MONTANA STATE UNIVERSITY

The topological complexity of Cr-diffeomorphisms with homoclinic tangency
by Brian Farley Martensen

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences
Montana State University
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Abstract:
Let F be a Cr--Diffeomorphism of a manifold M into itself with a saddle periodic point p and the property that branches of the stable and unstable manifolds of p exhibit a homoclinic tangency. Then Cr--Close to F is an F such that each non-empty relatively open set of the closure of the branch of the unstable manifold of p contains homeomorphic copies of all chainable continua. A non-local result is also included to illustrate that these chainable continua are quite large in this closure.

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APPROVAL

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Brian Farley Martensen

This dissertation has been read by each member of the dissertation committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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Signature Brian S. Martinson

Date 4/23/01

In loving memory of my father,
Jerry Thomas Moore
1950-1975

Dedicated to my parents,
Woody and Susan Martensen

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ABSTRACT

Let F be a C^r-diffeomorphism of a manifold M into itself with a saddle periodic point p and the property that branches of the stable and unstable manifolds of p exhibit a homoclinic tangency. Then C^r-close to F is an $\tilde{F}$ such that each non-empty relatively open set of the closure of the branch of the unstable manifold of p contains homeomorphic copies of all chainable continua. A non-local result is also included to illustrate that these chainable continua are quite large in this closure.

CHAPTER 1

INTRODUCTION

Non-Hyperbolic Dynamics and Homoclinic Bifurcations

It has been the goal of many studies in dynamical systems to describe the asymptotic behavior of systems with non-trivial recurrence. Much of the progress toward this end has been made in understanding hyperbolicity and has in many ways been limited to hyperbolic systems. Hyperbolicity was first introduced by Anosov ([A]) in his study of geodesic flows on negatively curved Riemannian manifolds. It was subsequently used by Smale to study other systems with non-trivial recurrence, leading to the study of Axiom A systems. It was hoped that these types of systems would be generic and thus an understanding of hyperbolicity might lead to an understanding of a generic system in the following sense. The recurrent sets for Axiom A systems, the hyperbolic basic sets, can be modeled by subshifts of finite type. Also, for a hyperbolic system, one can create a global model of the system and furthermore, this model persists for systems close to the original.

For non-hyperbolic systems, rarely can one find such a global model. Unfortunately, it seems hyperbolicity does not reign in the space of diffeomorphisms, and so it is important to understand the breakdown of hyperbolicity. For diffeomorphisms of two-dimensional manifolds, this breakdown between hyperbolicity and non-hyperbolicity has often been linked to the formation of homoclinic tangencies.

The motivation for this work comes from the study of invariant sets of surface diffeomorphisms, specifically, the attractors. In particular, we are interested in the topology of these attractors, since attractors give the forward asymptotic behavior of certain open sets in the manifold under the diffeomorphism. Much is known about the structure of these sets when the system is hyperbolic, but very little is known in the non-hyperbolic setting.

It is well known ([W]) that one-dimensional hyperbolic attractors are everywhere locally the product of a Cantor set and an arc. One would suspect that non-hyperbolic attractors displaying rich dynamics might in fact lead to rich local topological structure.

In this dissertation, we will show that the topology of certain invariant sets of diffeomorphisms exhibiting homoclinic tangencies must in fact be quite complex.

History

For a fixed point p, branches of the unstable and stable manifolds can create homoclinic orbits, orbits which are asymptotic to p in both forward and backward

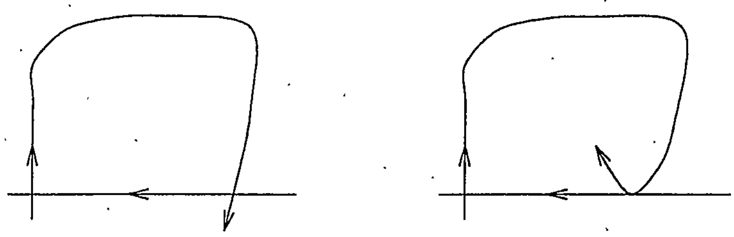


Figure 1. Transverse intersection vs. tangency.

time. These take two forms, transverse homoclinic intersections and homoclinic tangencies as depicted in Figure 1. In this dissertation we are concerned with homoclinic tangencies.

Homoclinic orbits were first studied by Poincaré ([P]) around 1889 while investigating the restricted 3-body problem. He was suprised by the apparent complexity involved in the dynamics when such orbits occur. Transverse homoclinic intersections were subsequently studied by Birkhoff and later by Smale. In 1935, Birkhoff ([Bi]) showed that a transversal homoclinic intersection is accumulated on by periodic orbits, of arbitrarily high periods. Thus, a map displaying a transverse homoclinic intersection has an infinite number of periodic orbits. In the 1960's, Smale ([Sm]) proved that a transverse homoclinic orbit is contained in a hyperbolic set. This set is a "horseshoe", in which the periodic orbits are dense.

Homoclinic tangencies lead to an even more complicated scenario and have been studied extensively. In particular, many people have studied the bifurcation process in the unfolding of a homoclinic tangency as a parameter evolves. This picture turns out to be much more complex than one might expect. For example, it has been known for some time that a homoclinic tangency is an accumulation point of other homoclinic tangencies. Homoclinic tangencies have shown themselves in many applications as well. They were studied by Cartwright and Littlewood ([CL]) in 1945 while considering the bifurcation process for highly non-linear forced Van der Pol equations.

Meanwhile, Gavrilov and Silnikov ([GS1], [GS2]) showed that there exists a sequence of saddle node bifurcations occurring arbitrarily close to homoclinic tangency. Thus, there are an infinite number of bifurcations occurring in the formation of a homoclinic tangency.

Newhouse ([Ne]) made perhaps the most startling discoveries about systems exhibiting homoclinic tangency. Utilizing the concept of "thick" Cantor sets, he introduced the concept of a wild hyperbolic set. This is an invariant set in which tangencies persist for small enough perturbations. He then showed that arbitrarily close to a locally dissipative diffeomorphism with homoclinic tangency, there exists a diffeomorphism displaying the Newhouse Phenomenon, characterized by the existence of wild hyperbolic sets and the property that the diffeomorphism has infinitely many periodic sinks. In particular, there are regions in the parameter space where homoclinic tangency is persistent (for the extension to parameter space, see [R]). Newhouse also found entire intervals of bifurcations in the parameter space.

