Students Use of Multiple Representations in Mathematical Problem Solving
by James William Ballard

A dissertation submitted in partial fulfillment of the requirement for the degree of Doctor of Education
Montana State University

Abstract:
This study investigated students use of mathematical representations and translations during probability problem solving. The purpose of the representation research was to identify the representations used by students, when they chose to use a representation, and how frequently and successfully they used the representations. The research looked for patterns of behavior that characterized students' successful and unsuccessful use of the representations. The research also identified the translations between the representations. The purpose of the translation research was to identify patterns of movement between representations that characterize the students' use of translations and identify the time spent using any particular representation. A qualitative research design was employed. The subjects (n = 21) were selected from a finite mathematics course. The data gathering included an interview to familiarize each volunteer with the research process, a problem-solving session in which each attempted to solve five probability problems while vocalizing the process, and a post-session interview in which students explained then-solution method.

This research demonstrates that there are distinct differences in the manner that successful and error-prone students use translations and representations. Successful problem solvers are able to analyze the problems and discover a solution method before translating, know when and how to use Venn diagrams, and use symbolic algebra competently. Error-prone problem solvers are often unable to discover a solution method and hence will initially use a Venn diagram (or other representations) to "discover" a solution method. Error-prone problem solvers do not separate, their use of translations and representations as distinctly (as do the successful solvers) and often do not finish a translation before attempting another translation. This indicates that the error-prone are unable to arrive at a conclusion on how to solve problems and do not understand (a) how a representation will clarify a problem, (b) the purpose of representations, and (c) which . representation to use.

Recommendations include the following: students need extensive practice translating between representations, need to understand what the various representations reveal, and need practice using many different representations. Further research is needed to identify when and how students should receive instruction with translations and representations.

STUDENTS' USE OF MULTIPLE REPRESENTATIONS IN MATHEMATICAL PROBLEM SOLVING

by

James William Ballard

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APPROVAL

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James William Ballard

This dissertation has been read by each member of the dissertation committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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Date

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Date 20 OCTOBER 2000

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ABSTRACT

This study investigated students use of mathematical representations and translations during probability problem solving. The purpose of the representation research was to identify the representations used by students, when they chose to use a representation, and how frequently and successfully they used the representations. The research looked for patterns of behavior that characterized students' successful and unsuccessful use of the representations. The research also identified the translations between the representations. The purpose of the translation research was to identify patterns of movement between representations that characterize the students' use of translations and identify the time spent using any particular representation. A qualitative research design was employed. The subjects (n = 21) were selected from a finite mathematics course. The data gathering included an interview to familiarize each volunteer with the research process, a problem-solving session in which each attempted to solve five probability problems while vocalizing the process, and a post-session interview in which students explained their solution method.

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CHAPTER 1

INTRODUCTION TO THE STUDY

Introduction to Representations and Mathematical Connections

Throughout the past century, the mathematics and mathematics education communities have noted the importance of multiple representations as tools for teaching and learning. In the early years of the twentieth century, for instance, the *Reorganization of Mathematics in Secondary Education Report* (National Committee on Mathematical Requirements of the Mathematics Association of America, 1923) recommended that students should develop the ability to use algebra, understand and interpret graphical representations, and have familiarity with geometric forms and the elementary properties and relations of these forms. Throughout the 1920's and 1930's (due, in part, to the influence of the 1923 MAA publication), teacher education journals contained numerous articles advocating and demonstrating the uses of multiple representations to solve geometric and algebraic problems (e.g. Nyberg, 1925; Bradshaw, 1925; or Haertter, 1931).

