switch equals the sum over the outgoing branches.

In figure 1 above, if $a_i$ is the weight associated to the branch $b_i$, the switch condition is that $a_1 + a_2 = a_3 + a_4 + a_5$. The purpose of a train track with a system of weights satisfying the switch condition is for us the construction of a measured foliation. Roughly speaking, we will lay rectangles along the edges of the track with widths given by the weights, and the switch condition allows us to glue them together nicely. This construction will be discussed at length in the next chapter.
DEFINITION 2.4. A foliation $F$ of a surface $M$ is a system of smooth charts taking $(0,1) \times (0,1)$ into open sets of $M$ which form a basis for $M$, so that for any two charts $\phi$ and $\psi$ mapping to overlapping neighborhoods, the vertical component of $\phi^{-1} \circ \psi$ is constant in the horizontal direction. A leaf of the foliation $F$ is an arc in $M$ consisting of points which can be joined by arcs all of which have constant vertical components under any inverse chart.

In practice we often identify foliations with the images of the charts, and likewise with leaves. The image of a chart looks like a standard flow box, and finite pieces of leaves look like trajectories in a flow box. Since leaves are locally arcs with no boundary, they are 1-1 smoothly immersed copies of the real line or the circle. The canonical example of a foliated surface is the torus, foliated by the projection of lines of a given irrational slope onto the torus, thinking of the torus as gotten by identifying points in the plane which differ by integral vectors.

DEFINITION 2.5. A foliation with singularities is a foliation with a finite number of points at which the foliation does not admit a system of charts as above, but rather admits as a chart $z \rightarrow z^{2/k}$ from the closed upper half of the complex plane.

Under this chart, the leaves are the images of the horizontal lines in the upper half plane. These points are known as $k$-prong singularities, where $k$ for us is at least 3. A picture of a $k$-prong singularity with $k = 3$ is given in figure 2 on the next page. A way to visualize the action of $z \rightarrow z^{2/k}$ is the following: take the upper
half plane foliated by horizontal lines and square it, to obtain a 1-prong singularity at the origin; and then taking the preimages of this under $f(z) = z^k$.

The Poincaré-Bohl-Hopf theorem is a famous result which states that the Euler characteristic of a surface is equal to the sum of the indices at the singularities of a foliation[6]. The Euler characteristic is a readily computable topological invariant for surfaces, and basic results of algebraic topology give us a formula to compute it for any closed surface without boundary. The index of a foliation at a fixed point is computed by taking a sufficiently small circle around the singularity, and defining a function from this circle to the unit circle by the direction of the tangent line to the
leaf of the foliation. The index is given by the (positive or negative) number of times the function takes the circle around the unit circle. One may check that the index of a $k$-prong singularity is $1 - \frac{k}{2}$. Since the only closed boundaryless surfaces with Euler characteristic zero are the torus and the Klein bottle, we see that there are only two surfaces with foliations which do not have singularities. Thus, unless it is otherwise stated, a foliation will henceforth mean a foliation with singularities. Now, leaves of such a foliation are circles or lines, as they are in the case of a foliation without singularities, with the exception of the singular leaves, which are those which end at a singularity of the foliation. These may now be closed arcs, if they begin and end at singularities, or rays, if they begin at a singularity and do not end.

**Definition 2.6.** A *partial foliation* is a foliation of a closed 2 dimensional submanifold of the surface, where the submanifold has piecewise smooth boundary. We moreover require that non-singular leaves which intersect the boundary of the submanifold must be either transverse to it or smooth components of it.

We consider leaves intersecting the cusps to be singular. Given a partial foliation of a manifold, we may easily recover a foliation by placing a point in each complementary region, and disjoint arcs from the point to the cusps of the complementary region, then collapsing the region down to these arcs. If the region had $k$ cusps, a $k$-prong singularity is formed, as in the picture on the next page.

**Definition 2.7.** A *lamination* on a surface is, in our context, a subset of the surface which in the generic case is everywhere locally the product of an arc with a
Cantor set. A *leaf* of a lamination is an arc component of the lamination. We will also allow as a special case a lamination which is locally the product of an arc with the interval, i.e. a foliation with no singularities. Leaves in this case are as they are in the definition of a foliation.

**Definition 2.8.** A *transverse measure on a foliation* is a function $m$ defined on the set of arcs transverse to the foliation with the following properties:

i) if $s$ and $t$ are such arcs having their respective endpoints on the same leaves, and are isotopic to each other through a set of arcs also with endpoints on the same leaves, $m(s) = m(t)$. 

Figure 3. Collapsing a Partial Foliation.
ii) if $s_1, s_2, s_3, \ldots$ is a set of arcs transverse to the foliation, with $s_i \cap s_j = \emptyset$ for $i \neq j, j \pm 1$, and $s_i \cap s_j = \partial s_i \cap \partial s_j$ for $i = j \pm 1$, then $\sum m(s_i) = m(s)$ where $s = \cup s_i$, if such an $s$ is measurable. (an arbitrary such union may fail to be transverse to the foliation.)

The intuitive idea of property one is that if we keep the endpoints of a transverse arc on given leaves, and slide it around in the foliation, we do not change the measure of the arc. Property two allows us to split transverse arcs into sub-arcs, and have the measure be additive. The combination of these allows us to split arcs by pushing them past cusps, for instance, and break them into smaller and smaller arcs while maintaining the sum of the measures.

The idea of a transverse measure on a lamination is similar to that of a transverse measure on a foliation. Both measures are a description of a foliation or lamination's thickness, which is invariant under "holonomy", that is, moving around in the foliation or lamination in the direction of the leaves.

**Definition 2.9.** A measure on a lamination is a function $\mu$ defined on the set of arcs transverse to the lamination, but with endpoints in the complement of the lamination such that:

i) if $s$ and $t$ are such arcs, isotopic through a set of such arcs, $\mu(s) = \mu(t)$.

ii) if $s_i \cap s_j = \emptyset$ for $i \neq j, j \pm 1$, and $s_i \cap s_j = \partial s_i \cap \partial s_j$ for $i = j \pm 1$, then

$$\sum \mu(s_i) = \mu(s) \text{ where } s = \cup s_i.$$ (Here again, if this $s$ is measurable.)

iii) the support of $\mu$ is the lamination in the sense that $\mu(s)$ is non zero if and
only \( s \) intersects the lamination.

We see a third qualifier in the definition of the measure on a lamination, that the support is the lamination. This will rule out certain examples which are laminations, but not measured laminations, as we will see in a few paragraphs, which will be appropriate, as these laminations cannot arise as in the construction we will give.

Definition 2.10. A \textit{measured geodesic lamination} is a measured lamination which has all leaves straight with respect to some hyperbolic metric on the surface.

Harer and Penner [9] showed that each isotopy class of laminations has a unique geodesic representative. Two laminations are said to be isotopic if there is an isotopy of the surface which maps one lamination to the other.

As an interesting aside, another description of geodesic laminations is that they are the (Hausdorff) limit of long simple closed geodesics. As such, we see that not all geodesic laminations support a measure, for if there is "spiraling" behavior in a lamination, as we may easily imagine could happen in this description, the lamination will not support a measure. This is made clear in figure 4, as by the holonomy property of a measure, the two arcs \( s_1 \) and \( s_2 \) must have the same measure, because we could isotope \( s_1 \) to \( s_2 \), but \( s_2 \) misses a piece of the lamination that \( s_1 \) hits. This forces a violation of the property that the support of a measure is the lamination. Again, Harer and Penner have shown that any measured lamination is supported by a weighted track, and that weighted tracks give rise via the above construction to measured laminations, so none of the laminations that we consider
can have this sort of behavior.

There are two ways of viewing the special case in which a lamination is not a Cantor set in cross-section. Many authors define laminations of surfaces to be locally zero-dimensional sets crossed with an arc. With this definition, the special case is a circle with atomic measure associated to it. This is the description which would be more coherent if we were approaching laminations from the viewpoint mentioned above, but they do not work well with our approach, via a measured foliation. Throughout the beginning part of the thesis we will mention more abstract ways of viewing things, but stick with a very constructive, geometric viewpoint. From our
viewpoint, this special case amounts to an arc crossed with a circle, the leaves being
the circles, and we will limit our consideration to Borel transverse measures on the
foliations which are the ones that arise in the construction starting from a weighted
track.

**Definition 2.11.** A *Whitehead/isotopy class* of foliations is a set of foliations
which are related by combinations of isotopies of the surface, and *Whitehead moves,*
defined by picture above.

A Whitehead move is in our context the collapse of singular leaf joining two
singularities, thus out of two singularities with $j$ and $k$ prongs, one $k + j - 2$ prong
singularity is born. We can see that as the index at a $k$-prong singularity is $1 - \frac{k}{2},$
we have $\left(1 - \frac{k}{2}\right) + \left(1 - \frac{j}{2}\right) = 2 - \frac{(k + j)}{2} = 1 - \frac{(k + j - 2)}{2},$ and the sum of the indices is
preserved by such an operation. See, for example, exposé 5 of Fathi, Laudenbach,
and Poénaru's tour de force on these topics [6].
Each Whitehead/isotopy class of foliations yields a unique isotopy class of laminations, [9] thus a unique measured geodesic lamination. The collapsing process described earlier in which a foliation is obtained from a partial foliation also respects Whitehead/isotopy classes. We now proceed to describe the construction of a measured lamination from a weighted train track, as put forth in Harer and Penner's book.
The first step in obtaining a measured lamination from a weighted track is to construct a bi-foliated neighborhood, $N$, of the track. This will be a closed 2-dimensional submanifold of the surface with piecewise smooth boundary and two transverse foliations. Foliations are transverse if they have a mutual set of singularities, and their leaves are transverse everywhere except on that set. First, we take rectangles with widths given by the weights on the branches of some weighted track, and lengths as appropriate for the lengths of the branches. We smoothly embed them lengthwise along the branches of the track and glue them end to end, over the switch points. The switch condition on the weights guarantees this can be done nicely at each switch; we know that the sum of the widths of the rectangles on the incoming branches is the same as the sum of the widths of those over the outgoing ones, as in the figure on the next page.

We now have a closed submanifold with piecewise smooth boundary in the surface, which we call a neighborhood of the track, with a number of cusp points arising from the various rectangles coming together at the switches. Two transverse foliations $F$ and $S$ are given by the horizontal and vertical lines in the rectangles, they
are clearly transverse everywhere except at their mutual set of singularities, which are the cusp points. The submanifold is not literally a neighborhood of the track, as can be seen in the picture above. It will be a neighborhood of the track after some nudging via isotopy to get the track inside of it, however, and the language is standard. We play a little bit loosely with these objects in the sense that we are interested in isotopy classes of laminations, so we are allowed to deform all these classes of objects as we wish without disturbing the class it lies in.