Recently, Benedicks and Carleson ([BC]) have shown the existence and abundance of chaotic, transitive non-hyperbolic one-dimensional attractors near a system with homoclinic tangency. In particular, they developed a calculus for 2-dimensional maps near 1-dimensional maps for the Hénon family:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 - a^2 + y \\ bx \end{pmatrix}.$$

Mora and Viana ([MV]) obtained similar results applied to homoclinic bifurcations by studying a generic unfolding a quadratic homoclinic tangency through one-parameter families of locally dissipative surface diffeomorphisms. Their method was to show that these families admit renormalizations which are Hénon-like and then use an extension of the Benedicks-Carleson method applied to these maps. The results have been generalized by Wang and Young ([WY]) to obtain checkable conditions on 2-dimensional maps near 1-dimensional maps for these attractors to exist.

Under different assumptions, Palis and Takens ([PT]) have shown the abundance of hyperbolicity, leading to the natural question: Of the Newhouse Phenomenon and strange attractors, which is generic?

Results

The above attractors of Benedicks and Carleson have the property that the attractor is the closure of the unstable manifold for the periodic point near tangency. This has led Barge to study the topology of the closure of the unstable manifold at homoclinic tangency. In [B], he has shown that generically this space is globally an indecomposable continuum; in particular, it contains uncountably many arc-components. Still, locally these closures may be the product of a Cantor set and an arc, except at finitely many points (as many computer pictures suggest). One would expect that the local structure is, in fact, much more complicated.

A result along this line is found in [BD], where Barge and Diamond show that if F is a C^∞-diffeomorphism of the plane with a hyperbolic fixed point p for which a branch of the unstable manifold, $W^u_+(p)$, has a same-sided quadratic tangency with the stable manifold, and if the eigenvalues of DF at p satisfy a generic non-resonance condition, then each non-empty relatively open set of $Cl(W^u_+(p))$ contains a copy of every continuum that can be written as the inverse limit space of a sequence of unimodal bonding maps. Thus, "hooks" appear densely in this closure; so that not only is the structure not locally a Cantor set of arcs, but it is, in fact, nowhere such a thing.

In this dissertation, we will use the terminology that a set which contains a homeomorphic copy of each element of a class of continua, W, is *universal with respect to* W, or *simply* W*-universal.* Thus the $Cl(W^u_+(p))$ above is everywhere locally universal with respect to unimodal continua.

The set of unimodal continua is a large class of continua. In particular, it is uncountable ([J]). But the result of [BD] leads to the natural question: How much more complicated might these closures be? That is, could they contain even richer structure still?

In this dissertation, we show that this is the case if we make a small perturbation to our diffeomorphism at homoclinic tangency. The class of unimodal continua is contained in a larger class called chainable continua. We show that this closure can contain a homeomorphic copy of each element in this class. In particular, we will be able to get complicated continua such as pseudoarcs, continua which are nowhere homeomorphic to an arc. The main result of this dissertation is stated as follows, where $\mathcal{C}$ is used to denote the class of chainable continua:

THEOREM 1. *Let F be a C^r-diffeomorphism of a 2-manifold M with a locally dissipative saddle periodic point p which exhibits a homoclinic tangency. Then, C^r-close to F is a diffeomorphism $\tilde{F}$ such that a branch of the closure of the unstable manifold, $Cl(W^u_+(p))$, is everywhere locally $\mathcal{C}$-universal.*

If p is a locally non-dissipative saddle, then the result above holds for a branch of the stable manifold.

Remark: For the case where $r = 1$, the condition that p be locally dissipative can be omitted so as to obtain a slightly stronger result. In general, this is also the case when the tangency is of order at least r. It will be noted why this is the case at the beginning of Chapter 6 as well as in Remark 4.7.

In order to prove the above theorem, we will need to first prove the following result:

THEOREM 2. *Let F be a C^r-diffeomorphism of a 2-manifold M with a locally dissipative saddle periodic point p exhibiting a homoclinic tangency. Then, C^r-close to F is a diffeomorphism $\tilde{F}$ such that it has a saddle periodic point $\tilde{p}$ (of higher period* *than p) with the closure of a branch of the unstable manifold of* $\tilde{p}$, $Cl(W_+^u(\tilde{p}))$, *being everywhere locally* C*-universal.*

If p is a locally non-dissipative saddle, then the result above holds for a branch of the stable manifold.

In fact, this new periodic point will have a period which is a multiple of the period of p and furthermore, $Cl(W_+^u(\tilde{p})) \subset Cl(W_+^u(p))$.

As was observed in [K], density results near homoclinic tangencies can be placed in further perspective by noting the Palis Conjecture ([PT], Chapter 7, § 1, Conjecture 2), which has been recently shown for C^1-approximations in [PS]:

CONJECTURE 1. *If* $\dim(M) = 2$, *then every* C^r*-diffeomorphism* $f \in \mathrm{Diff}^r(M)$ *can be approximated by a diffeomorphism which is either (essentially) hyperbolic or exhibits a homoclinic tangency.*

If this conjecture is true, then in the complement (in $\mathrm{Diff}^r(M)$) to the closure of the space of hyperbolic diffeomorphisms, every diffeomorphism can be C^r-approximated by those exhibiting the property of the main theorem of this dissertation.

Lastly, we will show that the above theorems lead to non-local results. That is, though our main theorem says that every non-empty relatively open subset of the closure of the unstable manifold contains all chainable continua, one might get the impression that we have only introduced tiny "wiggles" into our space. But in fact, we have:

THEOREM 3. *The diffeomorphism $\tilde{F}$ of the conclusion of Theorem 1 can be constructed so that for any non-degenerate chainable continuum, X, any arc in $W_+^u(p)$ can be approximated, with respect to the Hausdorff metric, by a continuum in $Cl(W_+^u(p))$ which is homeomorphic to X.*

Thus, these continua are quite large, and $W_+^u(p)$ itself can be approximated by subcontinua homeomorphic to any non-degenerate chainable continuum. In particular, the $Cl(W_+^u(p))$ is the Hausdorff limit of subcontinua homeomorphic with the pseudoarc.