Recently, the recommendations of the National Council of Teachers of Mathematics (NCTM, 1989, 1991, 1995, 2000) have prompted renewed interest in mathematical representations. In particular, there appears to be recognition of a relationship between students' facility using multiple representations and their success at problem solving. As an example, the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) state that the use of multiple representations empower students as problem solvers:

> Students who are able to apply and translate among different representations of the same problem situation or of the same mathematical concept will have at once a powerful, flexible set of tools for solving problems and a deeper appreciation of the consistency and beauty of mathematics. (p. 146)

In support of this belief, the *Standards* include a Mathematical Connections standard at each of the grade bands: K-4, 5-8, 9-12. In a departure from recommendations that focus on traditional mathematics content, the 9-12 Mathematical Connections standard recommends that representations become an object of explicit study:

> In grades 9 - 12, the mathematics curriculum should include investigation of the connections and interplay among various mathematical topics and their applications so that all students can --
> - recognize equivalent representations of the same concept;
> - relate procedures in one representations of the same concept;
> - use and value the connections among mathematical topics;
> - use and value the connections between mathematics and other disciplines. (NCTM, 1989, p. 148)

Moreover, the connections concept is woven into nearly all of the various other standards. The 9-12 Functions standard, for example, recommends that the mathematics curriculum include the continued study of functions so that all students can "represent and analyze relationships using tables, verbal rules, equations, and graphs [and] translate among tabular, symbolic, and graphical representations of functions (NCTM, 1989, p.154)."

More recently, a focus on representations is embodied in the problem-solving recommendations of the newest set of standards, the *Principles and Standards for School Mathematics* [1] (NCTM, 2000). For example, the Standards 2000 document states:

> Of the many descriptions of problem-solving strategies, some of the best known can be found in the work of Polya (1957). Frequently cited strategies include using diagrams, looking for patterns, listing all possibilities, trying special values or cases, working backwards, guessing and checking, creating an equivalent problem, and creating a simpler problem. (p.53-54)

Notice that each of the aforementioned strategies entails a translation and a "representation" of the problem. That is, each strategy involves the construction of an alternative representation of the problem information. As a teacher, the obvious question that comes to mind is; how should these strategies for representing mathematics problems be taught? In particular, should the construction of alternative representations of mathematics

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[1] Herein the NCTM Standards documents are collectively referred to as the *Standards* or the NCTM *Standards*. However, when referring to a particular Standard the document will be so designated, e.g. the *Principles and Standards for School Mathematics* is called *Standards* 2000.

information receive explicit attention, and, if so, how should this content be integrated into the mathematics curriculum?

In addition to implicit recommendations regarding the teaching of multiple representations, the *Standards 2000* document takes a very explicit stand on the subject. Specifically, *Standards* 2000 includes a separate Representations standard at each of the four grade bands (K-2, 3-5, 6-8, 9-12), rather than as part of the Connections standard as was done in the 1989 *Standards* document. It should be noted, however, that *Standards* 2000 recognizes that connections and representations are interrelated concepts and associates these two standards throughout its curricular recommendations[2].

To provide teachers with models of the classroom uses of mathematical connections and multiple representations, all versions of the *Standards* documents contain example problems and classroom vignettes. The 9-12 Representations standard of *Standards 2000*, for instance, examines typical student solutions to the following problem:

> A flight from SeaTac Airport near Seattle, Washington, to LAX Airport in Los Angeles has to circle LAX several times before being allowed to land. Plot a graph of the distance of the plane from Seattle against time from the moment of takeoff until landing. (Hughes-Hallet et al. 1994, p. 6)

In its discussion of problem-solving strategies, *Standards 2000* maintains that part of the teacher's role is to help students connect their personal images to more conventional representations:

> One very useful window into students' thinking is student-generated representations. To illustrate this point, consider the...problem [stated immediately above] that might be presented to a tenth-grade class: Students could work individually or in pairs to produce distance-versus-time graphs for this problem, and teachers could ask them to present and defend those graphs to the classmates. Graphs produced by this class, or perhaps by student in other classes, could be handed out for careful critique and comment. When they perform critiques, students get a considerable amount of practice in communicating mathematics as well as in

[2] In the mathematics education literature, representations are often represented as one branch of mathematical connections. *Standards* 2000 recognizes this relationship, but explicitly differentiates between the two for teaching, learning, and research.

> constructing and improving on representations, and the teacher gets information that can be helpful in assessment (p. 363).

Through the use of examples, such as the one above, the *Standards* suggest that connections between the descriptive (contextual), algebraic, graphical (geometric), and tabular formats facilitate concept development and mathematical problem solving.