The horizontal foliation $F$ will be our main object of interest. It is uniquely given by this construction, as long as the complementary domains of the track have
negative Euler characteristic, and we embed the rectangles with a uniform vertical scaling at the switches so that leaves match up as they ought to, given by the original heights of the rectangles and the height of each leaf within each rectangle [9]. It is not hard to write an exact description of the identifications necessary, but it is tedious and not particularly enlightening given the clarity of the picture. The vertical foliation, whose leaves we will refer to as ties, will be useful in some proofs, but of less interest. The remarks about the unicity of the class of foliations arising from the collapsing of complementary regions to get a foliation from a partial foliation do not apply to the tie foliation, for it is evident that there is no unique way, in the absence of more information, to “tie up the loose ends” so to speak. That is, we have not defined a structure in the transverse direction for S as we did for F, using vertical length, which would allow us to uniquely determine how to connect the ends of ties from different rectangles. This issue does not arise for F, since the leaves of F do not need to be joined under the collapse. There are many potential ways to collapse the complementary regions, giving rise to a variety of isotopy classes of foliations from S.

A Lamination

The next stage in the process could be thought of as “splitting” the foliation apart, starting at the cusps of the neighborhood, and opening it along the interior singular leaves. This is accomplished by two methods, depending upon the type of
singular leaf. In the case of an infinite (that is, non-compact) singular leaf, we will use a sequence of isotopies under which the images of the original neighborhood are new neighborhoods with the cusps pushed along the interior singular leaves. In the case of a finite (compact) singular leaf joining two singularities, we will completely separate (at least locally) the neighborhood at the singular leaf by introducing a long thin closed disk in place of the singular leaf and gluing appropriately. These two different methods correspond to the two different equivalence classes we wish to preserve, that of isotopy classes of foliations, and Whitehead classes. A more precise exposition of this splitting process is forthcoming, for now, let us think of the neighborhood as having a number of zippers which correspond to the interior singular leaves. Splitting the foliation completely is akin to unzipping the zippers all the way, starting at the cusps. For an intuitive understanding, see the picture on the following page.

As a transverse measure on our foliation $F$, we take Lebesgue measure $m$ in the vertical direction of the original rectangles we used to form the neighborhood. We will not need a measure on the tie foliation. The measure on the lamination gotten by splitting $F$ would be easy to describe if we restricted our attention to a smaller set of arcs — since the lamination sits inside the original neighborhood, we could take the measure on an arc transverse to the lamination to basically be that of an arc with endpoints on the appropriate leaves in the foliation. The definition of a lamination's measure however allows for "doubling back" in the complementary
regions, and such arcs are likely not transverse to the foliation, and may thus fail to be measurable. A precise description of the measure which arises will be given in the proof of the theorem at the end of this chapter, but for intuitive purposes we are well served by thinking of the measure of an arc in the lamination as being that of the arc in the foliation. This description is in fact true as long as the arc does not "double back" in complementary regions.
An Inverse Limit Description

Returning to our original bi-foliated neighborhood N, we will begin to formulate a more precise and controlled description of the splitting process; we will express the lamination in terms of an infinite intersection of nested neighborhoods which will in turn allow us to view it as an inverse limit, and give us a handle on the topology through that description. A preliminary reduction is in order. We assume for now that all of the interior singular leaves are infinite in length, which Hatcher [8] showed to be the generic case if and only if the surface is orientable. The measure referred to by “generic” is Lebesgue measure on the subspace of weight vectors which satisfy the switch condition on the track. We will address the case where a singular leaf is compact later on in the discussion for the sake of the argument’s flow.

**Definition 3.1.** Two leaves are said to be parallel if they admit globally homotopic parameterization.

**Lemma 3.2.** If the interior singular leaves of the foliation $F$ are infinite (non compact) they are each dense in the neighborhood.

**Proof.** Consider a component of the union of all open subsets of the neighborhood which some interior singular leaf fails to enter. Such a component will be a packet of parallel leaves. It is a basic fact from track theory that if the complementary regions have negative Euler characteristic, then leaves passing on opposite sides of a cusp do not allow such a parameterization. [9] Two nearby leaves are thus
parallel until they pass a cusp, so the boundary of such a region must consist in part of at least one interior singular leaf. If an interior singular leaf is on the boundary of such a region, however, the whole leaf is part of the boundary, for as mentioned, only a cusp destroys the parallelness of leaves, and the singular leaf which is on the boundary is parallel to leaves nearby in the region. This is a contradiction, for the boundary of such a region cannot have infinite length, and interior singular leaves are infinite. Thus, if the interior singular leaves are infinite, they are dense. □

The reason we do not start with the assumption of density is that we will have an easy criterion to verify which indicates they must be infinite.

**Unzipping Singular Leaves**

Now, let us assume we are in the generic setting of dense singular leaves, and deal with the unusual cases later. Choosing an arbitrary transversal $\ell$ to $F$ with distinct endpoints in the complement of $N$, because the interior singular leaves are dense, we may "split" the neighborhood so that each cusp of the neighborhood lies on $\ell$. "Splitting" the neighborhood is accomplished by "unzipping" an interior singular leaf, which we define by the following process:

**Definition 3.3.** Picking a cusp point $p$, and the interior singular leaf $k$, we *unzip* $k$ to a point $p'$ on $k$ by applying a smooth isotopy $H : M \times [0,1] \rightarrow M$ with the following properties:

i) $H(\cdot, 0)$ is the identity on $M$.

ii) $H(N,t) \subset H(N,s)$ whenever $s \leq t$ (monotonicity in the second factor)
iii) $H(p, 1) = p'$

iv) The support of $H$ in the first factor is contained in a small neighborhood of the piece of $k$ between $p$ and $p'$, small enough that no other cusps of the neighborhood are moved.

v) $H$ is also monotone in the first factor in the sense that transversality of the images of leaves with the ties (not the images of ties) is preserved for each $t \in [0, 1]$.

The first four conditions are fairly transparent, while condition v) guarantees that the images of each leaf hits each tie only once locally in the leaf topology. (i.e. the topology of the leaf considered as an embedding of the line, or in the case of an infinite singular leaf, as a ray, and not the topology of the surface) Thus, we do not introduce unwanted foldings in the picture, and a picture of the unzipping process is well represented by figure 2 on the next page.

So, we have now split open our neighborhood $N$ to a new neighborhood $N_0$, with the cusps of $N_0$ lying on the transversal $\ell$.

**Definition 3.4.** We call a component of $N_1 \setminus \ell$ a *strip* of the neighborhood $N_1$.

**Definition 3.5.** A foliation or lamination is said to be orientable if there is a continuous choice of a positive direction along leaves.

We make another reduction at this point by assuming that each strip meets $\ell$ on two sides of $\ell$. That is, fixing a point of view from which to view $\ell$, we think of one end of $\ell$ as the top, the other as the bottom, and a strip can be said to "leave"
Figure 8. Unzipping from $p$ to $p'$. 

\[
\begin{array}{c}
\text{\(\ell\) on the left, and "return" to \(\ell\) on the right. This situation occurs only in the case of an orientable foliation; to handle the more common situation, we will eventually describe a process of passing to the orientable double cover of a foliation, and then proceed with the following process. A note of caution to the reader is in order.}
\
\text{We refer to orientability throughout in two different ways — first, as mentioned, the orientability of a surface, and its implications for the types of laminations the surface supports, and second, the orientability of the foliation or lamination itself.}
\
\text{The two issues are unrelated — non-orientable surfaces have oriented foliations, and non-orientable foliations can live on an oriented surface. The context should make}
\end{array}
\]
clear the orientability we are referring to at various times.

The Splitting Process

We have a new measure on $N_0$, call it $m_0$, given by $m_0(s) = m(s')$ where $s'$ is the arc which isotoped to $s$ under the splitting isotopy. Then, we have a new vector of weights $x_0$ sitting in $n$-space, where $n \leq m$. Isotoping to this new neighborhood gives us a track with one super-switch, and $n$ branches, some of the others possibly having been eliminated by the splitting. Basically, we have gotten rid of the redundancy in some of the switch conditions which were not linearly independent, see figure 9 for an intuitive picture of how this occurs. The new weight vector is given by the measure of the $n$ strips, from bottom to top, on the left side of $\ell$. So, $x_{0_1}$ is the thickness of the bottom strip on the left, $x_{0_2}$ the second from the bottom on the left, etc.

**DEFINITION 3.6.** An *interval exchange transformation* is a piecewise isometry of the interval.

Notice now as in figure 10 that this neighborhood is the suspension of an interval exchange transformation.

Interval exchanges are commonly represented by a positive vector whose entries represent the lengths of the subintervals upon which the map is an isometry, a permutation describing the order in which the subintervals are rearranged, and a vector of +'s and -'s indicating whether the strip is a Möbius component or not. We will call upon results of Keane, [11,12] Veech [19] and others regarding these
Figure 9. Losing Redundancy in the Switch Conditions.

transformations.

Considering the neighborhoods from this viewpoint makes it easy to see that the singular leaves will be infinite if the subinterval lengths are not rationally related [11], as long as the neighborhood is not orientable. This criterion gives us a set of zero measure in the space of allowable weight vectors in which there can possibly be finite singular leaves, at least in the case the surface is orientable.

Let us call the permutation associated to this transformation $\sigma$. A strip leaves from the $n_{th}$ position and returns in the $\sigma(n)_{th}$, numbered bottom to top on the right as well. We split the neighborhood $N_0$ to a new neighborhood $N_1$ by unzipping the topmost cusp along its singular leaf until it returns to $\ell$. Thus, if $x_{0_{\sigma^{-1}(n)}} \geq x_{0_n}$ we will split to the right, see figure 11, since the strip coming in on the right is wider than on the left. In the other case, we split to the left, figure 12.
We see that if the split goes to the left, the $n_{th}$ strip is split apart, and if it splits to the right, the $\sigma^{-1}(n)_{th}$ strip is split apart. The new neighborhood $N_1$ will have again $n$ strips, with labels given by the convention that the strip which caused the split (the narrower of the strips on the top) retains its label, and the bottom portion of the split strip retains its original label. With this convention, under the inclusion map from $N_1$ into $N_0$, with a left split, the $\sigma^{-1}(n)_{th}$ strip of $N_1$ traverses both the $\sigma^{-1}(n)_{th}$ and the $n_{th}$ strip of $N_0$, and with a right split, the $n_{th}$ strip of $N_1$
traverses the \( n_{th} \) and \( \sigma^{-1}(n)_{th} \) strips of \( N_0 \).