Structure of this Dissertation

This dissertation is organized in the following way. Chapter 2 gives a brief introduction to continuum theory and inverse limit spaces. We then prove a theorem which allows us to express chainable continua as inverse limit spaces using a finite family of smooth bonding maps (Theorem 2.1). This chapter also provides two lemmas and a theorem (from [Br]) which will be used to decide how two inverse limit spaces relate to one another. Chapter 3 gives a brief outline of the proof of Theorem 2. In Chapter 4, we perform a series of perturbations on a diffeomorphism exhibiting a homoclinic tangency. In Chapter 5, we prove that the diffeomorphism obtained in Chapter 4 exhibits the properties of the conclusion of Theorem 2. Next, in Chapter 6, we use the construction of Theorem 2 to prove the main theorem of this dissertation, Theorem 1. And lastly, in Chapter 7, we give an application of our main theorem which provides for a non-local result.

CHAPTER 2

HOMEOMORPHIC INVERSE LIMIT SYSTEMS

In this chapter, we introduce some definitions and conventions which will be used throughout this dissertation. We then turn to describing each chainable continua as the inverse limit space of interval maps (Theorem 2.1). Next, we examine conditions under which two inverse limit spaces are homeomorphic, state an extremely useful result of Brown, and end this chapter with an embedding lemma (Lemma 2.3) and a homeomorphism lemma (Lemma 2.4) which will be needed in Chapter 5.

Continua

A *continuum* X is a non-empty compact connected metric space. A *chain* in X is a non-empty, finite, indexed collection, $\mathcal{C} = \{U_1, ..., U_n\}$, each U_i open in X, such that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$. An *ϵ-chain* is a chain $\mathcal{C}$ with the mesh$(\mathcal{C}) < \epsilon$ (i.e. $\max\{\text{diam}(U_i)\} < \epsilon$). A continuum is said to be *chainable* if it is contained in an ϵ-chain for each ϵ.

Suppose that $\{X_i\}_{i=0}^{\infty}$ is a collection of compact metric spaces and for each i, $f_{i+1} : X_{i+1} \to X_i$ is a continuous map, often referred to as a bonding map. The *inverse* *limit space* of $\{X_i, f_i\}_{i=1}^{\infty}$(or simply, of $\{f_i\}_{i=1}^{\infty}$) is

$$X_\infty = \left\{ \underline{x} = (x_0, x_1, ...) | \underline{x} \in \prod_{i=0}^{\infty} X_i, f_{i+1}(x_{i+1}) = x_i, i \geq 0 \right\}$$

and has metric $\underline{\mathrm{d}}$ given by

$$\underline{\mathrm{d}}(\underline{x}, \underline{y}) = \sum_{i=0}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$$

where for each i, d_i is a metric for X_i bounded by 1. It is well known that if each X_i is a continuum, then X_∞ is also (see, for example, [Na]). For each i, π_i will denote the restriction, to X_∞, of the usual projection map from $\prod_{i=0}^{\infty} X_i$ into X_i.

In this dissertation, a map is meant to be a continuous transformation. An interval map is a map from the unit interval, $I = [0, 1]$, back into itself. Jolly and Rogers ([JR]) have shown that there are four interval maps such that each chainable continuum is homeomorphic to the inverse limit of interval bonding maps, where each bonding map is taken to be one of these four maps. Using a result of Jarník and Knichal([JK]), Cook and Ingram ([CI]) have reduced the number of bonding maps to two. We state this result, but add the additional condition that the two maps be C^∞-differentiable:

THEOREM 2.1. *There exist maps $\hat{f}_0$ and $\hat{f}_1$, each C^∞ and mapping I to I, such that if X is any chainable continuum, X is homeomorphic to an inverse limit of interval maps, with each map coming from $\{\hat{f}_0, \hat{f}_1\}$.*

PROOF. It is a well known fact that any chainable continuum, X, can be written as the inverse limit of interval maps([F], [M]).

We follow closely to [I], giving a brief outline of the proof, while making the appropriate changes to achieve the C^∞-differentiability. The space of all mappings of I into itself is separable so there is a countable sequence of C^∞-maps, $\{f_i\}_{i\in\mathbb{N}}$, such that if f is a interval map and $\epsilon > 0$, there is an i such that $||f_i - f||_0 < \epsilon$. By an approximation theorem of Brown ([Br], see Theorem 2.2 below), there is a subsequence $\{f_{n_i}\}_{i\in\mathbb{N}}$ such that X is homeomorphic to the inverse limit space of $\{I, f_{n_i}\}_{i\in\mathbb{N}}$. The goal is to construct maps $\hat{f}_0$ and $\hat{f}_1$ so that each f_i above can be written as the composition of $\hat{f}_0$ and $\hat{f}_1$.

To that end, denote $I_1 = [\frac{1}{2}, 1]$, $I_2 = [\frac{1}{8}, \frac{1}{4}]$, $I_3 = [\frac{1}{32}, \frac{1}{16}]$, ... as a sequence of copies of I with $\limsup_{n\to\infty} I_n = \{0\}$. For each i, let $j_i > i^2 + i$ and be large enough so that $M_i < 2^{2j_i - 2i^2 - i - 1}$, where M_i is a bound on the C^i-norm of f_i (the definition of this norm is given in Chapter 4). Let $J_i = \frac{1}{4^{j_i}} I_i = I_{i+j_i}$. Let $\hat{f}_0$ be the homeomorphism of I onto I_1 given by $\hat{f}_0(x) = \frac{x+1}{2}$. Let $\alpha : I \to I, \beta : I \twoheadrightarrow I$, and $\gamma : I \twoheadrightarrow I$ be defined by $\alpha(x) = \frac{x}{4}, \beta(x) = 4x$ for $x \in [0, \frac{1}{4}]$ and $\beta(x) = 1$ for $x \in [\frac{1}{4}, 1]$ and $\gamma(x) = 0$ for $x \in [0, \frac{1}{2}]$ and $\gamma(x) = 2x - 1$ for $x \in [\frac{1}{2}, 1]$. Note that $\alpha(I_i) = I_{i+1}, \beta|_{I_{i+1}} = (\alpha|_{I_i})^{-1}$ and $\gamma|_{I_1} = \hat{f}_0^{-1}$. Let $\hat{f}_1$ be a C^∞-extension to a map of $[0, 1]$ onto $[-\xi, 1 + \xi]$ which places a "copy" of α over I_1(that is, $\hat{f}_1(x) = \frac{2x-1}{4}$ for $x \in I_1$), a "copy" of β over I_2, a "copy" of γ over I_3 and a scaled down "copy" of f_i from I_{i+3} into J_{i+3} for all $i \in \mathbb{N}$.