Overall, it can be concluded that the Mathematical Connections and Representations standards focus on two primary areas: (1) the connections that can be made among mathematical topics and (2) connections that can be made between mathematics and other disciplines.

In addition, two classes of connections are generally discussed in the research literature. The first is the connections that are made between real-life problems and mathematics (often called "mathematical modeling"). The second class of connections are those that are made between two or more mathematical representations of a problem. Both types of connections are illustrated in Figure [1.1].

With the newfound curricular interest in multiple representations, there is newfound research interest in the topic. Research by Skemp (1987), Janvier (1987), and Janvier, Girardon, and Morand (1987) refer to each form of representation (e.g., contextual, algebraic, geometric) as a "mode" and the process of transferring between two modes as a "translation." The representations research will be reviewed and presented in the subsequent review of literature chapter, as will formal definitions of the terminology used in the research.

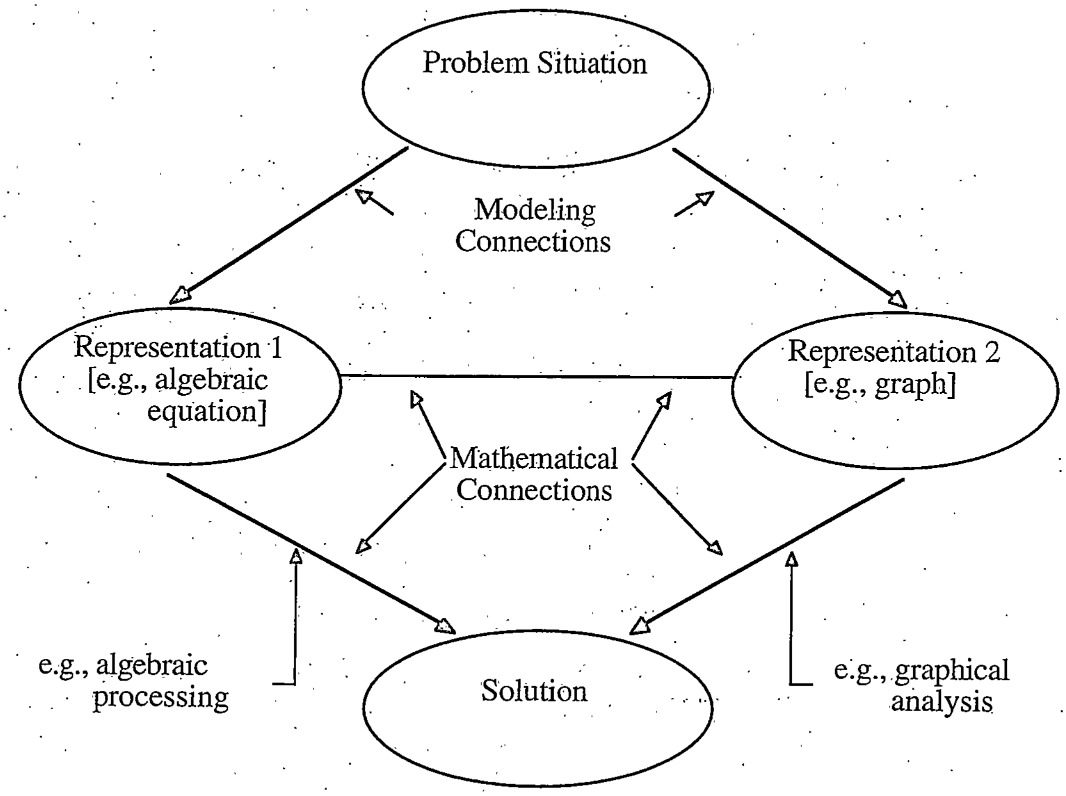

Figure 1.1 Connections
Two general types of connections (NCTM, 1989 p. 148)

Although the *Standards* have helped to focus attention on the translations and representations, the Standards naturally oversimplifies the processes by which students construct and use representations. Several related studies (Behr, Lesh, Post, & Silver, 1983; Janvier, 1987; Lesh, Landau, & Hamilton, 1983) provide comprehensive descriptions of many of the representation and translation concepts. Lesh, Landau, and Hamilton (1983), for example, examined the development and use of student's conceptual models in the setting of investigating children's rational-number understanding and problem-solving behaviors in realistic problem-solving situations. Lesh and his colleagues define the conceptual model as