This information is captured by matrices having ones in the diagonal, and, in the case of a left split, a one in the \((n, \sigma^{-1}(n))\) position, or a one in the \((\sigma^{-1}(n), n)\) position in the case of a right split. This may be the transpose of the matrix one would expect, as the information could be read "the strip given by the second coordinate splits the strip given by the first". However, the matrix also expresses the action on the weight vectors. The neighborhood \( N_1 \) has a new vector \( x_1 \) given by the
measure of the strips, and if we call the above matrix $A_1$ we have $A_1 x_1 = x_0$. This follows since, in the case of a left split, there are ones in the $(n, \sigma^{-1}(n))$ position and the $(n, n)$ position, and it was the $\sigma^{-1}(n)_{th}$ strip which split apart the $n_{th}$, breaking it into two pieces of width $x_{1_n}$ and $x_{1_{\sigma^{-1}(n)}}$. Each of the other rows has a one in the diagonal position only, and the vectors $x_1$ and $x_0$ do not differ in other than the $n_{th}$ coordinate. Likewise with a right split, the difference is only in the $\sigma^{-1}(n)_{th}$ coordinate, and the matrix reflects this.
Our new neighborhood $N_1$ is also a bi-foliated neighborhood, given by taking the image of the original foliation $F$ under the splitting isotopy, and in the transverse direction, the intersection of the original ties with $N_1$. It is equipped with a new transverse measure $m_1$ just as we obtained the measure $m_0$ from $m$ above when we split $N$ to $N_0$. If we denote by $p_i$ the projection which collapses ties in $N_i$ to a point, the image of $p_i$ is in either case a wedge of $n$ circles, joined at the image of $\ell$. Denoting by $X_i$ the wedge of $n$ circles so obtained, we have a map $f_1 : X_1 \rightarrow X_0$ induced under the projection by the inclusion map $\iota_1 : N_1 \rightarrow N_0$. The map $f_1$ will wrap each circle of $X_1$ around the corresponding one in $X_0$, and will also wrap the circle corresponding to the strip which caused the splitting around the circle corresponding to the strip which was split. Thus the following diagram commutes.

\[
\begin{array}{ccc}
N_0 & \xrightarrow{\iota_1} & N_1 \\
\downarrow p_0 & & \downarrow p_1 \\
X_0 & \underset{f_1}{\leftarrow} & X_1 
\end{array}
\]

Notice now, that the 1-dimensional homology of $X_i$ is $\mathbb{Z}^n$, and if we think of the circles as the generators of the homology, label them as we labeled the strips, and orient them as the strips are oriented, our matrix $A_1$ also is the induced mapping in homology.
The Full System Arising from Splitting

We unzip all of the singular leaves completely by the convention of always unzipping the topmost singular leaf until it again intersects $\ell$. At each stage of the unzipping, all proceeds as in the last section except that the topmost strips will not necessarily be the $n_{th}$ and $\sigma^{-1}(n)_{th}$. Rather, it will be the $i_{th}$ or $\sigma^{-1}(i)_{th}$ strip which is split apart, if the $i_{th}$ is topmost at that stage. Thus, the matrices which arise will be the identity with a one in the $(i, \sigma^{-1}(i))$ entry, or vice versa. Thus, we have the following commutative diagram, with $p_i$ the tie collapsing projection of $N_i$ onto $X_i$, $\iota_i$ the inclusion mapping of $N_i$ into $N_{i-1}$, and $f_i$ the induced map of $X_i$ onto $X_{i-1}$.

![Diagram](image)

The inverse limit of a system $\{X_i, f_i\}$, denoted $\lim(X_i, f_i)$, is a topological space defined as follows if $f_i$ maps $X_i$ into $X_{i-1}$:

**Definition 3.7.** $\lim(X_i, f_i) = \{x = (x_0, x_1, x_2, \ldots) \text{ where } f_i(x_i) = x_{i-1} \text{ for } i \geq 1\}$, with the product topology. We denote by $\pi_k$ the projection onto the $k_{th}$ coordinate space of an inverse limit space. The metric $d(x, y) = \sum d_i(x_i, y_i)/2^{i+1}$ will generate the topology of $\lim(X_i, f_i)$.

The first theorem may now be stated as follows.
**Theorem 3.8.** If we begin with a weighted track, construct a bi-foliated neighborhood as in the construction in the first section of this chapter, and the $N_i$ denote the sequence of neighborhoods gotten by isotopy of the horizontal foliation as in the previous section, then $\Lambda = \bigcap_{k>0} N_i$ is a lamination homeomorphic to $\lim(X, f_i)$. Moreover, $\Lambda$ supports a transverse measure $\mu$ induced by the measure $m$ on the original foliation.

**Proof.** First, we must verify that $\Lambda$ is a lamination. We are in the generic setting in which the interior singular leaves are infinite and dense. To see that $\Lambda$ has locally the structure of an arc crossed with a Cantor set, examine a cross section by any transversal $s$ (i.e. an arc which was transverse to the foliations of the neighborhoods). We clearly have a Cantor set structure, for $\Lambda \cap s$ is closed, as it can be seen as the intersection of the closed neighborhoods with $s$. Every point of the cross section is a limit point by the density of the singular leaves. Also by the density of the singular leaves we have no open sets left in the cross section. In the longitudinal direction, associated to any point of $\Lambda \cap s$ is an arc given by the leaf the point was on. Thus, what we have is a lamination.

Next, $\Lambda$ is homeomorphic to $\lim(N_i, \iota_i)$. This is evident, since any point $n_0$ in $\lim(N_i, \iota_i)$ is of the form $(n_0, n_0, n_0, \ldots)$ since the $\iota_i$'s are just inclusion mappings. Thus, every point of $\lim(N_i, \iota_i)$ corresponds directly to a point in the intersection of the $N_i$'s, so they are setwise the same. Note also that the metric given for an inverse limit agrees with the metric on $\Lambda$ as a subset of $N_0$. 
Now we must see that $\ell^\infty(N_i, \nu_i)$ is homeomorphic to $\ell^\infty(X_i, f_i)$. Let us define a mapping $h : \ell^\infty(N_i, \nu_i) \to \ell^\infty(X_i, f_i)$ by $h(n) = h((n, n, n, \ldots)) = (p_0(n), p_1(n), p_2(n), \ldots)$. This point is in $\ell^\infty(X_i, f_i)$ because the diagram commutes. $h$ is continuous since the $p_i$'s are continuous, and open because the $p_i$'s are open. $h$ is 1-1, for if $h(n) = h(n')$, then $p_i(n) = p_i(n')$ for every $i$. Notice that $p_i^{-1}(n)$ is a tie in the $i$th neighborhood for every $i$, and the diameter of the ties is going to zero as we split, because the singular leaves are dense. Thus $n = n'$, and $h$ is 1-1.

Last, $h$ is onto. To prove this, pick an $x \in \ell^\infty(X_i, f_i)$. If we normalize the diameter of $X$, we can choose a sequence $\{n_k\}$ in $\ell^\infty(N_i, \nu_i)$ so that $d(h(n_k), x) < 1/2^k$ as follows: take $n_k \in p_k^{-1}(x)$, and $n_k = (n_k, n_k, n_k, \ldots)$. Since the ladder diagram with the $\ell'_i$'s, $p'_i$'s, and $f'_i$'s commutes, $n_k \in p_j^{-1}(x)$ for each $j < k$, so $p_j(n_k) = x_j$ for $j < k$. Thus they can disagree only after the $k$th coordinate, and $d(h(n_k), x))$ is at most $1/2^k$. Then, since $\ell^\infty(N_i, \nu_i)$ is compact, the sequence $\{n_k\}$ has a limit point, $n$, and by continuity, $h(n) = x$. $h$ must be a homeomorphism, as it is a 1-1, onto, continuous and open map.

We are also ready to give a precise description of the measure which $\Lambda$ supports in terms of the neighborhoods in the splitting. Any arc which is transverse to $\Lambda$ is transverse to one of the neighborhoods $N_k$, since the neighborhoods limit in on $\Lambda$. Each of the neighborhoods is equipped with a transverse measure $m_k$ equivalent to the original measure via the splitting isotopies. We define $\mu(s) = m_k(s \cap N_k)$. \[\square\]
A More General Approach

It is not really necessary to resort to the language of train tracks to get the inverse limit description. It does, however, provide such a geometrically appealing and intuitive picture that it seems worthy of inclusion in the thesis. Harer and Penner [9] have shown that combinatorial classes of weighted tracks, (There are certain "combinatorial moves" on tracks corresponding to our splittings which generate an equivalence class of tracks.) Whitehead/isotopy classes of measured foliations and partial foliations, isotopy classes of measured laminations, and measured geodesic laminations are all in 1-1 correspondence with each other, so we neither lose nor require any extra information by starting from this viewpoint.

If, however, we wished to be rid of the machinery of train tracks, here is a sketch of how we could proceed. Starting with a measured geodesic lamination, we identify the finitely many asymptotic $k$-cycles of leaves in the lamination. These are a set of $k$ leaves which arise from opening up $k$-prong singularities of a foliation. It is a well known fact from hyperbolic geometry that each complementary region of a measured geodesic lamination has finitely many leaves as a boundary. We call them asymptotic cycles of leaves as if we split each leaf into two half leaves, the pairs of half leaves are asymptotic to each other in a cyclic manner. That is, half leaves of the $k$ leaves admit parameterizations $\psi_i$ and $\phi_i$ for which $\psi_i(t)$ approaches $\phi_{i+1}(t)$ as $t$ goes to $\infty$ for $1 \leq i \leq k - 1$ and $\psi_k(t)$ approaches $\phi_1(t)$. (Depicted in figure 13...
is an asymptotic 3-cycle of leaves.)

Putting a point in each complementary region, we could identify a set of points, one on each of the boundary leaves, which are closest to the point in the complementary region in the sense of being able to connect them and the given point with the shortest geodesic arcs inside the complementary region. Gluing these points together, and gluing the resultant pairs of half leaves together around the cycle by the parameterizations mentioned above, we have turned the cycle into a $k$-prong
singularity of a foliation, as all of the complementary regions of the lamination have been collapsed.

Now that we have a foliation, with the obvious induced measure, we can define a splitting process. We split open the singularities by inserting a closed disk with the appropriate number of cusps to get a partial foliation. Taking an arc transverse to the foliation with ends on some boundary leaves, we push the cusps to the arc and proceed as before. The process is obviously not unique, but irrespective of what arc we begin with, the splitting process will yield the same lamination.