In order to smoothly connect the function between the I_i intervals, it may be necessary for the range to dip below 0 or above 1 and extend our domain to slightly larger than 1. This is why we have extended I by adding the ξ-terms above. Then one can check that $f_i = \hat{f}_1 \circ (\hat{f}_1 \circ \hat{f}_0)^2 \circ \hat{f}_0 \circ (\hat{f}_1^2 \circ \hat{f}_0^2)^{i+j_i+2} \circ \hat{f}_1 \circ (\hat{f}_1 \circ \hat{f}_0)^{i+2} \circ \hat{f}_0$.

Utilizing this fact, we define the sequence $\{g_i : g_i \in \{\hat{f}_0, \hat{f}_1\}\}_{i\in\mathbb{N}}$ cofinal with f_i. That is, inductively, for each i, there exist j_i such that $f_i = g_{j_{i-1}+1} \circ ... \circ g_{j_i}$. Then the inverse limit of $\{g_i\}$ is homeomorphic to the inverse limit of $\{f_i\}$ since the inverse limit of cofinal sequences are homeomorphic (See the Corollary 1.7.1 in [I]).

It remains to show that the above functions can be made C^∞. Due to the choice of I_i and J_i, $\hat{f}_1$ is C^∞-flat at zero. To see this, first note that:

$$\frac{|J_i|}{|I_i|^i} = \frac{(2^{2i-1})^{i-1}}{2^{2j_i}} < \frac{1}{2^{2j_i-2i^2-1}} < \frac{1}{2^{2i-1}} \to 0,$$

as $i \to \infty$. Secondly, on I_i, the kth derivative of $\hat{f}_1$ is bounded above by

$$\frac{|J_i|}{|I_i|^k} M_i < \frac{|J_i|}{|I_i|^i} M_i < \frac{1}{2^{2j_i-2i^2-1}} 2^{2j_i-2i^2-i-1} = \frac{1}{2^i} \to 0$$

as $i \to \infty$. Similarly, between I_i and I_{i+1}, we place a smooth function whose range need not be bigger than $[0, \frac{1}{2^{2(i+j_i)+1}}]$, where the upper endpoint is the upper endpoint of J_i. The ratio of the range to the i-th power of the domain is then bounded above by $2^{-(2j_i-2i^2)} \to 0$ as $i \to \infty$ and thus the function $\hat{f}_1$ is C^∞-flat at 0. Everywhere else, this function is obviously C^∞ due to our extensions, as is $\hat{f}_0$. Lastly, we rescale the functions so that they map I to I while maintaining the relationship between f_i, $\hat{f}_0$ and $\hat{f}_1$. □

Approximation Results

Suppose we have a sequence of maps, $\{f_i : X_i \to X_{i-1}\}_{i\in\mathbb{N}}$. We now wish to consider the question: What conditions can we place on a sequence of maps, $\{g_i : X_i \to X_{i-1}\}_{i\in\mathbb{N}}$, so that the inverse limits of the two sequences are homeomorphic? A powerful result in this direction is an approximation theorem given by Brown in [Br] as Theorem 3.

THEOREM 2.2. (*Brown*) *Let S be the inverse limit of $\{X_i, f_i\}_{i=1}^{\infty}$, where X_i are compact metric spaces. For $i \geq 2$, let K_i be a nonempty collection of maps from X_i into X_{i-1}. Suppose for each $i \geq 2$ and $\epsilon > 0$, there is $g \in K_i$ such that $||f_i - g||_0 < \epsilon$. Then there is a sequence of g_i where $g_i \in K_i$ and S is homeomorphic to the inverse limit of $\{X_i, g_i\}_{i=1}^{\infty}$.*

This tells us that the above sequence of f_i determines a sequence of ϵ_i such that as long as $||g_i - f_i||_0 < \epsilon_i$, for each i, then the inverse limit of the two sequences are homeomorphic. We now ask the question: What conditions can we place on a sequence of maps f_i and g_i, where the f_i and X_i are not fixed, but rather are inductively defined along with g_i so that their inverse limit spaces are homeomorphic? The following two lemmas provide results in this direction, and will be needed in Chapter 5.

The first, from Barge and Diamond ([BD]), will be useful in building particular spaces as subcontinua of $Cl(W_+^u(\tilde{p}))$. It gives conditions under which one inverse limit space can be embedded into another. The second requires slightly stronger conditions on the spaces, but gives the conditions under which the spaces are homeomorphic. The proofs of these lemmas are identical with the exception that one must prove additional requirements of surjectivity and the existence of a continuous inverse for the homeomorphism in the second. We therefore use the first lemma to prove most of the second. Given a sequence of maps $\{f_n : X_n \to X_{n-1}\}_{n\in\mathbb{N}}$, $f_{i,n}$ will denote the map $f_i \circ ... \circ f_n : X_n \to X_{i-1}$ for $n \geq i$, where $f_{i,i} = f_i$.

