The topology of laminations
by Luther William Johnson

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences
Montana State University

Abstract:
A lamination of a surface is a one dimensional subset of the surface, generically locally the product of a Cantor set with an arc. Some laminations arise in conjunction with psuedo-Anosov maps of the surface, others are more general. This thesis addresses the question of detecting which laminations are topologically equivalent. By following the construction of Harer and Penner, we realize the lamination as an inverse limit on wedges of circles. Then, taking two such laminations which are topologically equivalent, a “nearly commuting” diagram is obtained, which induces a weak equivalence diagram in the homology maps. This in turn enables us to partially classify such laminations- if such laminations are equivalent, then the systems of matrices which arise are weakly equivalent. Another incomplete invariant which turns out to be equivalent to that of weak equivalence in this setting, is the relationship of the transverse measures the laminations support by a square integer matrix with determinant plus or minus one. Ergodicity of laminations is discussed, and the result is further refined in the case of uniquely ergodic laminations to the relationship of a unique weight vector associated to either lamination by such a. matrix.

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STATEMENT OF PERMISSION TO USE

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ABSTRACT

A lamination of a surface is a one dimensional subset of the surface, generically locally the product of a Cantor set with an arc. Some laminations arise in conjunction with psuedo-Anosov maps of the surface, others are more general. This thesis addresses the question of detecting which laminations are topologically equivalent. By following the construction of Harer and Penner, we realize the lamination as an inverse limit on wedges of circles. Then, taking two such laminations which are topologically equivalent, a "nearly commuting" diagram is obtained, which induces a weak equivalence diagram in the homology maps. This in turn enables us to partially classify such laminations- if such laminations are equivalent, then the systems of matrices which arise are weakly equivalent. Another incomplete invariant which turns out to be equivalent to that of weak equivalence in this setting, is the relationship of the transverse measures the laminations support by a square integer matrix with determinant plus or minus one. Ergodicity of laminations is discussed, and the result is further refined in the case of uniquely ergodic laminations to the relationship of a unique weight vector associated to either lamination by such a matrix.

CHAPTER 1

INTRODUCTION

The goal of this thesis is to topologically classify certain one dimensional subsets of surfaces called measured laminations by giving algebraic invariants for topological classes of these subsets. Generically, a lamination is locally the product of a zero dimensional set with an arc. As such, they have much in common with the generalized solenoids of Williams [20] and the matchbox manifolds Fokkink [7] studied in his thesis. Our laminations, however, are constrained to live on a surface, while many of the above mentioned objects do not admit an embedding into a surface, although they are the same as laminations locally. We limit our investigation to laminations on compact surfaces without boundary. The theory admits a straightforward extension to other surfaces, but the overhead of extra definitions and cases is awkward and we avoid it.

The study of laminations as a class of objects was begun in the 70's by Thurston [18] in his investigation of surface automorphisms — certain laminations can be viewed as the closure of the unstable manifold of the fixed points of pseudo-Anosov mappings, split apart. An Anosov map is a hyperbolic map which leaves invariant a pair of transverse foliations, (without singularities) known as the stable and unstable foliations. Invariance is meant in the sense that leaves are mapped to leaves, and the map contracts or expands the measures on these foliations by a factor known as the dilatation factor, the log of which is the topological entropy of the map. The canonical examples of such maps on surfaces are the maps of the torus given by projecting the action on the plane of integer matrices with determinant 1 and distinct real eigenvalues, where the invariant foliations are lines parallel to the eigenvectors.

Since there are few surfaces supporting foliations without singularities, the class of mappings investigated was broadened to pseudo-Anosov maps, which are hyperbolic except at finitely many singular points. These maps also have associated to them a pair of transverse foliations with singularities which are invariant under the mapping in the same sense as above, and are expanded and contracted in measure by some dilatation factor. Thurston's famous classification theorem states that within each homotopy equivalence class of irreducible diffeomorphisms of a surface there is a map which is either periodic or pseudo-Anosov. Here, irreducible means that the map doesn't fix any homotopy class of simple closed curves. If a map is reducible, then some power of the map will, modulo homotopy, send the complementary regions of the fixed curve to themselves, and the theorem applies to this iterate of the map on the complementary regions. Thus, a reducible map fixes some classes of simple closed curves, and a power of this map is homotopic to a map which is periodic or pseudo-Anosov on each complementary region of the surface with the curves removed [18].