**Finite Singular Leaves**

In the case of an orientable surface, there is a measure zero subset of the allowable weights on a track for which the foliation arising has finite singular leaves. This may or may not correspond to a packet of parallel finite leaves, but any packet of finite leaves definitely entails finite singular leaves, as discussed earlier in the proof of lemma 3.2. In any case, as many authors have noted, the set of foliations on an orientable surface with packets of parallel finite leaves is of measure zero. (The measure is either that gotten by Lebesgue measure on the subspace of weight vectors satisfying switch conditions on a track, or equivalent to it.) In the case of a non-orientable surface, Hatcher [8] has shown there is an open set of weight vectors which give rise to parallel components — in certain cases the packet of parallel circles forming a Möbius band is stable under any perturbation of the weight vector.
In any case, we need to formulate a method for dealing with finite singular leaves interior to the neighborhood. We would like the unzipping of a finite singular leaf to completely split apart the neighborhood as pictured below. This changes the topology of the neighborhood, so our method for splitting infinite singular leaves is not going to do the job, since isotopies respect the topology of the various objects they act on. It isn’t so desirable to need two separate means of unzipping leaves, but a way of justifying this is that we are trying to preserve Whitehead/isotopy classes of (partial) foliations. The unzipping of infinite singular leaves, via isotopy, preserves Whitehead/isotopy classes, and the method described below for finite singular leaves will preserve the Whitehead class, if not the isotopy class, of the foliation.

To accomplish the unzipping of a finite singular leaf, we do the following. Remove the singular leaf, then pry apart the neighborhood enough to accept the insertion of a thin rectangle in place of the singular leaf. We do this in such a manner that the top and bottom of the rectangle match up smoothly with the incoming exterior singular leaves, and the two arcs forming the top and bottom of the rectangle will be the replacements for the singular leaf, the singularities involved are gone.

From now on, the complete splitting process will be to first unzip of all the finite singular leaves. Each of these unzippings may or may not split the neighborhood into different components, and these components may be packets of parallel circles, or partial foliations with dense leaves. We then split apart completely the remaining neighborhoods to obtain laminations consisting of various components which will be
either packets of parallel circles (recall that no splitting takes place, as there are no
singularities to split apart in this case, and the definition of laminations allows for
these packets) or objects which are Cantor sets in cross section.
CHAPTER 4

TOPOLOGICAL INVARIANTS OF LAMINATIONS

Notation

With the inverse limit description of laminations we developed above, we are now ready to examine the main question of this thesis. That is, given two laminations, are they homeomorphic? We cannot answer in the affirmative, but give a nice criteria which will show when they cannot be. Before stating the main result, several notational conventions must be established.

Let \( \mathcal{M} = \mathcal{M}(x_0, \sigma, \epsilon) \) be the set of Borel probability measures on the arc \( \ell \) which are invariant under the interval exchange on \( \ell \) induced by the foliated neighborhood \( N_0 \). Here \( x_0 \) is the vector of weights on \( \ell \), \( \sigma \) is the permutation of the intervals of \( \ell \) induced by the foliated neighborhood \( N_0 \) and \( \epsilon \) is a vector of signs, (± or -) indicating whether each strip has a Möbius twist or not. These three things are typically used to denote a particular interval exchange as they capture all the necessary information.

Let \( \Delta_{n-1} \) denote the unit simplex in \( \mathbb{R}^n \), those non-negative vectors of \( \mathbb{R}^n \) with 1-norm equal to 1.

Let \( E : \mathcal{M} \rightarrow \Delta_{n-1} \) be given by \( E(m) = (x_1, x_2, x_3, \ldots, x_n) \) where \( x_i \) is the measure of the \( i_{th} \) subinterval of \( \ell \). We refer to \( E(m) \) as the evaluation of \( m \).
Let \( \tau \) denote the interval exchange transformation induced on \( \ell \) by the neighborhood \( N_0 \).

Finally, a projective linear map on \( \Delta_{n-1} \) is given by \( A(x) = \frac{Ax}{||Ax||} \) where \( A \) is a non-negative matrix such as those which arise from the splittings. Notice the distinction between \( A(x) \) and \( Ax \), one being the projective linear map, the other matrix multiplication. An example of the action of a projective linear mapping on \( \Delta_2 \) is given on the next page for the elementary matrix having a one in the (3,1) position.

**Lemma 4.1.** If \( (A_k) \) is the sequence of matrices arising from the splitting of \( N_0 \), then \( E(\mathcal{M}(x_0, \sigma, \epsilon)) = \bigcap_{k>0} (A_1 \circ A_2 \circ A_3 \circ \ldots \circ A_k)(\Delta_{n-1}) \). That is, the invariant Borel probability measures for the interval exchange induced by the neighborhood “are” exactly the asymptotic range of the sequence of splitting matrices for the neighborhood.

**Proof.** Let us first define \( \ell_k \) to be the component of \( \ell \cap N_k \) containing the cusps of the neighborhood \( N_k \). Thus, \( \ell_k \) is the interval of \( \ell \) on which the \( k \)th exchange induced by \( N_k \) is taking place. This language is also used in the context of interval exchanges, where an equivalent process is known as “inducing on a subinterval”. Let \( \ell_{k,i} \) be the \( i \)th subinterval of \( \ell_k \), where the \( n \) labels of subintervals are assigned as in the discussion of the splitting process.

If \( x_0 \) is the evaluation of \( m \in \mathcal{M} \), then the evaluation \( x_1 \) of \( m \) on \( \ell_1 \) satisfies \( A_1x_1 = x_0 \) as discussed in the section on splitting. (If we do not projectivize the
action of the matrix onto the simplex.) Then, if we normalize $x'_1$ so that it lives in $\Delta_{n-1}$, and call it $x_1$, and repeat the process, we obtain inductively a sequence $(x_0, x_1, x_2, \ldots)$ of vectors each of which is the normalized evaluation of $m$ on the corresponding $\ell_k$, and which satisfy $A_i(x_i) = x_{i-1}$. Thus, $x_0 \in \bigcap_{k>0} (A_1 \circ A_2 \circ A_3 \circ \ldots \circ A_k)(\Delta_{n-1})$, since we have produced a sequence of preimages of $x_0$ all in $\Delta_{n-1}$.

The other inclusion remains to be shown.

Supposing now that $x_0 \in \bigcap_{k>0} (A_1 \circ A_2 \circ A_3 \circ \ldots \circ A_k)(\Delta_{n-1})$, we define a function
\( \mu_x \) on certain subintervals \( \ell_{0,i} \) of \( \ell_0 \) by \( \mu_x(\ell_{0,i}) = x_i \). We then recursively define \( \mu_x \) on all \( \ell_{k,i} \) by the formula \( \mu_x(\ell_k) = A_{k+1}\ell_{k+1} \), with \( \ell_k \) the vector with entries \((\ell_{k,i})\). If we now also set the function \( \mu_x \) to zero on the cusp points and endpoints of the interval, define \( \mu_x \) by equality on sets holononomous to the \( \ell_{k,i} \), and impose the proper additive properties for a measure, we have a suitable function defined on open sets of \( \ell \) which has a unique extension to a Borel measure by elementary measure theory. We conveniently call this measure \( \mu_x \). \( \mu_x \) is obviously invariant under the interval exchange \( \tau \) on either open or closed sets of \( \ell \), by its definition, and invariance on any set follows. A slicker way of seeing the invariance is the following: set \( \nu(A) = \mu_x(\tau(A)) \). \( \mu_x \) and \( \nu \) coincide on open sets, hence on Borel sets, and are the same because the extension to a Borel measure is unique. Thus \( \mu_x(A) = \nu(A) = \mu_x(\tau(A)) \).

\[ \square \]

**Ergodicity**

Upon consideration of the action of the elementary matrices which arise in the course of a generic splitting, it seems quite reasonable that the asymptotic range of the sequence of matrices should be the singleton initial vector \( x_0 \), which was the evaluation of Lebesgue measure. Each of the matrices which occurs shrinks the volume of \( \Delta_{n-1} \) by a factor of two, and because the singular leaves are dense, each strip is split apart infinitely many times, and each strip induces infinitely many splittings. In the case in which the sequence of matrices is periodic, we have a
pseudo-Anosov foliation, as per [13] and others. Multiplying the periodic stretch of matrices together, we have an eventually positive matrix repeated infinitely often [10], and by the Perron-Frobenius theorem, in this case any vector in $\Delta_{n-1}$ converges to the unique positive normalized eigenvector. It has been shown that in some special cases, however, the asymptotic range is not trivial.

Every invariant measure is a convex combination of ergodic invariant measures, as is well known from ergodic theory. Keane [11] showed in 1975 that all minimal exchanges on 2 or 3 sub-intervals were in fact uniquely ergodic. Minimal exchanges are those for which each singular orbit is dense. (Singular points are the discontinuity points of the interval exchanges which are in direct correspondence to the cusp points of the neighborhood which are on $\ell$ in our setting, and singular orbits being dense corresponds to singular leaves being dense.) In the case of 2 subintervals, he noted that a minimal interval exchange was basically an irrational rotation on the circle, known for some time to be uniquely ergodic. (This result is “Weyl’s Theorem” [3]) It turns out that the case of 3 subintervals reduces to the case of two. He conjectured then that in fact, any minimal exchange was uniquely ergodic.

Roughly two years later, counter examples were produced with 5 subintervals. Keane [12] used different methods to improve the counter example to 4 subintervals later that year, and revised his conjecture to “almost all minimal exchanges are uniquely ergodic”, noting that the counter examples were very special number theoretically. This was in fact proved by Veech [19] and others in 1982. In terms of the
sequence of splitting matrices, Kerckhoff [13] has shown that “simplicial processes are generically normal”, that is, that every finite block of matrices which can occur does occur, infinitely often, in almost every case. It is easy to see how this implies unique ergodicity, as the asymptotic range of the sequence is crushed to a singleton by one repeating block which yields a positive matrix. Interspersing with more matrices will not increase the size of the asymptotic range. What seems to go wrong in the measure zero case of non-unique ergodicity is that there is an exponentially diminishing frequency with which certain strips are split and with which others cause splitting, and the directions corresponding to these which are not crushed out by the action of the matrices. For an example, again, see Keane’s second paper [12], and for good intuition on the genericity of the usual case, Kerckhoff’s paper [13] is very nice.

From ergodic theory, the other formulation of what is occurring is that almost always, every leaf of the lamination or foliation asymptotically distributes according to the original Lebesgue measure on the interval, see [15] or any number of other books with some basic ergodic theory. In the non generic case, leaves fail to do this, even when the system is minimal. It is fascinating to ponder the nature of the minimal and non-uniquely ergodic laminations — imagine the leaves as being fragile — perturbing the weight vectors in the direction of the sub-simplex of invariant measures, one preserves the topological structure, the ordering of the leaves. Any perturbation in the generic case immediately breaks the leaves as they have to jump
over each other to match the new weight vector.