LEMMA 2.3. *Let $G_n : X_n \to X_{n-1}$ and $g_n : Y_n \to Y_{n-1}$ be sequences of maps of compact metric spaces and $i_n : Y_n \to X_n$ a sequence of embeddings. There is a sequence of positive numbers $\{\kappa_n\}_{n\in\mathbb{N}}$, with κ_n depending only on $i_0, ..., i_{n-1}, g_1, ..., g_{n-1}, G_1, ..., G_{n-1}$, such that if $||G_n \circ i_n - i_{n-1} \circ g_n||_0 < \kappa_n$ for $n \in \mathbb{N}$, the map $\hat{\imath} : Y_\infty \to X_\infty$ defined by $(\hat{\imath}(\underline{y}))_n = \lim_{k\to\infty} G_{n+1,n+k} \circ i_{n+k}(y_{n+k})$ is a well-defined embedding.*

PROOF. We follow [BD] almost exactly. The proof is included here for completeness. Let $\gamma_1 > 0$ be arbitrary, and, for $i \geq 2$, let $\gamma_i > 0$ be small enough so that if $|x - x'| < \gamma_i$, then $|G_{j,i-1}(x) - G_{j,i-1}(x')| < \frac{1}{2^i}$ for all j such that $1 \leq j \leq i-1$. We will show that if $|G_n \circ i_n - i_{n-1} \circ g_n| < \gamma_n$ for all $n \in \mathbb{N}$, then $\hat{\imath}$ is well-defined and continuous.

<u>$\hat{\imath}$ is well-defined:</u> Let $\underline{y} = (y_0, y_1, ...) \in Y_\infty$. For $k, l \geq 1$,

$$
\begin{aligned}
&|G_{n+1,n+k+l} \circ i_{n+k+l}(y_{n+k+l}) - G_{n+1,n+k} \circ i_{n+k}(y_{n+k})| \\
\leq &|G_{n+1,n+k+l} \circ i_{n+k+l}(y_{n+k+l}) - G_{n+1,n+k+l-1} \circ i_{n+k+l-1}(y_{n+k+l-1})| \\
+&|G_{n+1,n+k+l-1} \circ i_{n+k+l-1}(y_{n+k+l-1}) - G_{n+1,n+k+l-2} \circ i_{n+k+l-2}(y_{n+k+l-2})| \\
+&... + |G_{n+1,n+k+1} \circ i_{n+k+1}(y_{n+k+1}) - G_{n+1,n+k} \circ i_{n+k}(y_{n+k})|
\end{aligned}
$$

$$< \sum_{j=1}^{l} \frac{1}{2^{n+k+j}} < \frac{1}{2^{n+k}}.$$

Then the sequence $\{G_{n+1,n+k}\}$ is Cauchy for each $n \geq 1$, hence convergent, and $(\hat{\imath}(\underline{y}))_n$ is well-defined. The fact that $G_n((\hat{\imath}(\underline{y}))_n) = (\hat{\imath}(\underline{y}))_{n-1}$ is trivial and hence $\hat{\imath}$ is well-defined.

$\underline{\hat{\imath} \text{ is continuous:}}$ Let $\epsilon > 0$. Let $\delta' > 0$ be chosen so that $\underline{x}, \underline{\hat{x}} \in X_\infty$ with $|x_N - \hat{x}_N| < \delta'$ implies $|\underline{x} - \underline{\hat{x}}| < \epsilon$. Choose k large enough so that $\frac{1}{2^{N+k}} < \delta'/3$, and $\delta'' > 0$ so that if $|y_{N+k} - \hat{y}_{N+k}| < \delta''$, then

$$|G_{N+1,N+k} \circ i_{N+k}(y) - G_{N+1,N+k} \circ i_{N+k}(\hat{y})| < \delta'/3.$$

Lastly, choose $\delta > 0$ so that if $\underline{y}, \underline{\hat{y}} \in Y_\infty$ with $|\underline{y} - \underline{\hat{y}}| < \delta$, then $|y_{N+k} - \hat{y}_{N+k}| < \delta''$. Then $|\underline{y} - \underline{\hat{y}}| < \delta$ implies

$$\begin{aligned} |\hat{\imath}(\underline{y})_N - \hat{\imath}(\underline{\hat{y}})_N| &\leq |(\hat{\imath}(\underline{y})_N - G_{N+1,N+k} \circ i_{N+k}(y_{N+k})| \\ &+ |G_{N+1,N+k} \circ i_{N+k}(y_{N+k}) - G_{N+1,N+k} \circ i_{N+k}(\hat{y}_{N+k})| \\ &+ |G_{N+1,N+k} \circ i_{N+k}(y_{N+k}) - (\hat{\imath}(\underline{\hat{y}})_N| \\ &< \delta'/3 + \delta'/3 + \delta'/3 = \delta' \end{aligned}$$

which in turn implies $|\hat{\imath}(\underline{y}) - \hat{\imath}(\underline{\hat{y}})| < \epsilon$. Thus $\hat{\imath}$ is continuous.

Before proving that the map is one-to-one, we prove the following claim:

Claim: Given $\delta > 0$ and $n \in \mathbb{N}$, there is a sequence $\nu_{n,k}(\delta) > 0, k = n+1, n+2, \ldots,$ and $\lambda_n = \lambda_n(\delta) > 0$ such that:

(i) $\nu_{n,k}$ depends only on δ, i_n and G_j for $j = n+1, \ldots, k-1$ and

(ii) if $|G_k \circ i_k - i_{k-1} \circ g_k| < \nu n, k$ for all $k \geq n+1$, and if $m \geq n+1$ and $y, y' \in Y_m$ are such that $|G_{n+1,m} \circ i_m(y) - G_{n+1,m} \circ i_m(y')| < \lambda_n$, then $|g_{n+1,m}(y) - g_{n+1,m}(y')| < \delta$.

Proof of Claim: Let $\lambda_n > 0$ be small enough so that if $|y - y'| \geq \delta$, then $|i_n(y) - i_n(y')| \geq 3\lambda_n$ (recall i_n is an embedding). Let $\nu_{n,n+1} = \lambda_n/2$. If both $|G_{n+1} \circ i_{n+1} - i_n \circ g_{n+1}| < \nu_{n,n+1}$ and $|G_{n+1} \circ i_{n+1}(y) - G_{n+1} \circ i_{n+1}(y')| < \lambda_n$, then

$$
\begin{aligned}
|i_n \circ g_{n+1}(y) - i_n \circ g_{n+1}(y')| &\leq |i_n \circ g_{n+1}(y) - G_{n+1} \circ i_{n+1}(y)| \\
&+ |G_{n+1} \circ i_{n+1}(y) - G_{n+1} \circ i_{n+1}(y')| \\
&+ |G_{n+1} \circ i_{n+1}(y') - i_n \circ g_{n+1}(y')| \\
&\leq \nu_{n,n+1} + \lambda_n + \nu_{n,n+1} < 3\lambda_n,
\end{aligned}
$$

so that $|g_{n+1}(y) - g_{n+1}(y')| < \delta$.