> an adaptive structure consisting of (a) within-concept networks of relations and operations that the student must coordinate in order to make judgments concerning the concept; (b) between-concept systems that link and/or combine within-concept networks; (c) systems of representations (e.g. written symbols, pictures, and concrete materials), together with coordinated systems of translations among and transformations within modes; and (d) systems of modeling processes; that is, dynamic mechanisms that enable the first three components to be used, or to be modified or adapted to fit real situations (p. 264).

The research then focuses on the manner in which middle-grade students solved problems that were presented in a variety of formats. Toward this end, the researchers report that in applied problem-solving, important translation and/or modeling processes include

> (a) simplifying the original problem situation by ignoring 'irrelevant' characteristics in a real situation in order to focus on other characteristics;
> (b) establishing a mapping between the problem situation and the conceptual model(s) used to solve the problem;
> (c) investigating the properties of the model in order to generate information about the original situation; and
> (d) translation (or mapping) the predictions from the model back into the original situation and checking whether the results 'fit' (p. 270).

The models presented by Lesh and his colleagues (1983) describe how students use representation and translation. However, Lesh et al. made no attempt to (a) explain what effect the use of inappropriate representations or incorrect translations has on problem solving success, (b) examine how more mathematically mature students use translations and representations during problem solving, or (c) illustrate the types of mistakes made by the subjects as they used representations and translate between representations.

In the dissemination of his doctoral study, Hodgson (1993) describes aspects and patterns in students' errors in translation between representation, in particular, the procedural nature of the translation tasks. The translation Hodgson investigated is illustrated in Fig 1.2 by the arrow pointing from the Algebraic format to the Visual or Graphical format.

Hodgson's research identifies and characterizes the systematic procedural errors students make as they translate algebraic expressions into Venn diagrams. Note that Hodgson investigated translation tasks and characterized errors as the students translated in one direction, from algebra to Venn diagrams. Hodgson made no attempt to investigate students efforts to translate in the other direction (from the Venn diagrams back to the algebraic representation) or between other representations.

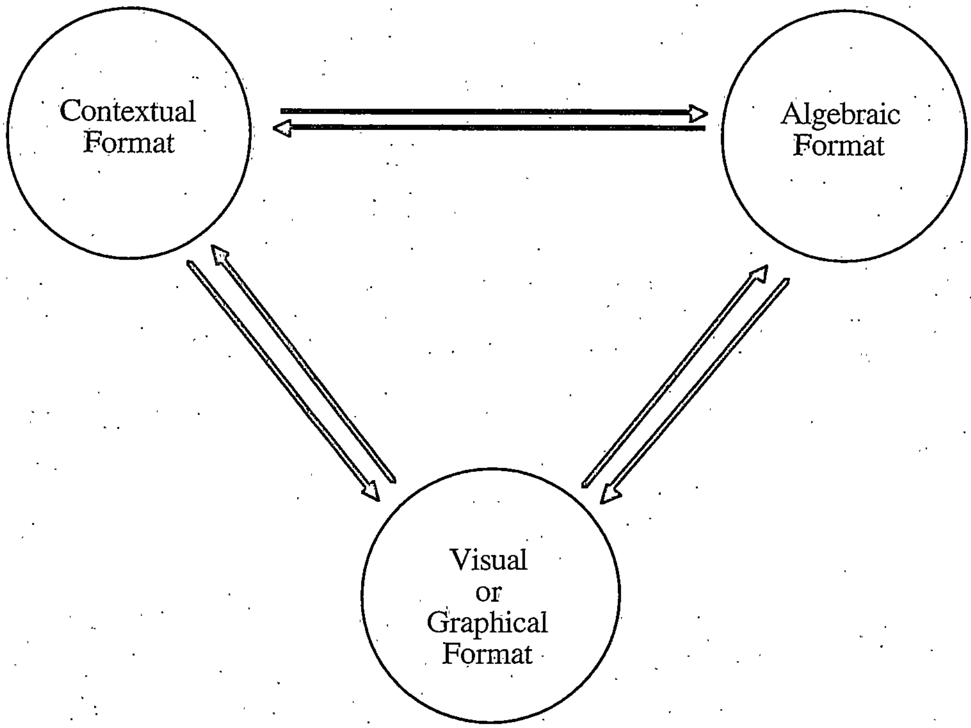

Figure 1.2: Translations
Directionality of Translations between Representations during Problem Solving and Concept Development Activities