Keane shows [11] that there are no more than \( n + 2 \) ergodic invariant probability measures for a minimal exchange on \( n \) intervals, and remarks that by a refinement of the argument, one may decrease this bound to \( n \). The sharpest bound comes in at \( n/2 \), given by Mane, [15] and it is not known whether this is sharp for all even integers or not, clearly it is for \( n \) equal to 2 and 4, since for \( n = 2 \) and 3, we have Lebesgue measure uniquely ergodic, and Keane gave examples with \( n = 4 \).

**The Invariants**

**Definition 4.2.** Two sequences of integral matrices \((A_i)\) and \((B_i)\) are said to be *weakly equivalent* if there exist sequences \((S_i)\) and \((T_i)\) of non-negative integral matrices and increasing sequences \((n_i)\) and \((m_i)\) so that \( S_i \circ T_i = A_{n_{i+1}} \circ \ldots \circ A_{n_i} \) and \( T_i \circ S_{i+1} = B_{m_{i+1}} \circ \ldots \circ B_{m_i} \) for \( i > 1 \). That is, a commuting diagram as follows exists.

\[
\begin{array}{cccccccccc}
Z^n & A_1 & A_2 & \cdots & A_{n_1} & A_{n_1+1} & \cdots & A_{n_2} & A_{n_2+1} & \cdots \\
\downarrow S_1 & \downarrow T_1 & \downarrow S_2 & \downarrow T_2 & \downarrow S_3 & \cdots \\
Z^n & A_1 & A_2 & \cdots & A_{n_1} & A_{n_1+1} & \cdots & A_{n_2} & A_{n_2+1} & \cdots \\
B_1 & B_2 & B_{m_1} & B_{m_1+1} & B_{m_2} & B_{m_2+1} & \cdots & B_{m_3} & B_{m_3+1} & \cdots
\end{array}
\]

We are now ready to apply the machinery we have developed, and state the main results of this thesis as Theorem 4.3 and its corollaries, namely;
Theorem 4.3. If \( \Lambda = \Lambda(x, \sigma, \epsilon) \) is homeomorphic to \( \Lambda' = \Lambda'(x', \sigma', \epsilon') \), and they are orientable, the corresponding sequences of splitting matrices \( (A_i) \) and \( (B_i) \) are weakly equivalent.

Here, recall that an interval exchange is defined by the data \( (x, \sigma, \epsilon) \) where \( x \) is the weight vector describing the lengths of the subintervals, \( \sigma \) is the permutation by which the subintervals are reordered, and \( \epsilon \) is a vector of +’s and -’s describing whether the suspension twists the subinterval or maintains its order.

We delay briefly the proof of the theorem in order to state several corollaries which alongside Theorem 4.3 are the main results of the thesis.

Corollary 4.4. If the orientable lamination \( \Lambda = \Lambda(x, \sigma, \epsilon) \) is homeomorphic to \( \Lambda' = \Lambda'(x', \sigma', \epsilon') \), there is a matrix \( M \) in \( GL_n(\mathbb{Z}) \) so that \( M(E(M(x, \sigma, \epsilon))) = E(M(x', \sigma', \epsilon')) \).

Proof. Recall that \( GL_n(\mathbb{Z}) \) is the set of invertible \( n \times n \) integer matrices, thus it is the set of \( n \times n \) integral matrices with determinant \( \pm 1 \). Take \( M = B_1 \circ B_2 \circ B_3 \circ \ldots \circ B_j \circ S_1^{-1} \) in the weak equivalence diagram. \( M \) must take one asymptotic range to the other because the diagram commutes. We could back the asymptotic range up as far as we wanted on the top level, push across by one of the connecting matrices, and then push forward. The further \( \Delta_{n-1} \) gets pushed forward in the sequence, the closer it is to the asymptotic range of the sequence, so in the limit we have containment, but since the diagram commutes, we have containment to start with. Then, since the argument is wholly reversible, we have
the other containment as well. Note that each $S_i$ and $T_i$ have determinant ±1 by the multiplicativity of determinants and the fact that they are integral matrices, so $M$ also has determinant ±1.

\textbf{Corollary 4.5.} \textit{In case the asymptotic range is a singleton, the first corollary may be read: if $\Lambda \cong \Lambda'$ is orientable there is an $n \times n$ integer matrix $M$ with determinant ±1 so that $M(x) = x'$.}

Let us denote by $Q[x]$ the extension field over the rationals by the entries of the vector $x$.

\textbf{Corollary 4.6.} \textit{If the asymptotic range is a singleton, and $\Lambda \cong \Lambda'$ is orientable, then $Q[x] = Q[x']$.}

\textbf{Proof.} This corollary follows from $M(x) = x'$. Then, $b_{i,1}x_1 + b_{i,2}x_2 + \ldots + b_{i,n}x_n = \lambda x'_i$ for some scalar $\lambda$ which is killed by the projective part of the projective linear map $B$. Now since $\lambda$ is simply the sum of the entries of $Bx$, it is clear $x'_i$ is in the extension field $Q[x]$, and the logic is the same with $B^{-1}$ for the other inclusion. Both these corollaries are true for laminations on non-orientable surfaces as well as long as the asymptotic range is a singleton. □

Thus, we have several algebraic invariants for given topological classes of laminations: weak equivalence of splitting matrices, a special matrix relating invariant measures supported by the laminations, and the unicity of the extension field over the rationals by the entries of the weight vectors. The proof of the main result is aided by the following notational definitions and series of lemmas.
Let $X_m$ denote the $m$th coordinate space of $\lim(X_i, f_i) \cong \Lambda$ and $X'_m$ the $m$th coordinate space of $\lim(X'_i, f'_i) \cong \Lambda'$. Each of these is a wedge of $n$ circles, where $n$ is the number of strips in the neighborhoods. The number of circles is the same in either case because the Čech cohomology must agree, and in our setting is just $\mathbb{Z}_n$. We fix from the beginning bounded convex metrics on each $X_m$ and $X'_m$. We label the branch points of $X_m$ and $X_m'$ by $p_m$ and $p'_m$.

Let $\Psi$ be the homeomorphism induced between $\lim(X_i, f_i) \cong \Lambda$ and $\lim(X'_i, f'_i) \cong \Lambda'$ by the homeomorphism between $\Lambda$ and $\Lambda'$.

Let $X_{n,i}$ denote the 1-cell which is the $i$th circle of $X_n$ less the branch point $p_n$, and similarly $X'_{n,i}$.

Notice that there are monotone functions $f^{-1}_{n,i} : X_{n,i} \to X_{n+1,i}$ with $f_{n,i} \circ f^{-1}_{n,i}$ equal to the identity on $X_{n,i}$, because at each stage of the splitting, the map $f_n$ induced by inclusion on the nested neighborhoods maps each circle onto the corresponding circle at the next level. Likewise we have such maps on $X'_{n,i}$.

Let $i_n : X_n \setminus \{p_n\} \to \lim(X_i, f_i)$ be given by $i_n(x) = ((f_1 \circ \cdots \circ f_n)(x), \ldots, f_n(x), x, f_{n,i}^{-1}(x), (f_{n+1,i}^{-1} \circ f_{n,i}^{-1})(x), \ldots)$ for $x \in X_{n,i}$. We define $i'_n$ in the analogous fashion, taking $X'_{n,i} \setminus \{p'_n\} \to \lim(X'_i, f'_i)$.

Let $\psi_{n,k,\epsilon} : X_n \setminus B_\epsilon(p_n) \to X'_k$ be given by $\psi_{n,k,\epsilon}(x) = (\pi_k \circ \Psi \circ i_n)(x)$, and similarly $\psi^{-1}_{n,k,\epsilon} : X'_n \setminus B_\epsilon(p'_n) \to X_k$ by $\psi^{-1}_{n,k,\epsilon}(x) = (\pi_k \circ \Psi^{-1} \circ i'_n)(x)$.

**Lemma 4.7.** Given $\delta$ small, $k$, there is an $N$ such that if $n \geq N$ there is an $\epsilon_n$ such that we may define $\Psi_{n,k,\epsilon} : X_n \to X'_k$ by extending $\psi_{n,k,\epsilon}$ across the gap...
$B_\epsilon(p_n)$, with the property that $\text{diam}(\Psi_{n,k,\epsilon}(B_\epsilon)) < \delta$ if $\epsilon < \epsilon_n$. (This is for any $\epsilon$-ball in $X_n$.)

**Proof.** First, we show that there is an $\epsilon_n$ such that if $\epsilon < \epsilon_n$, $\psi_{n,k,\epsilon}$ takes balls of radius $\epsilon$ inside balls of radius $\delta$.

i) $p'_k$ and $\Psi$ are continuous, so there is an $\xi$ such that if $d(x,y) < \xi$ then $d'_k((\pi'_k \circ \Psi)(x), (\pi'_k \circ \Psi)(y)) < \delta/4$.

ii) We take $N$ large enough so that $\sum_{k \geq n} \text{diam}(X_i)/2^{i+1}$ is less than $\xi/2$ for any $n \geq N$. We then take $\epsilon_n$ so small that $d_0((f_1 \circ \ldots \circ f_n)(x), (f_1 \circ \ldots \circ f_n)(y)) < \frac{\epsilon}{2(n+1)}$, $d_1((f_2 \circ \ldots \circ f_n)(x), (f_2 \circ \ldots \circ f_n)(y)) < \frac{2\epsilon}{2(n+1)}$, $\ldots$, $d_n(x,y) < \frac{2^n \epsilon}{2(n+1)}$ if $d_n(x,y) < \epsilon$ for any $\epsilon < \epsilon_n$. We may do this by the continuity of each of the compositions. Then, for $x,y$ with $d_n(x,y) < \epsilon < \epsilon_k$, we have $d(i_n(x), i_n(y)) < \xi$ since by choice of $\epsilon_k$, the first $n+1$ coordinates yield a difference of at most $\xi/2$ and we have chosen $k$ large enough so that the tail ends of the sequences differ by at most $\xi/2$. Then, by the first item in this proof, we have that $\psi_{n,k,\epsilon}$ takes any ball of diameter $\epsilon$ inside a ball of diameter $\delta/2$ for every $\epsilon < \epsilon_n$.

In particular, since the endpoints of $X_n \setminus B_\epsilon(p_n)$ are within $2\epsilon$ of each other, they are taken to within $\delta$. Thus, we may extend $\psi_{n,k,\epsilon}$ across $B_\epsilon(p_n)$ to $\Psi_{n,k,\epsilon}$ by first defining $\Psi_{n,k,\epsilon}(p_n)$ to be the same as $\psi_{n,k,\epsilon}$ of some endpoint of $X_n \setminus B_\epsilon(p_n)$, and extending linearly on the arcs of $B_\epsilon(p_n)$. This is well defined for small $\delta$, since all the endpoints land in a $\delta$ ball. $\Psi_{n,k,\epsilon}$ then has the desired property of taking $\epsilon$-balls inside of $\delta$-balls. \qed
Lemma 4.8. Given $k, \delta$, there is an $N$ so that for $n > N$ there is $\epsilon_n$ such that

if $\epsilon < \epsilon_n$, for any $x \neq p_n$ there is $\epsilon' < \epsilon$ so that $d_k(\Psi_{n,k,\epsilon}(x), \Psi_{n,k,\epsilon'}(x)) < \delta$.