Continuing, for $k > n+1$, choose $\nu_{n,k}$ small enough so that if $|x - x'| < \nu_{n,k}$, then $|G_{n+1,k-1}(x) - Gn+1, k-1(x')| < \frac{\lambda_n}{2(k+1)-(n+1)}$. Now suppose that $|G_k \circ i_k - i_{k-1} \circ g_k| < \nu_{n,k}$ for $k = n+1, ..., m$ and $|G_{n+1,m} \circ i_m(y) - G_{n+1,m} \circ i_m(y')| < \lambda_n$ for some $m \geq n+2$. Then,

$$
\begin{aligned}
|i_n \circ g_{n+1,m}(y) - i_n \circ g_{n+1,m}(y')| &\leq |i_n \circ g_{n+1,m}(y) - G_{n+1} \circ i_{n+1} \circ g_{n+2,m}(y)| \\
&+ |G_{n+1} \circ i_{n+1} \circ g_{n+2,m}(y) - G_{n+1} \circ G_{n+2} \circ i_{n+2} \circ g_{n+3,m}(y)| \\
&+ ... + |G_{n+1,m-1} \circ i_{m-1} \circ g_m(y) - G_{n+1,m} \circ i_m(y)| \\
&+ |G_{n+1,m} \circ i_m(y) - G_{n+1,m} \circ i_m(y')|
\end{aligned}
$$

$$+|G_{n+1,m}\circ i_m(y')-G_{n+1,m-1}\circ i_{m-1}\circ g_m(y')|$$

$$+\ldots+|G_{n+1,m-1}\circ i_{m-1}\circ g_m(y')-i_n\circ g_{n+1,m}(y')|$$

$$<\frac{\lambda_n}{2}+\frac{\lambda_n}{4}+\ldots+\frac{\lambda_n}{2^{m-n}}+\lambda_n+\frac{\lambda_n}{2^{m-n}}+\ldots+\frac{\lambda_n}{2}<3\lambda_n.$$

Thus $|g_{n+1,m}(y)-g_{n+1,m}(y')|<\delta$ and so the claim is proved.

Continuing the proof of the lemma, let $\delta_0=1$ and $\kappa_1=\min\{\gamma_1,\nu_{0,1}(\delta_0)\}$. Choose δ_1 small enough so that if $|y-y'|<\delta_1$, then $|g_1(y)-g_1(y')|<\delta_0/2$. Define $\kappa_2=\min\{\gamma_2,\nu_{0,2}(\delta_0),\nu_{1,2}(\delta_1)\}$. Let δ_2 be small enough so that if $|y-y'|<\delta_2$, then $|g_2(y)-g_2(y')|<\delta_1/2$ and $|g_{1,2}(y)-g_{1,2}(y')|<\delta_0/4$. Define $\kappa_3=\min\{\gamma_3,\nu_{0,3}(\delta_0),\nu_{1,3}(\delta_1),\nu_{2,3}(\delta_2)\}$. More generally, define $\kappa_{k+1}=\min\{\gamma_{k+1},\nu_{0,k+1}(\delta_0),\ldots,\nu_{k,k+1}(\delta_k)\}$, with κ_k small enough so that if $|y-y'|<\kappa_k$, then $|g_{l,k}(y)-g_{l,k}(y')|<\frac{\delta_{l-1}}{2^{k-l+1}}$ for all $1\le l\le k$.

<u>$\hat{\imath}$ is one-to-one:</u> Suppose $\hat{\imath}(\underline{y})=\hat{\imath}(\underline{\hat{y}})$. If $\underline{y}\neq\underline{y}'$, there is n such that $y_n\neq y_n'$. Choose m large enough so that $|y_n-y_n'|>\frac{\delta_n}{2^{m-n}}$ and $l\ge m$ large enough so that $|G_{m+1,l}\circ i_l(y_l)-G_{m+1,l}\circ i_l(y_l')|<\lambda_m=\lambda_m(\delta_m)$ of the claim. Then $|g_{m+1,l}(y_l)-g_{m+1,l}(y_l')|<\delta_m$, from which it follows that $|g_{n+1,m}g_{m+1,l}(y_l)-g_{n+1,m}g_{m+1,l}(y_l')|<\frac{\delta_n}{2^{m-n}}$. That is $|y_n-y_n'|<\frac{\delta_n}{2^{m-n}}$, a contradiction. Thus, $\hat{\imath}$ is one-to-one.

Lastly, note that since $\gamma_k<\kappa_k$, $\hat{\imath}$ is well-defined and continuous. $\square$

LEMMA 2.4. *Let $f_n:X_n\to X_{n-1}$ and $g_n:Y_n\to Y_{n-1}$ be sequences of maps of compact metric spaces and $h_n:Y_n\to X_n$ a sequence of homeomorphisms. There is a sequence of positive numbers $\{\epsilon_n\}_{n\in\mathbb{N}}$, with ϵ_n depending only on $h_0,\ldots,h_{n-1}$, $g_1,\ldots,g_{n-1}$, $f_1,\ldots,f_{n-1}$, such that if $||f_n\circ h_n-h_{n-1}\circ g_n||_0<\epsilon_n$ for $n\in\mathbb{N}$, the map*

$\hat{\jmath} : Y_\infty \to X_\infty$ defined by $(\hat{\jmath}(\underline{y}))_n = \lim_{k\to\infty} f_{n+1,n+k} \circ h_{n+k}(y_{n+k})$ is a well-defined homeomorphism.