Need for the Study

Probability problems provided an ideal context for examining students use of multiple representations because the students have opportunity to use graphical, symbolic algebra, and language representations. The research herein extends the work of Hodgson (1993) and other researchers in that it (a) identifies and characterizes the translation process between more than one representation and (b) investigates translations in both directions between the representations. This study differs from Hodgson's in that it does not attempt to identify students' procedural errors. Rather, the study identifies the representations used, whether the student was able to successfully translate between representations, and why the student choose to translate. Figure 1.2 illustrates the modes and possible translation paths that were investigated. The author found no research that examines (in detail) translation processes between these three representations. Moreover, no existing study examines the patterns of behavior in the use of translations by student problem-solvers.

The present study, therefore, contributes to the existing research base in that it elaborates on and extends the existing problem-solving and multiple representation research (e.g., Gfeller, Niess, & Hamilton, 1999; Hodgson, 1993; Lesh, Landau, & Hamilton, 1983; Schoenfeld, 1985; and Janvier, 1987). It is believed that the study herein will contribute information that provides researchers with a better understanding of how students use multiple representations in problem solving, as well as information as to where and how problem solvers have difficulty using multiple representations. In doing so, the study will provide improved and informed direction for both teaching and curriculum design of instruction involving translations and representations in mathematical problem solving.

Introduction to Data Collection and Methodology

The data from the student interviews and probability problem solutions were gathered at Indiana University in Bloomington, Indiana. The analysis of this research took place at Montana State University-Bozeman and at Oregon Institute of Technology. The representation and translation data were collected from interviews with a group of student volunteers. For the study, five probability problems were administered to students ($n = 21$) from various sections of a finite mathematics course. Each of the probability problems can be solved using elementary algebra, Venn diagrams, or by using both of these representations. Each participant was video taped while attempting to solve the problems and their solution methods were analyzed to determine their use of multiple representations and translations.

For the purpose of this research, the term *successful problem solver* describes an individual that was competent at solving the assigned probability problems, that is, the student consistently obtained correct solutions to probability problems that require translations between several representations.

Likewise, *unsuccessful problem solver* describes an individual that was not competent at solving the assigned probability problems, that is, the student did not consistently obtain correct solutions to probability problems that require translations between several representations.

The identification as to whether the student was a successful or unsuccessful problem solver was made post hoc, on the basis of whether the student was able to successfully solve a set of probability problems.

Each subject demonstrated his or her solution method(s) as he or she attempted to solve probability problems like that of Example 1. The participants were asked to verbalize their thinking and each solution attempt was video and audio recorded.

Example 1

At Giant State University a survey of students taking college mathematics found that 40% took college algebra, 30% took statistics, and 42% took neither course. If one of the students from the survey is chosen at random, what would be the probability that she or he took either algebra or statistics but not both courses?

The video taped "out loud" problem-solving sessions were analyzed to identify and characterize the subject's use of representations and translations. The methods of data collection and analysis are further described in the Methodology chapter. Appendix A provides a description of the rubric used to identify (a) when a subject was using a particular representation and (b) when a subject was translating between representations. Appendix B provides an example of how the rubric was used to identify and characterize a subject's use of translations and representations

In particular, this research focused on (a) the subjects use of multiple representations and (b) on the cognitive links the subjects made between the representations (translations). This research identifies (a) what representations successful and unsuccessful students use, (b) when they choose to use representations, (c) how frequently they use each representation, and (d) how successful they are at using the representations. Further, this research identifies (e) patterns of movement between the representations that characterize the successful and unsuccessful students use of translations and (f) patterns in the time spent using each representation (between translations) that characterize the successful and unsuccessful students use of translations. The research questions are introduced in the following section.