In his work on surface automorphisms, starting with a pseudo-Anosov map of the surface, Thurston introduced an object called a weighted train track, which is a special one dimensional graph on the surface upon which the action of the mapping gives a Markov transition matrix, and a system of weights satisfying a "switch condition". The (log of the) Perron-Frobenius eigenvalue of the matrix gives the topological entropy of the mapping, which is the minimal entropy of any mapping in its homotopy class, and the associated eigenvector gives a system of weights associated to the edges of the track which will give rise to the transverse measure on the unstable foliation of the map.

Los [13] discusses an algorithm for producing the Markov matrix for a map of a surface based upon fitting an invariant train track to the action of the map. The Perron eigenvector of the incidence matrix he produces in a pseudo-Anosov case yields a weight vector satisfying the switch conditions on the track, and he uses this to characterize when a given mapping class has a pseudo-Anosov element in it, as opposed to a reducible or periodic element, in which case his algorithm does not produce a track.

We should not become too attached to the historical and dynamical origins of the train tracks however, as the study of laminations has broadened to consideration of sets which are not connected in any way to surface automorphisms. Starting with a train track on some surface, a weight vector which satisfies the "switch condition" (which we will discuss at length later) gives rise to a measured foliation on the surface. We develop a method for "splitting open" the singularities of such a foliation in such a fashion that we arrive at a measured lamination. Roughly speaking, the foliation is a surface of fabric, with leaves ending at singularities corresponding to infinite zippers, and our splitting procedure will amount to unzipping the zippers completely. The subset of laminations arising in this fashion which correspond to pseudo-Anosov mappings is important as a special class because much is known about pseudo-Anosov foliations and laminations, and examples are easy to come by in this class, but the set of vectors which corresponds to these laminations is of measure zero in the appropriate setting, which we will also discuss.

A means of obtaining the results of this paper is discussed later on which avoids the machinery of train tracks, but we begin with train tracks in spite of this, because of the ease of presentation train tracks afford, and because the concrete, constructive nature of the development offers a nice geometric intuition for the laminations arising from them. Therefore, the first section of the paper is devoted to developing the language of train tracks, measured foliations, and measured laminations.

Following this, we develop a splitting procedure which opens up a measured foliation by splitting apart the singular leaves of the foliation, the result of which is a measured lamination. By carefully controlling this splitting procedure, we realize the lamination as a nested intersection of a sequence of foliated subsets arising from the weighted track. From this we are afforded a description of the lamination as an inverse limit in a common procedure when a set can be written as an infinite intersection in a nice way.

The description of a lamination as an inverse limit allows us a good handle to grasp the topology of the lamination by, and we quickly arrive at the main result of the thesis: If two laminations gotten in such a manner are homeomorphic, their weight vectors are related by a square integer matrix with determinant 1. This result is arrived at by first getting the result that in case two such laminations are homeomorphic, the two sequences of "splitting matrices" are weakly equivalent. The splitting matrices arise during the splitting process, and essentially capture the order in which singular leaves wind around in the foliation, which determines the manner in which each of the successive foliated subsets injects into the preceding one. Weak equivalence of sequences of matrices has been studied fruitfully as a topological invariant for a variety of similar systems by a number of authors of late; see Barge and Diamond [2] for the weak equivalence of matrices associated to the inverse limit of unimodal interval maps, or Anderson and Putnam [1] for an invariant of substitution tilings, for example.

In the course of the development of the invariants in this paper, we also discuss the ergodicity of laminations, and the transverse measures they can possibly support. It turns out to be the case that almost every lamination is uniquely ergodic, that is, the transverse measure which descends from the weight vector on the track is the only invariant measure for the lamination under holonomy. After stating the main result and proving it, we examine some examples, and discuss further the special case of pseudo-Anosov laminations. Finally, we will talk about work other authors have been pursuing, and the relationship of this thesis to some of their results.

CHAPTER 2

DEFINITIONS

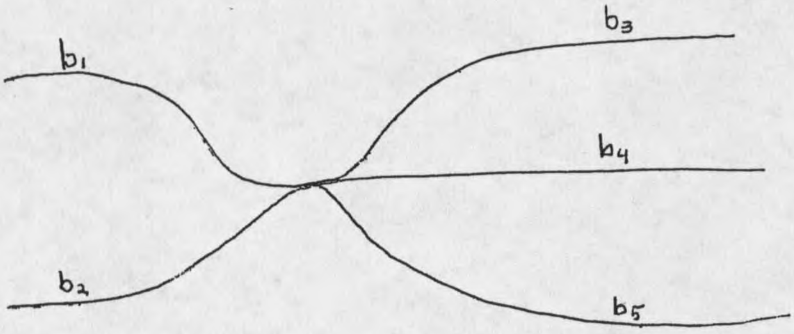
Train Tracks

DEFINITION 2.1. A *branched one manifold* is a one dimensional manifold except at finitely many points, where it has the topological type of a neighborhood of a point with finitely many rays emanating from it.