Proof. We take $N$ large enough, and $\epsilon_n$ small enough, as in the previous lemma, so that the diameter of any $\epsilon$ ball under $\Psi_{n,k,\epsilon}$ is less than $\delta/2$. Let

$$\Psi_{n,k,0} = p'_k \circ \Psi \circ \iota_n : X_n \setminus \{p_n\} \to X'_k.$$ Note that the proof of the last lemma also says that $\text{diam}(\Psi_{n,k,0}(B_\epsilon)) < \delta$ for the same $n, \epsilon < \epsilon_n$ as $\Psi_{n,k,\epsilon}$. For $x \neq p_n$ in $B_\epsilon(p_n)$, we take $\epsilon' < \epsilon$ small enough so that $x \notin B_{\epsilon'}(p_n)$. Then,

$$d_k(\Psi_{n,k,\epsilon}(x), \Psi_{n,k,\epsilon'}(x)) \leq d_k(\Psi_{n,k,\epsilon}(x), \Psi_{n,k,\epsilon}(y)) + d_k(\Psi_{n,k,\epsilon}(y), \Psi_{n,k,0}(y)) + d_k(\Psi_{n,k,0}(y), \Psi_{n,k,0}(x)) + d_k(\Psi_{n,k,0}(x), \Psi_{n,k,\epsilon'}(x))$$

where $y$ is an endpoint of $B_\epsilon(p_n)$ within $\epsilon$ of $x$. The first and third quantities are made small by lemma 2 and the remark above regarding $\text{diam}(\Psi_{n,k,0}(B_\epsilon))$. The second and last are zero, since off $B_\epsilon(p_n)$, the maps $\Psi_{n,k,\epsilon}$ and $\Psi_{n,k,0}$ coincide, and off $B_{\epsilon'}(p_n)$, $\Psi_{n,k,\epsilon'}$ and $\Psi_{n,k,0}$ coincide. The lemma is thus proved, since as mentioned, for $x \notin B_\epsilon(p_n)$ the inequality is trivial. \(\square\)

Let $f_{m,n}$ denote the composition $f_m \circ f_{m-1} \circ \ldots \circ f_{n+1}$.

Lemma 4.9. Given $n, \delta$, we can find $k, \epsilon_k, M$ so that for every $m \geq M$ we can find $\epsilon_m$ so that if $\epsilon_1 < \epsilon_k$ and $\epsilon_2 < \epsilon_m$, the following diagram commutes to within $\delta$. 
Proof. The previous lemmas apply to all the analogously defined maps from the $X'_k$ to the $X_n$. We take $k$ large enough as in lemma 2, and also large enough so that $d_n((\Psi_{k,n,e_1}^{-1} \circ \Psi_{m,k,e_2})(x), f_{m,n}(x)) < \epsilon/4$ for $x \notin B_{e_2}(p_m) \cup (\Psi_{m,k,e_2})^{-1}(B_{e_1}(p'_k))$.

To do this, note that $\Psi_{k,n,e_1}^{-1} \circ \Psi_{m,k,e_2}(x) = \pi_n \circ \Psi^{-1} \circ \pi'_k \circ \Psi \circ i_m(x)$ when $x \notin B_{e_2}(p_m) \cup (\Psi_{m,k,e_2})^{-1}(B_{e_1}(p'_k))$.

i) $d'(i'_k \circ \pi'_k(\Psi(y)), \Psi(y))$ is uniformly small dependent on $k$, since they differ only in the tails of the sequences.

ii) $d(\Psi^{-1}((i'_k \circ \pi'_k \circ \Psi)(y)), \Psi^{-1}(\Psi(y))) = d(\Psi^{-1}((i'_k \circ \pi'_k \circ \Psi)(y)), y)$ may be made small by uniform continuity of $\Psi^{-1}$.

iii) $d_n(\pi_n \circ \Psi^{-1} \circ i'_k \circ \pi'_k \circ \Psi(y), \pi_n(y))$ may be made small by uniform continuity of $\pi_n$.

Thus, by appropriate choice of $k$ the distance between $(\pi_n \circ \Psi^{-1} \circ i'_k \circ \pi'_k \circ \Psi \circ i_m)(x)$ and $(\pi_n \circ i_m)(x)$ may be made as small as wished for $x \notin B_{e_2}(p_m) \cup (\Psi_{m,k,e_2})^{-1}(B_{e_1}(p'_k))$, independent of $m$, and note that $\pi_n \circ i_m = f_{m,n}$. 


We choose $\epsilon_k$ as in the preceding two lemmas so that if $\epsilon_1 < \epsilon_k$, $\text{diam}(\Psi_{k,n,\epsilon_1}^{-1}(B_{\epsilon_1})) < \delta/4$ and for $x \neq p'_k$ we can find $\epsilon' < \epsilon_1$ so that $d_k(\Psi_{n,k,\epsilon_1}(x), \Psi_{n,k,\epsilon'}(x)) < \delta/4$.

We get $M$ as in lemma 4.9, and for any $m \geq M$, $\epsilon_2 < \epsilon_m$ given by that lemma so that $\text{diam}(\Psi_{m,k,\epsilon_2}(B_{\epsilon_2})) < \epsilon_1$ if $\epsilon_2 < \epsilon_m$, also small enough so that if $d_m(x,y) < \epsilon_2$, $d_n(f_{m,n}(x), f_{m,n}(y)) < \delta/4$.

Now we consider several cases for $d_n(\Psi_{k,n,\epsilon_1}^{-1} \circ \Psi_{m,k,\epsilon_2}(x), f_{m,n}(x))$.

1) If $x \in B_{\epsilon_2}(p_m) \setminus \{p_m\}$ and there is a point $y \notin B_{\epsilon_2}(p_m)$ within $\epsilon_2$ of $x$ which is not taken to $B_{\epsilon_1}(p'_k)$ by $\Psi_{m,k,\epsilon_2}$, then $d_n((\Psi_{k,n,\epsilon_1}^{-1} \circ \Psi_{m,k,\epsilon_2})(x), f_{m,n}(x)) \leq d_n((\Psi_{k,n,\epsilon_1}^{-1} \circ \Psi_{m,k,\epsilon_2})(x), (\Psi_{k,n,\epsilon_1}^{-1} \circ \Psi_{m,k,\epsilon_2})(y)) + d_n((\Psi_{k,n,\epsilon_1}^{-1} \circ \Psi_{m,k,\epsilon_2})(y), f_{m,n}(y)) + d_n(f_{m,n}(y), f_{m,n}(x))$.

The first quantity is less than $\delta/4$ by the control lemma 4.7 gives us on diameters of images of $\epsilon_2$-balls, based upon appropriate choices of the parameters.

The second is less than $\delta/4$ by choosing $k$ as per i), ii) and iii) at the beginning of this proof, since $y \notin B_{\epsilon_2}(p_m) \cup (\Psi_{m,k,\epsilon_2})^{-1}(B_{\epsilon_1}(p'_k))$. The last quantity is made less than $\delta/4$ by choosing $\epsilon_2$ based upon the uniform continuity of $f_{m,n}$.

2) If $x \in B_{\epsilon_2}(p_m) \setminus \{p_m\}$ but there is no point $y$ within $\epsilon_2$ of $x$ which is not taken to $B_{\epsilon_1}(p'_k)$, we may shrink $\epsilon_1$ to $\epsilon'$ so that there is a point $y \notin B_{\epsilon_2}(p_m)$ within $\epsilon_2$ of $x$ which is not taken to $B_{\epsilon'}(p'_k)$. This may be done since $\Psi_{m,k,\epsilon}$ is a local homeomorphism off $B_{\epsilon}(p_m)$. We see that this is the case because all the $f_k$ are strictly increasing, so $i_m$ takes arcs in $X_m$ to arcs of $\ell_{\text{lim}}(X_i, f_i)$, which map under $\Psi$ to arcs of $\ell_{\text{lim}}(X'_i, f'_i)$, which if they are sufficiently small, map
to arcs of $X_k'$ under $\pi'_k$. Thus, we may choose a point $y$ outside $cl(B_e(p_m))$ but within $\varepsilon_2$ of $x$ which does not map to $p'_k$, hence missing $B_{e'}$ for small enough $e'$. (We take $y \notin cl(B_e(p_m))$ because then even if we should happen to pick a point which maps to $p'_k$ other points arbitrarily close do not.)

Then, for such an $x$, we pick $y, e'$ in such a fashion that $d_n((\Psi_{k,n,\varepsilon_1}^{-1} \circ \Psi_{m,k,\varepsilon_2})(x), f_{m,n}(x)) \leq d_n((\Psi_{k,n,\varepsilon_1}^{-1} \circ \Psi_{m,k,\varepsilon_2})(x), (\Psi_{k,n,\varepsilon_1}^{-1} \circ \Psi_{m,k,\varepsilon_2})(y)) + d_n((\Psi_{k,n,\varepsilon_1}^{-1} \circ \\
\Psi_{m,k,\varepsilon_2})(y), (\Psi_{k,n,\varepsilon_1}^{-1} \circ \Psi_{m,k,\varepsilon_2})(y)) + d_n((\Psi_{k,n,\varepsilon_1}^{-1} \circ \Psi_{m,k,\varepsilon_2})(y), f_{m,n}(y)) + \\
\text{These quantities are small as in the previous case, except for the second, which is controlled by choice of parameters as in the last lemma.}

3) The case of $x \notin B_{e_3}(p_m)$ but which maps to $B_{e_1}(p'_k)$ admits the same triangle inequality as the last case.

4) Finally, off $B_{e_2}(p_m) \cup (\Psi_{m,k,\varepsilon_2})^{-1}(B_{e_1}(p'_k))$, as mentioned earlier, $d_n((\Psi_{k,n,\varepsilon_1}^{-1} \circ \\
\Psi_{m,k,\varepsilon_2})(x), f_{m,n}(x)) \leq \delta/4$ by choice of $k$.

Thus, we have the advertised $\delta$-commuting diagram everywhere except at $p_n$, and it is easy to see that a diagram of continuous maps which $\delta$-commutes on a dense set $\delta$-commutes everywhere. \[\square\]

Thus girded with lemmas, we proceed to the proof of Theorem 4.3.