For the purpose of this research (a) the contextual format is probability problems written in English; (b) the algebraic format is Symbolic Algebra at a college introductory level; and (c) the visual or graphical format is usually Venn diagrams, although two students attempted (unsuccessfully) to solve the problems using Tree diagrams.

Research Questions:

The researcher entered the study with four research questions, two that were considered "Primary" and two "Secondary" questions. Primary research questions focus on the mechanics of students' representation use, whereas secondary questions focus on the role of representation use in problem solving.

Primary Research Questions:

1. Is there a difference in successful and unsuccessful students' use of multiple representations to solve probability problems?
2. Is there a difference in the patterns exhibited by successful and unsuccessful students with regard to their translation between representations?

Secondary Research Questions:

3. With regard to the problem solving process as a whole, what benefits (if any) do students gain through the use of multiple representation?
4. What difficulties (if any) do students encounter in their use of multiple representations?

CHAPTER TWO

A REVIEW OF RELATED LITERATURE

Introduction

Through a detailed investigation of students' solutions to probability problems (and their solution processes), this study examined students' use of mathematical representations and their translation between representations. Specifically, the students' use of Venn diagrams, symbolic algebra, and contextual (English) language, and translations between these representation modes, were analyzed in reference to two primary (Questions 1 and 2) and two secondary (Questions 3 and 4) research questions:

1. Is there a difference in successful and unsuccessful students' use of multiple representations to solve probability problems?
2. Is there a difference in the patterns exhibited by successful and unsuccessful students with regard to their translation between representations?
3. With regard to the problem solving process as a whole, what benefits (if any) do students gain through the use of multiple representation?
4. What difficulties (if any) do students encounter in their use of multiple representations?

In this chapter, research related to this investigation is reviewed in order to lay a theoretical foundation for an examination of students' use of representations and translations during problem solving. Namely, the fundamental elements of translation and representation activities are presented and reviewed in light of existing research literature. Secondly, theoretical and empirical considerations are examined regarding the use of translations and representations. In particular, this review seeks to identify the utility and limitation of translations and representations during problem solving. The chapter concludes with a brief review of the research literature regarding learning styles, with a particular focus on problem solvers' use of visual and analytic strategies.

Mathematical Problems, Translations, and Representations

Problem solving, a topic that has long been of interest to educators and psychologists, served as the basis of this study. According to Lester and Charles (1985), problem-solving situations are those in which an individual or group (a) must solve some type of problem; (b) the problem may be presented either as a "real life" situation or in some other mode, such as an English or mathematical question; and (c) the solver does not have ready access to a schemata or algorithm to apply to the problem or question. That "the problem solver does not have a solution schemata" does not imply that he or she doesn't possess the necessary cognitive and manipulative tools to solve the problem. However, answering the question is only an "exercise" if the individual possesses a solution schemata, algorithm, or knows immediately how to "do" the problem.

In the study described herein, students were administered five probability problems. The problems were "true" problems in that they met Lester and Charles (1985) criteria of problem solving, as mentioned above. However, many of the students, especially the more mathematically mature students, were able to solve the problems. Yet, the method of solution was not immediately apparent nor did a solution algorithm exist. Rather, students solved the problems through the use of a broad class of tools called mathematical "connections."

In Chapter One, it was noted that two classes of mathematical connections are discussed in the research literature. The first class of connections involves mathematical modeling, or the translation of information (generally relational in nature) observed in the "real world" into some alternative expression. In the setting of mathematics education, the "expression" of the event is generally made in context (i.e., in the spoken or written language native to the individual expressing the relationship), as a set of algebraic symbols and expressions, as a Cartesian or data graph, or as some combination of these representations.

The second class of connections includes those that can be made between two or more expressions. That is, this class of connections includes those connections that can be made between contextual descriptions, graphics, or algebraic expressions--or any other of the several descriptors of real-world events. Specific descriptors may include geometric figures, Venn diagrams, tree diagrams, Cartesian graphs, tables, lists, flow charts, letters and numerals in descriptive patterns, written descriptions, of any mix of these (Bell, 1976). This research focused on this second class of connections: those that can be made between the various models or representations of real-world events, but not on the real-world events.