DEFINITION 2.2. A *train track* τ on a smooth surface M is a branched one manifold with continuously varying tangent directions and no relative boundary, smoothly immersed in M, whose complementary regions have negative Euler characteristic (see below for an explanation of this). We call the finitely many points at which τ is not locally an arc the *switches*, and the components of τ less the switches are called *branches*.

That the complementary domains have negative Euler characteristic is a technical requirement which will guarantee the construction of the foliation given in the next section is well defined. Put more concretely, this requirement is that no complementary domain is a disk with zero, one or two cusps, or an annulus with one. The restriction that the tangent direction vary continuously gives us a local picture at a switch like the following, where by way of an example, a 5-prong switch with branches $b_1, b_2, \ldots, b_5$ is depicted.

Figure 1. Train Tracks.



At each switch point we arbitrarily pick a positive direction in the tangent space, and refer to branches ending at the switch as either incoming or outgoing, depending upon the direction they approach the switch from.

DEFINITION 2.3. A *weight* on a train track is a function from the branches of the track to the non-negative reals which satisfies the *switch condition*, that the sum of the function values over the incoming branches at each switch equals the sum over the outgoing branches.

In figure 1 above, if a_i is the weight associated to the branch b_i, the switch condition is that $a_1 + a_2 = a_3 + a_4 + a_5$. The purpose of a train track with a system of weights satisfying the switch condition is for us the construction of a measured foliation. Roughly speaking, we will lay rectangles along the edges of the track with widths given by the weights, and the switch condition allows us to glue them together nicely. This construction will be discussed at length in the next chapter.

Foliations and Laminations

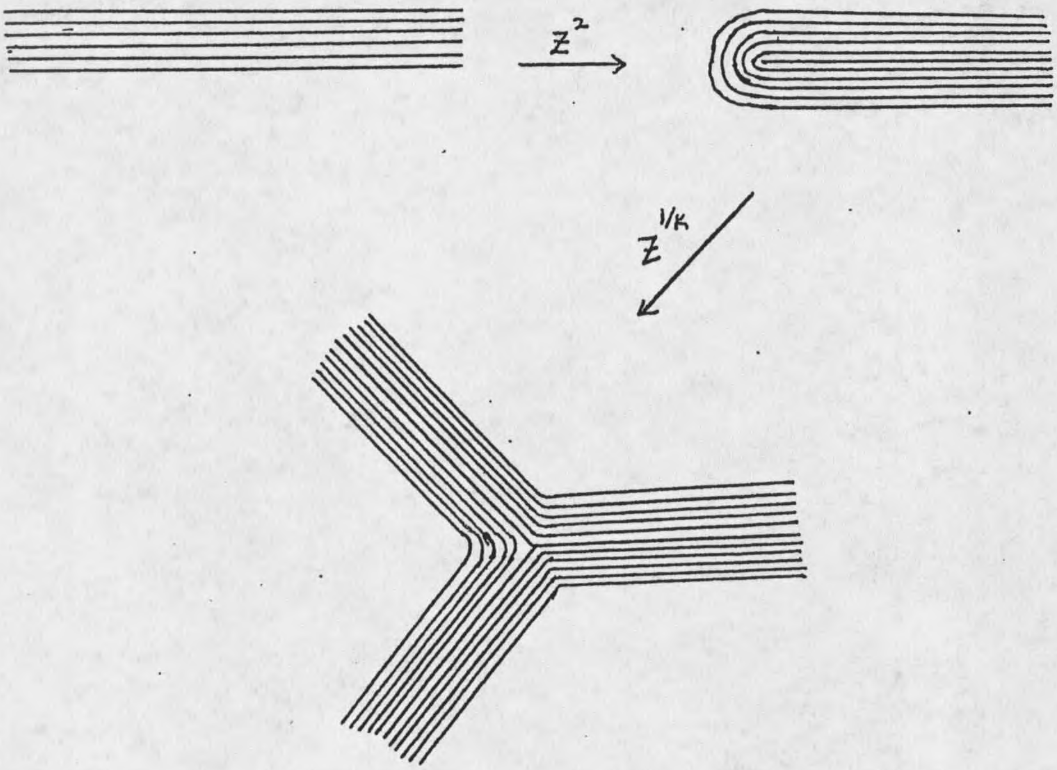
DEFINITION 2.4. A *foliation* F of a surface M is a system of smooth charts taking (0,1) $\times$ (0,1) into open sets of M which form a basis for M, so that for any two charts ϕ and ψ mapping to overlapping neighborhoods, the vertical component of $\phi^{-1} \circ \psi$ is constant in the horizontal direction. A *leaf* of the foliation F is an arc in M consisting of points which can be joined by arcs all of which have constant vertical components under any inverse chart.