PROOF. We note that $\hat{\jmath}$ is well-defined, continuous and one-to-one by Lemma 2.3 above. Continuing as in the proof of that lemma:

$\underline{\hat{\jmath}}$ <u>is onto:</u> Suppose $\underline{x} = (x_o, x_1, \ldots) \in X_\infty$. Let $\underline{y}$ be defined by $y_k = \lim_{n\to\infty} g_{k+1,n} \circ h_n^{-1}(x_n)$. Fix a k and $n \geq 1$ and let δ be such that $|y - \hat{y}| < \delta$ implies $|h_{k+m}(y) - h_{k+m}(\hat{y})| < \gamma_{n+k}$, where γ_i is as in the proof of Lemma 2.3. Choose $l \geq 1$ so that $|y_k - g_{k+1,k+n+l} \circ h^{-1}_{k+n+l}(x_{k+n+l})| < \delta$. Then,

$$\begin{aligned}
|x_k - f_{k+1,k+n} \circ h_{k+n}(y_{k+n})| &= |f_{k+1,k+n+l}(x_{k+n+l}) - f_{k+1,k+n} \circ h_{k+n}(y_{k+n})| \\
&< |f_{k+1,k+n+l}(x_{k+n+l}) - f_{k+1,k+n} \circ h_{k+n} \circ g_{k+n+1,k+n+l} \circ h^{-1}_{k+n+l}(x_{k+n+l})| \\
&+ |f_{k+1,k+n} \circ h_{k+n} \circ g_{k+n+1,k+n+l} \circ h^{-1}_{k+n+l}(x_{k+n+l}) - f_{k+1,k+n} \circ h_{k+n}(y_{k+n})| \\
&< \frac{1}{2^{k+n}} + \frac{1}{2^{k+n}}.
\end{aligned}$$

Thus, $x_k = \lim_{n\to\infty} f_{k+1,n} \circ h_n(y_n)$ and therefore $\hat{\jmath}$ is onto.

<u>$\hat{\jmath}^{-1}$ is continuous:</u> This follows from the fact that **h** is a continuous, one-to-one, and onto map from a compact space to a Hausdorff space, and thus has a continuous inverse. □

Remark 2.5: Lemma 2.4 is really just a rephrasing of Theorem 2.2. As such an alternate proof of Lemma 2.4 is to note that there exists ϵ_i (depending only on previous choices of maps) such inverse limit of $\{X_i, f_i\}_{i\in\mathbb{N}}$ is homeomorphic to that of $\{X_i, h_{i-1} \circ g_i \circ h_i^{-1}\}_{i\in\mathbb{N}}$ by Theorem 2.2. But the latter is cofinal with the inverse limit of

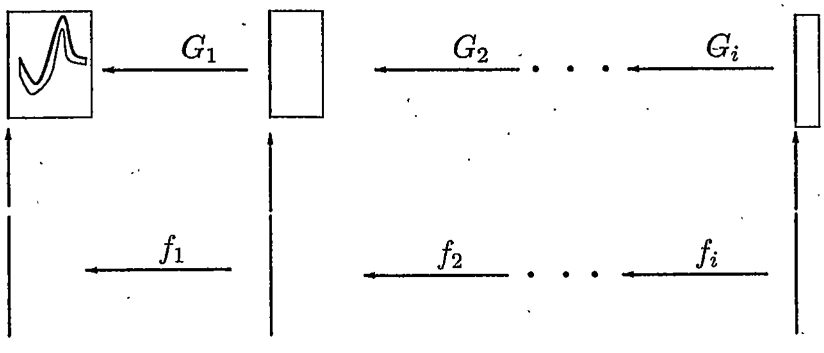
$\{Y_i, g_i \circ h_i^{-1} \circ h_i\}_{i\in\mathbb{N}} = \{Y_i, g_i\}_{i\in\mathbb{N}}$ and thus they are homeomorphic. For our purposes, however, it is convenient to include this lemma as our maps and spaces will be defined inductively and thus it is difficult to cite Theorem 2.2 directly.

CHAPTER 3

OUTLINE OF THE PROOF OF THEOREM 2

We now describe the idea behind the proof of Theorem 2 which will be done in detail in subsequent chapters.

Key to this proof is the use of Lemma 2.3 which allows us to view inverse limits on intervals as intersections. To see this, consider the inverse limit system determined by the sequence $\{f_i\}_{i\in\mathbb{N}}$. If we think of thickening up each of the intervals of domain to boxes, B_i, with the interval being the left edge of the box, these functions induce maps on boxes which mimic f_i in the sense that the projection to the left edge agrees with the original function. In particular, the following diagram κ_i-commutes (for κ_i from the lemma) if the thickness of the boxes is chosen small enough:



with the graph of f_1 pictured in Figure 2 to illustrate the induced map G_1.

But since each G_i is an embedding, the inverse limit space of $\{G_i\}_{i\in\mathbb{N}}$ is homeomorphic to $\bigcap_{i\in\mathbb{N}} G_i(B_i)$. Thus Lemma 2.3 allows us to view the inverse limit of the

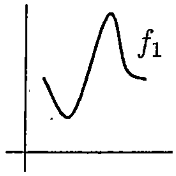


Figure 2. The graph of f_1 of the commuting diagram above.

f_i's as embedded in this intersection. We will use this technique of constructing boxes to build continua in the closure of the unstable manifold of Theorem 2.

In Chapter 4, we begin with a C^r-diffeomorphism, F, exhibiting a homoclinic tangency. All of our initial perturbations are geared toward constructing a linearized neighborhood and toward the crucial step of creating a new periodic point which exhibits an r-th order tangency between a branch of its unstable and stable manifolds. In a linearized neighborhood of this new periodic point we can make the stable and unstable manifolds the x-axis and y-axis, respectively. We find a point of tangency on the x-axis, $q = (q_x, 0)$, and a neighborhood V in which we can express the segment of the unstable manifold above the stable as the graph of $y = (x - q_x)^{r+1}$ as in Figure 3.

We intend to modify this segment of the unstable manifold lying in V. Since our perturbations must be small up to order r, it is important that the tangency has been changed to one of order r. The fact that the unstable manifold will be coming into the stable C^r-flat is what allows us to modify the unstable manifold as we desire while affecting the C^r-norm very little. In Chapter 4, we describe the modificaiton and go into detail as to how to perform the perturbations to achieve the desired

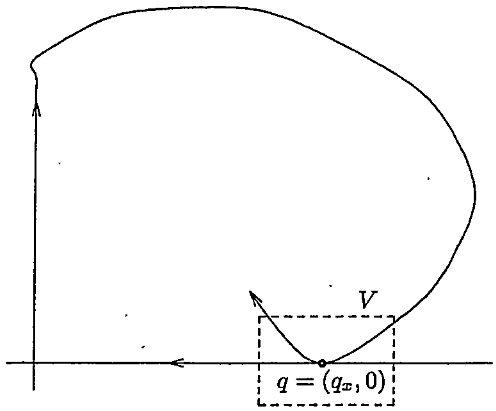


Figure 3. The linearized neighborhood of the new point of tangency.