As a word, representation is open to many interpretations. In the context of this study, as in most mathematics education research, representation refers to any observable embodiment that symbolizes or expresses an event or relationship. It should be noted that this definition represents a departure from that employed by Bell (1976), Janvier (1987), and some other researchers. Namely, this study did not restrict its focus to expressions directly related to real-world events.

An additional issue concerning representation research, and an important issue that needs to be addressed, is the need to observe the representation. Mental imagery is an essential component of thought. As such, Lesh, Post, and Behr (1987) remark that it is both naive and restrictive to require a representation to be observable. In particular, these researchers maintain that a representation may be an internal conceptualization and, therefore, unobservable to the researcher. The research literature (e.g., Behr, Lesh, Post, & Silver, 1983; Lesh, 1981; Lesh, Landau, & Hamilton, 1983) identifies at least six distinct types of representations used by problem solvers. These include (a) manipulative models such as arithmetic blocks and fraction bars; (b) graphics, pictures, and diagrams; (c) experience-based "scripts" wherein students' knowledge is organized around "real-world" events or experiences that serve as context to describe and solve other problems; (d) specialized forms of spoken languages, such as used by mathematicians and logicians; (e) spoken or written language as used in context by lay persons, i.e., non-scientists and non-mathematicians; and (f) written symbols and phrases--including algebraic equations or set-logic expressions. Note that this study focused on three of the aforementioned representations--graphs, spoken language, and algebra--and the translations and transformations among and between these three modes of representation.

The six representation modes listed above, together with the translation processes that link them, are essential features of the steps in mathematical modeling. Lesh and his colleagues (1987) identify these modeling steps as (a) simplifying the original situation to reveal the most important aspects of the model, (b) building a map (i.e., a translation) between the original situation and the model, (c) investigating the model to understand the original situation, (d) translating the relevant information from the model back to the original situation, and (e) checking that the translated information is a sensible representation of some observable elements of, and relevant to, the original situation. With regard to this research, the original situation presented to each research subject was a probability problem in written English. That is, the "context" of the problem was written English (as opposed to spoken English). For most of the subjects, the representation model chosen was a Venn diagram representation of the problem information, symbolic algebra, or some combination of the two.

Lesh and colleagues (1987) suggest that the modeling action (i.e., those representations reached through acts of translation and transformation) tend to be *plural, unstable,* and *evolving*. Moreover, these three attributes play important roles in the evolution of concepts and representations during the course of problem-solving sessions.

An act of representation is *plural* in that problem solvers commonly use more than one representational system (with appropriate acts of translation) to attain the necessary understanding of the problem. In particular, Lesh and colleagues found that problem solvers typically employ several different representations when solving a problem, and the various representations used may be incompatible with one another. However, the researchers noted that each of the representations help the student to understand or solve some aspect of the problem.

The modeling action may also be plural in the sense that the problem solver may use a connected series of representations while attempting to solve a problem. For example, with regard to a probability problem, a student might translate contextual information into a Venn diagram, solve the Venn by translating information into a symbolic algebra mode, and then translate information back to the Venn and into the contextual mode. Each of the various representations may depict only a portion of the modeled system, and several serial acts of translation may be needed before the problem solver gains a good understanding of the problem. Furthermore, Lesh and his colleagues (1987) assert that good problem solvers tend to be sufficiently flexible in their use of a variety of relevant representational systems that they instinctively switch to the most convenient representation to emphasize at any given point in the solution process. It is interesting to note that the present study does not support the assertion that good problem solvers instinctively switch between representations.

Acts of representation are said to be *evolving* when the representational mode is used by the problem solver not only to represent and solve some aspect of a problem, but also as a tool to gain understanding of the basic nature of the problem. Through the use of representations, a problem solver may discover and understand relationship inherent in the problem that were previously not accessible or not noticed.

The act of representation is *unstable* in that the problem solver may focus on a particular representation of the problem and, in so doing, forget the "big picture" or other important aspects of the problem. Alternatively, problem solvers that attempt to focus on the "big picture" may forget to account for important aspects of the problem.