In practice we often identify foliations with the images of the charts, and likewise with leaves. The image of a chart looks like a standard flow box, and finite pieces of leaves look like trajectories in a flow box. Since leaves are locally arcs with no boundary, they are 1-1 smoothly immersed copies of the real line or the circle. The canonical example of a foliated surface is the torus, foliated by the projection of lines of a given irrational slope onto the torus, thinking of the torus as gotten by identifying points in the plane which differ by integral vectors.

DEFINITION 2.5. A *foliation with singularities* is a foliation with a finite number of points at which the foliation does not admit a system of charts as above, but rather admits as a chart $z \longrightarrow z^{2/k}$ from the closed upper half of the complex plane.

Under this chart, the leaves are the images of the horizontal lines in the upper half plane. These points are known as k-prong singularities, where k for us is at least 3. A picture of a k-prong singularity with $k = 3$ is given in figure 2 on the next page. A way to visualize the action of $z \longrightarrow z^{2/k}$ is the following: take the upper

Figure 2. A 3-Prong Singularity.



half plane foliated by horizontal lines and square it, to obtain a 1-prong singularity at the origin; and then taking the preimages of this under $f(z) = z^k$.

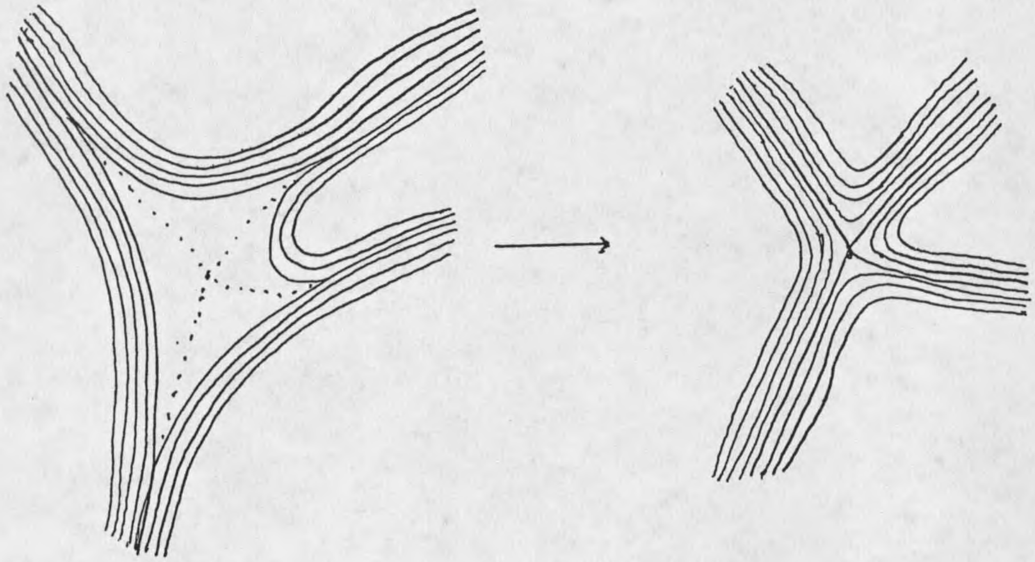
The Poincaré-Bohl-Hopf theorem is a famous result which states that the Euler characteristic of a surface is equal to the sum of the indices at the singularities of a foliation[6]. The Euler characteristic is a readily computable topological invariant for surfaces, and basic results of algebraic topology give us a formula to compute it for any closed surface without boundary. The index of a foliation at a fixed point is computed by taking a sufficiently small circle around the singularity, and defining a function from this circle to the unit circle by the direction of the tangent line to the leaf of the foliation. The index is given by the (positive or negative) number of times the function takes the circle around the unit circle. One may check that the index of a k-prong singularity is $1-\frac{k}{2}$. Since the only closed boundaryless surfaces with Euler characteristic zero are the torus and the Klein bottle, we see that there are only two surfaces with foliations which do not have singularities. Thus, unless it is otherwise stated, a foliation will henceforth mean a foliation with singularities. Now, leaves of such a foliation are circles or lines, as they are in the case of a foliation without singularities, with the exception of the singular leaves, which are those which end at a singularity of the foliation. These may now be closed arcs, if they begin and end at singularities, or rays, if they begin at a singularity and do not end.