**Proof.** We construct the following "nearly commuting" diagram, that is, given any sequence $(\varepsilon_i) \rightarrow 0$ we can produce a diagram which commutes to within $\varepsilon_i$ on the $i_{th}$ triangles. (The $i_{th}$ triangles being $f_{n_i,n_{i-1}+1} \approx s_i \circ t_i$ and $f'_{m_i,m_{i-1}+1} \approx t_{i-1} \circ s_i$)
We produce this diagram by repeated use of the last lemma: Starting with $n_0 = 0$ as $n$ in the lemma, and $e_1$ as $\delta$, we choose $m_1$ as $k$ in the lemma, and are handed $e_1$ and $M_1$ so that any choice of $n_1 \geq M_1$ yields a $e_1$ for which there is a first triangle in the diagram with $s_1 = \Psi^{-1}_{m_1,n_0,e_1}$ and $t_1 = \Psi_{n_1,m_1,e_1}$ which $e_1$ commutes.

There is nothing preferential in the proofs of the preceding lemmas, so we may now turn to the next triangle, using $m_1$ as $n$ and $e_2$ as $\delta$, and fix $n_1 \geq M_1$ as $k$ in the lemma. We then get the parameters for the correct $t_1$ and a lower bound on what $m_2$ will be. We keep making these choices as indicated, and obtain the diagram as claimed.

Now, by choosing our sequence $(e_i)$ so that $e_i$ is small relative to the diameter of the various circles of $X_{m_i-1}$ and $X'_{m_i}$ which we fixed at the beginning, the following commuting diagram is induced in homology, with the $S_i$ and $T_i$ non negative.
The diagram commutes because homology does not detect a difference between maps which vary from each other by a small amount relative to the diameter of the cycles, as ours are chosen to do. To see that the $S_i$ and $T_i$ are non-negative, consider that except possibly on the $e$-balls about the branch points, the $s_i$ and $t_i$ are orientation preserving, since $\tau_k$, $\Psi$, and $i_k$ are. (If $\Psi$ isn't orientation preserving, we choose a $\Psi$ which is, or reverse the orientation on one of the laminations so that it is.) Thus, cycles are taken by $s_i$ and $t_i$ to non-negative combinations of cycles, since the sets on which they are not orientation preserving are small relative to the diameter of the cycles. Then, the matrices are non-negative as claimed. This is the weak equivalence diagram we were to produce, and the proof of the main theorem is complete. 

Another corollary unrelated contextually to the others, but easy to obtain is the following:

**Corollary 4.10.** There are uncountably many topological classes of measured laminations.

This is evident as the orbit of a vector under $GL_n(\mathbb{Z})$ is countable, and there
are uncountably many weight vectors to choose from.

Incompleteness

A natural question arises as to the completeness of the invariant. That is, given two laminations with weight vectors related by $GL_n(\mathbb{Z})$, are they necessarily homeomorphic? The answer is no, as shown in the following example.

**Example 4.1.** Let us take as a starting point two interval exchanges with rational weight vectors both $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})$, permutations (124) and (134), and no twists, that is, $\epsilon = (+, +, +, +)$. Figure 19 on the next page depicts the splitting, in which we see that the first yields an annulus, while the second separates into two annuli.

This example is special number-theoretically. For one, the leaves are all finite because the components of the weight vector are all rationally related. See figure 16 on the next page. In such a case, the exchange is not uniquely ergodic, as any number of invariant measures may be produced given so many periodic orbits. We might then ask, in the generic setting in which the exchange is minimal, and uniquely ergodic, is the invariant complete? The answer to this question is also no, as seen from the following considerations.

Recall that an asymptotic $k$-cycle of leaves is a collection $\{l_1, l_2, l_3, \ldots l_k\}$ of leaves of a lamination which are related by their half leaves' asymptotic behavior. These come from splitting open the interior singular leaves of the foliations, and this is the only way leaves in our laminations can be asymptotic, since the denseness of
Figure 16. Non-Homeomorphic Laminations from the Same Weight Vector.
the interior singular leaves guarantees that any two locally parallel leaves of the foliation have a singular leaf between them which kills any chance of asymptotic behavior between resulting leaves of the lamination. Asymptotic cycles of leaves are clearly topological; any homeomorphism of laminations must take $k$-cycles to $k$-cycles. The relevant point of this is that it is easy to see the $k$-cycles in our foliated neighborhood, or in the suspension of an interval exchange. If we trace around a boundary component and record the number of cusps, this gives rise to an asymptotic cycle of the same order as the number of cusps, as each cusp splits into a pair of leaves asymptotic in one direction.

It is also easy then to produce non-homeomorphic laminations with the same weight vectors, by changing the permutation, as it turns out the number and order of the cycles is determined solely by the permutation. For example, if we choose the permutation given by $\sigma(i) = i + 1$ and $\sigma(n) = 1$, we end up with a bunch of 2 cycles. On the other hand, if we take the permutation $\sigma(i) = i + 1, i > 2, \sigma(1) = 3, \sigma(2) = 2$, we get a 6-cycle, and the rest 2 cycles as $n$ allows.

Since our neighborhoods are orientable, we end up with even ordered cycles, and any combination of these is realizable by choosing the correct permutation. Another way of thinking of this whole issue is that to every interval exchange is associated a natural surface, given by the inverse of the splitting process — starting at a cusp of the suspension of an exchange, we sew the foliation back together, "zipping" together the exterior singular leaves to form $k$-prong singularities from $k$ cusped
boundary components of the suspension. The surface produced will be a rough sort of topological invariant of the exchange, since by the Poincaré-Bohl-Hopf Theorem we obtain an oriented manifold of genus \( \sum (1 - \frac{i_j}{2}) \) where \( i_j \) is the number of prongs at the \( j \)th singularity. Thus if two exchanges are homeomorphic, they must give rise to the same surface.

Of course the converse fails to hold, since unrelated weight vectors with the same permutations give rise to the same surface. It does however serve to illustrate the incompleteness of our invariant by easily detecting non-homeomorphic laminations with the same weight vector.

**Non-Orientable Foliations**

In many cases, when we start with a weighted train track and construct a partial foliation, we cannot choose a transverse arc so that the foliation can be realized as the suspension of an interval exchange transformation, because the foliation does not have an orientation. Likewise, the lamination arising from such a foliation is not orientable. This will be expressed in the situation of a strip leaving and returning to the proposed distinguished transversal on the same side. For this case, we construct an orientable double covering of the foliation, which will split to an orientable double cover of the lamination. Homeomorphisms of the laminations will lift to homeomorphisms of the double covers, and our invariant will still be good in this setting.
Constructing the Double Cover

A non-orientable foliation will manifest its lack of an orientation in the unfortunate situation of strips "leaving" and "returning" on the same side of any transversal. To correct this, we take two copies of the foliated neighborhood, cut any of the problematic strips in half, and glue them to the opposite half of their twins in the obvious way. More precisely, we number the strips from 1 to \( n \) in the first copy of the neighborhood, and mimic this order from \( n+1 \) to \( 2n \) in the second copy. Then, we parameterize each strip of the first copy by \((x, t) \in [0, \ell_i] \times (0, 1)\), and each strip of the second by \((x', t) \in [0, \ell_i] \times (0, 1)\), in such a way that \((x, t)\) and \((x', t)\) are twins in the two copies. See figure 17 on the following page.

Next, we cut the strips which leave and return on the same side at \( t = 1/2 \). Completing the half strips at \( t = 1/2 \), we have 2 points, \((x_1, 1/2)\) and \((x_2, 1/2)\) which come from every point \((x, 1/2)\) of the strips, which if identified, give us our original neighborhood. Likewise we have \((x'_1, 1/2)\) and \((x'_2, 1/2)\). We connect the two copies by the identification \((x_1, 1/2) \sim (x'_2, 1/2)\) and \((x_2, 1/2) \sim (x'_1, 1/2)\).

The resulting object is orientable — leaving from the distinguished arc in the first copy of the neighborhood, if the strip was not orientable, one proceeds to the same side of the other copy of the distinguished arc, travels around some number of times on already oriented strips, until another non-oriented strip is encountered, this time leaving from the opposite side of the arc, hence returning to the opposite side of the original arc.
Figure 17. Building a Double Cover.
To get the sort of neighborhood we want for our splitting process, we push the singularities from the second copy to the first copy. We now have the suspension of an interval exchange on $2n - 1$ intervals. This happens since there are $2(n - 1)$ cusps for an interval with $n$ strips, and we double the number of cusps to $2(2(n - 1)) = 4n - 4$. Then, the number of strips associated with an interval exchange with this many singular points is $(4n - 4)/2 + 1 = 2n - 1$.

**Results for the Non-Orientable Case**

Having now gotten a neighborhood in the form of the ones in the discussion of the splitting, we need to check a few things. First, we will see that the lamination arising from the new neighborhood is an orientable double cover of the lamination which would have arisen from splitting the original neighborhood. Then we verify that this double cover is the same as the one Fokkink describes in his thesis [7], wherein he also shows that homeomorphisms of "matchbox manifolds" lift to homeomorphisms of their double covers. Fokkink moreover proves that minimal non-orientable laminations have minimal double covers. Then, starting with two homeomorphic non-orientable laminations, we have homeomorphic double covers, and the theorem and corollaries have the analogous results denoting by $(x'_d, \sigma'_d, \epsilon'_d)$ the data of the interval exchange given by the orientable double cover:

**Theorem 4.11.** If $\Lambda$ is homeomorphic to $\Lambda'$, and they are not orientable, there is a weak equivalence diagram connecting the sequences of splitting matrices for the orientable double covers.
**Corollary 4.12.** If the non-orientable lamination $\Lambda$ is homeomorphic to $\Lambda'$, there is a matrix $M$ in $GL_n(\mathbb{Z})$ so that $M(E(M(x_d, \sigma_d, \epsilon_d))) = E(M(x'_d, \sigma'_d, \epsilon'_d))$.

**Corollary 4.13.** In case the asymptotic range is a singleton, the first corollary may be read: if $\Lambda \cong \Lambda'$ is not orientable there is an $n \times n$ integer matrix $M$ with determinant $\pm 1$ so that $M(x_d) = x'_d$.

**Corollary 4.14.** If the asymptotic range is a singleton, then $Q[x_d] = Q[x'_d]$.

As the double covers fit the hypotheses of the theorems at the beginning of the chapter, we need not offer proofs. Thus we proceed to prove the statements made concerning the double cover. First off we have:

**Theorem 4.15.** The double cover of the neighborhood $N_0$ given by the construction in this section splits open to an orientable lamination which is a double cover of the lamination which would result from splitting $N_0$.