DEFINITION 2.6. A *partial foliation* is a foliation of a closed 2 dimensional submanifold of the surface, where the submanifold has piecewise smooth boundary. We moreover require that non-singular leaves which intersect the boundary of the submanifold must be either transverse to it or smooth components of it.

We consider leaves intersecting the cusps to be singular. Given a partial foliation of a manifold, we may easily recover a foliation by placing a point in each complementary region, and disjoint arcs from the point to the cusps of the complementary region, then collapsing the region down to these arcs. If the region had k cusps, a k-prong singularity is formed, as in the picture on the next page.

DEFINITION 2.7. A *lamination* on a surface is, in our context, a subset of the surface which in the generic case is everywhere locally the product of an arc with a

Figure 3. Collapsing a Partial Foliation.



Cantor set. A *leaf* of a lamination is an arc component of the lamination. We will also allow as a special case a lamination which is locally the product of an arc with the interval, i.e. a foliation with no singularities. Leaves in this case are as they are in the definition of a foliation.

DEFINITION 2.8. A *transverse measure on a foliation* is a function m defined on the set of arcs transverse to the foliation with the following properties:

i) if s and t are such arcs having their respective endpoints on the same leaves, and are isotopic to each other through a set of arcs also with endpoints on the same leaves, m(s) = m(t).

ii) if $s_1, s_2, s_3, \ldots$ is a set of arcs transverse to the foliation, with $s_i \cap s_j = \emptyset$ for $i \neq j, j \pm 1$, and $s_i \cap s_j = \partial s_i \cap \partial s_j$ for $i = j \pm 1$, then $\sum m(s_i) = m(s)$ where $s = \cup s_i$, if such an s is measurable. (an arbitrary such union may fail to be transverse to the foliation.)

The intuitive idea of property one is that if we keep the endpoints of a transverse arc on given leaves, and slide it around in the foliation, we do not change the measure of the arc. Property two allows us to split transverse arcs into sub-arcs, and have the measure be additive. The combination of these allows us to split arcs by pushing them past cusps, for instance, and break them into smaller and smaller arcs while maintaining the sum of the measures.

The idea of a transverse measure on a lamination is similar to that of a transverse measure on a foliation. Both measures are a description of a foliation or lamination's thickness, which is invariant under "holonomy", that is, moving around in the foliation or lamination in the direction of the leaves.

DEFINITION 2.9. A *measure on a lamination* is a function μ defined on the set of arcs transverse to the lamination, but with endpoints in the complement of the lamination such that:

i) if s and t are such arcs, isotopic through a set of such arcs, $\mu(s) = \mu(t)$.

ii) if $s_i \cap s_j = \emptyset$ for $i \neq j, j \pm 1$, and $s_i \cap s_j = \partial s_i \cap \partial s_j$ for $i = j \pm 1$, then $\sum \mu(s_i) = \mu(s)$ where $s = \cup s_i$. (Here again, if this s is measurable.)

iii) the support of μ is the lamination in the sense that $\mu(s)$ is non zero if and only s intersects the lamination.

We see a third qualifier in the definition of the measure on a lamination, that the support is the lamination. This will rule out certain examples which are laminations, but not measured laminations, as we will see in a few paragraphs, which will be appropriate, as these laminations cannot arise as in the construction we will give.

DEFINITION 2.10. A *measured geodesic lamination* is a measured lamination which has all leaves straight with respect to some hyperbolic metric on the surface.

Harer and Penner [9] showed that each isotopy class of laminations has a unique geodesic representative. Two laminations are said to be isotopic if there is an isotopy of the surface which maps one lamination to the other.

As an interesting aside, another description of geodesic laminations is that they are the (Hausdorff) limit of long simple closed geodesics. As such, we see that not all geodesic laminations support a measure, for if there is "spiraling" behavior in a lamination, as we may easily imagine could happen in this description, the lamination will not support a measure. This is made clear in figure 4, as by the holonomy property of a measure, the two arcs s_1 and s_2 must have the same measure, because we could isotope s_1 to s_2, but s_2 misses a piece of the lamination that s_1 hits. This forces a violation of the property that the support of a measure is the lamination. Again, Harer and Penner have shown that any measured lamination is supported by a weighted track, and that weighted tracks give rise via the above construction to measured laminations, so none of the laminations that we consider