**Proof.** The logic on this is straightforward. The lamination $\Lambda$ arising from the original neighborhood $N_0$ projects naturally onto $N_0$ in a 1-1 fashion except on interior singular leaves, which are projected to in 2-1 fashion by the half leaves of the asymptotic cycles. Likewise, the neighborhood $N_{0,d}$ which is the orientable double cover neighborhood we constructed, projects onto $\Lambda_d$. We denote the first projection by $p$, the second by $p_d$, and the 2-1 covering of $N_0$ by $N_{0,d}$ by $\pi$. Then the covering $\pi_\lambda$ of $\Lambda_d$ onto $\Lambda$ is easy to describe everywhere except on the asymptotic cycles, where the projections $p$ and $p_d$ are not 1-1. We define $\pi_\lambda(x) = p^{-1}(\pi(p_d(x)))$, for those points of $\Lambda_d$ which do not project to interior singular leaves of $N_{0,d}$, as these
points will be taken by \( \pi \) to points of \( N_0 \) which are also not on interior singular leaves, and there is no choice to be made by \( p^{-1} \). In the case of the other points, however, a consistent choice is not hard to describe: Moving "in" along a leaf of an asymptotic cycle in \( L_d \), we come eventually to a place where \( p_d \) is 1-1, having passed from the interior of the neighborhood out through the cusp it originates at, to one of two singular leaves on the boundary of \( N_d \). This tells us which leaf of the asymptotic cycle is the appropriate choice for \( p_1 \) in the definition of \( \pi_\lambda \). (Note that because every leaf of a lamination is dense in its component except in the uninteresting case, the definition of \( \pi_\lambda \) is gotten with no work on the asymptotic cycles, but we present the argument anyway in the spirit of the constructive and intuitive nature of this thesis.)

Now we turn to Fokkink's description of the double cover as given in his thesis [7].

**Definition 4.16.** A *matchbox manifold* \( X \) is a set locally homeomorphic to a zero-dimensional set crossed with an arc. A *matchbox* is such a neighborhood. An *orientation* on a matchbox is a continuous choice of positive direction on the matchbox.

Thus, a matchbox may be oriented in infinitely many ways, as the orientation must be constant in the arc direction, but given any partition of the zero dimensional set into clopen subsets, we may arbitrarily assign + or - to the elements of the partition.
Definition 4.17. A matchbox manifold $X$ is orientable if it admits a coherently oriented basis of matchboxes. (That is, two overlapping matchboxes induce the same orientation on every matchbox in their intersection.)

Definition 4.18. The orientable double cover of $X$, $\tilde{X} = \{(x, o_x)| x \in X\}$ where $o_x$ represents an orientation at $x$. There are two such orientations, given by the equivalence classes of orientated neighborhoods whose orientations agree in a neighborhood of $x$. The set $\tilde{X}$ is topologized by taking $V$ to be a matchbox of $X$ with orientation $o_V$. Then $\tilde{V}$ is a matchbox neighborhood in $\tilde{X}$, defined to be $\{(x, o_x)| x \in V, o_x \text{ is induced by } o_V\}$. The covering projection taking $(x, o_x)$ to $x$ is denoted by $\pi$.

Proposition 4.19. The double cover we construct, $\Lambda_d$, is homeomorphic to Fokkink's $\tilde{\Lambda}$.

Proof. Each oriented matchbox neighborhood of $\Lambda$ has two inverse images under the projection from $\tilde{\Lambda}$, which project down with opposite orientation. Likewise with the double cover we defined, $\Lambda_d$. We see that they are homeomorphic by producing a correspondence of their bases — Given an oriented matchbox neighborhood of $\tilde{\Lambda}$, we project down to $\Lambda$, and then pick the neighborhood of $\Lambda_d$ which projects down under $\pi_\Lambda$ with the correct orientation. To understand this, it helps to free ourselves of the notions we have about $\Lambda_d$ as an oriented lamination, and think of it as just being orientable. Then, we can arbitrarily orient neighborhoods and establish this correspondence. Having established it, we have made concrete how
Fokkink's double cover is oriented, the nature of which is somewhat obscure from his definitions.

The one thing to check to make the above correspondence good is that oriented matchbox neighborhoods of $A$ have two inverse images in $A_d$ which project down under $\pi_A$ with opposite orientation. This is clear from the picture of the construction of $A_d$. Looking at the picture, draw the two arcs with positive orientation in the double cover of the foliation which project to the same arc in the foliation. (The projection was denoted in the proof of the previous theorem by $\pi$.) They project down with opposite orientations. Given the way in which the definition of $\pi_A$ arises from the foliations, the same property holds for arcs of $A_d$. □

**Weak Equivalence**

An area of research that has seen some recent activity is the investigation of weak equivalence of matrices.

**Definition 4.20.** Two primitive integral matrices $A$ and $B$ are said to be *weakly equivalent* if there are non-negative integral matrices $S_i, T_i$ and sequences of positive integers $(m_i), (n_i)$ such that the following infinite diagram commutes:
Note that this is nearly the same diagram as in the proof of the main result of this thesis in the case that the splitting sequence is periodic. This equivalence has generated such interest because of results by Barge, Diamond, Jacklitch, Anderson and Putnam [2,10,1] and others indicating that homeomorphisms of various classes of 1-dimensional inverse limit spaces or tiling substitution spaces imply weak equivalence of sequences of associated transition matrices.

Swanson and Volkmer [17] have recently shown that for primitive (positive under some iterate) invertible integral $n \times n$ matrices $A$ and $B$, weak equivalence is the same as the existence of an invertible integral $n \times n$ matrix $S$ which satisfies:

1) $S$ maps the eigenvectors associated to the Perron eigenvalue of $A$ to the eigenvectors associated to the Perron eigenvalue $B$.

2) For each $n \in N$ there is $m \in N$ so that $B^{-n}S A^m$ is an integral matrix.

3) For each $m \in N$ there is $n \in N$ so that $A^{-m}S^{-1}B^n$ is an integral matrix.

What their theorem says in our case, is that for pseudo-Anosovs, there is no difference between the existence of an integral matrix relating the asymptotic ranges
and weak equivalence of the sequences. We do not need 2) and 3) since our splitting matrices all have integral inverses. In fact, the proof given by Swanson and Volkmer extends to our larger class of sequences of matrices with little change, and we have the following theorem:

**Theorem 4.21.** If \((A_i)\) and \((B_i)\) are sequences of matrices which arise as in our splitting process, then the sequence \((A_i)\) is weakly equivalent to \((B_i)\) (defined for this more general case below) if and only if there is an invertible integral matrix \(S\) taking the asymptotic range of \((A_i)\) to the asymptotic range of \((B_i)\).

Recall that two sequences \((A_i)\) and \((B_i)\) of matrices such as those which arise from splitting a foliation are weakly equivalent as per definition 4.2 if a commuting diagram of the following sort exists:

\[
\begin{array}{cccccccc}
Z^n & A_1 & Z^n & A_2 & \cdots & A_{n_1} & A_{n_1+1} & \cdots & A_{n_2} & A_{n_2+1} & \cdots \\
& S_1 & & & & T_1 & & & & & \\
& & S_2 & & & & T_2 & & & & \ \\
B_1 & Z^n & B_2 & Z^n & B_{m_1} & Z^n & B_{m_1+1} & \cdots & B_{m_2} & Z^n & B_{m_2+1} & \cdots & B_{m_3} & Z^n & B_{m_3+1} & \cdots \\
\end{array}
\]

A diagram which commutes is easy to obtain, one could take \(T_1\) to be \(S^{-1}B_1\) and similarly for the rest, to get a sequence of matrices which commute. However, there would be no guarantee these were non-negative, and a little bit more is needed for this. We prove the theorem in the general case, but note that in the generic uniquely
ergodic case, as the analog of the Perron eigenvectors, we have the singleton weight vectors which are the asymptotic range of the sequence.

**Proof.** Take \( k_1 \) large enough so that \((B_1 \circ \ldots \circ B_{k_1})(\Delta_{n-1})\) is close to the asymptotic range of \((B_i)\). Then, since \(S^{-1}\) maps the asymptotic range of \((B_i)\) to the asymptotic range of \((A_i)\), \((S^{-1} \circ B_1 \circ \ldots \circ B_{k_1})(\Delta_{n-1})\) will be positive. This is true because given any weight vector in the asymptotic range, we get the same topological lamination, so none of the asymptotic ranges include points on the boundary of \(\Delta_{n-1}\), which correspond to zero measure on a strip, which in turn would violate the property of the support of a lamination's measure being the whole lamination. Thus the closeness we need for the product \(B_1 \circ \ldots \circ B_{k_1}\) above is that which takes the image of the simplex near enough the asymptotic range to be all positive. This implies that \(T_1 = S^{-1} \circ B_1 \circ \ldots \circ B_{k_1}\) is a non-negative matrix, since it takes all the standard basis vectors to positive vectors. Similarly, we can take \(l_1\) large enough so that \((A_1 \circ \ldots \circ A_{l_1})(\Delta_{n-1})\) is very close to the asymptotic range of \((A_i)\). If it is close enough, then \((S \circ A_1 \circ \ldots \circ A_{l_1})(\Delta_{n-1})\) will be very near the asymptotic range of \((B_i)\) so that \((B_{k_1}^{-1} \circ \ldots \circ B_1^{-1} \circ S \circ A_1 \circ \ldots \circ A_{l_1})(\Delta_{n-1}) \subset \Delta_{n-1}\), hence \(S_1 = B_{k_1}^{-1} \circ \ldots \circ B_1^{-1} \circ S \circ A_1 \circ \ldots \circ A_{l_1}\) is also non-negative. Similarly we produce sequences \((S_i)\) and \((T_i)\), all non-negative integral matrices for which the diagram commutes. \(\square\)

**Decidability of Weak Equivalence**

A last note is in order. Bratteli, Jorgensen, Kim and Roush recently showed
that the weak equivalence of a stationary system is \textit{decidable}, that is, that there is an algorithm which can check in finitely many steps whether two given primitive integer matrices are weakly equivalent [4]. In their setting, weak equivalence is referred to as $C^*$-equivalence. It appears to be unknown whether the weak equivalence of two general sequences of matrices such as those which laminations induce is decidable or not. Perhaps an easier question to answer will be the decidability of whether two vectors are in the same orbit of $GL_n(Z)$, which would imply the weak equivalence of the related sequences as per our theorem above.

Given the recent interest in weak equivalence, this is a nice place to end the thesis, with an exact picture of where the invariant we developed, the relation of weight vectors by square integral matrices with determinant \( \pm 1 \), fits into the scheme of things, and a hope that further related results may be obtained by investigating the orbit structure of $GL_n(Z)$. 
REFERENCES CITED