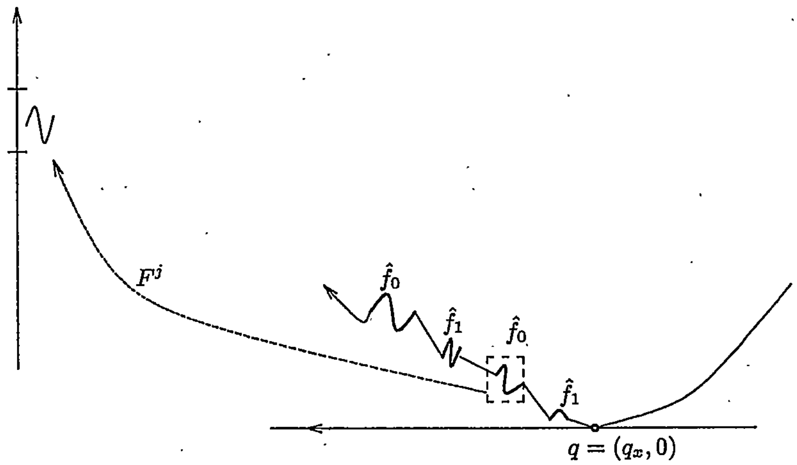


Figure 4. Approximating intervals with graphs of f_0.

modification. The process is to consider the maps of Theorem 2.1 which generate all chainable continua. We intend to place scaled graphs of these two maps into the segment of the unstable manifold above. We place an infinite number of "copies" of the graphs of each of these two maps with the ratio of their placement (as well as the ratio of the heights to widths) of each such that the graphs of each map, under the linearization of F, are densely mapped up the unstable manifold in the linearized neighborhood. That is, for any interval on the y-axis in the linearized neighborhood, a graph of either function can be made to approximate this interval (see Figure 4).

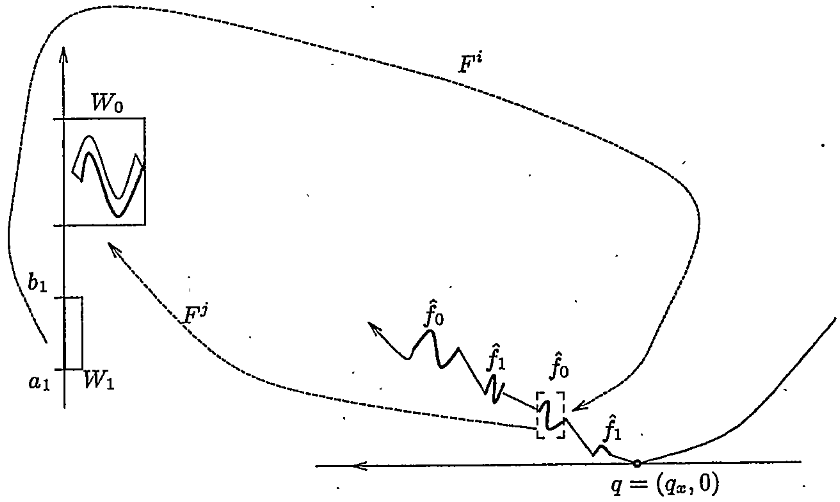


Figure 5. Thickening intervals to induce maps on W_1.

For any chainable continuum X, X is homeomorphic to an inverse limit space with bonding maps, f_i, chosen from $\{\hat{f}_0, \hat{f}_1\}$. Thus if W is an open set which intersects the closure of the unstable manifold, we intend to show we can embed this inverse limit space in this intersection. We do this by finding a box $W_0 = [0, \eta_0] \times [a_0, b_0]$ in the linearized neighborhood so that W_0 has the y-axis as the left edge as in Figure 5 and so that W_0 is eventually mapped into W under our diffeomorphism. Then, we can place a scaled graph of f_1 in W_0, since f_1 is one of the two maps whose graph is placed in the unstable manifold above. Then, we can find an interval on the y-axis $[a_1, b_1]$ so that this interval is mapped by our diffeomorphism to that segment of the unstable manifold which is the graph of f_1. Then, there is a η_1 so that $W_1 = [0, \eta_1] \times [a_1, b_1]$

embeds into W_0. Furthermore, there is an induced map from $[a_1, b_1]$ to $[a_0, b_0]$ which closely mimics f_1 and κ_0 commutes with the map from W_1 to W_0 under the appropriate inclusion mappings, where κ_0 is from Lemma 2.3.

Continuing in this way, we get a sequence of embeddings from W_{i+1} into W_i and maps from $[a_{i+1}, b_{i+1}]$ to $[a_i, b_i]$ mimicking f_i which κ_i commutes. Thus, the inverse limit space determined by the f_i sequence is embedded in W_0 by Lemma 2.3. Furthermore, the Hausdorff distance of this inverse limit space to the unstable manifold is shown to go to zero. Under the diffeomorphism it is then mapped into the intersection of W with the closure of the unstable manifold. Thus, the intersection of W with the closure of the unstable manifold is $\mathcal{C}$-universal. Since W was arbitrary, the closure of the unstable manifold is everywhere locally $\mathcal{C}$-universal.

CHAPTER 4

PERTURBATIONS FOR C^r-DIFFEOMORPHISMS EXHIBITING HOMOCLINIC TANGENCY

We will consider an arbitrary C^r-diffeomorphism F exhibiting a homoclinic tangency. In what follows, we will assume a saddle point p exhibiting the tangency is a fixed point of F (that is, $F(p) = p$), noting that we may simply replace F by F^P for p of period P. We will also use the convention that a C^r-perturbation is taken to mean an arbitrarily small C^r-perturbation. In this chapter, we perform a series of C^r-perturbations on F to obtain a new C^r-diffeomorphism which will be shown (see Chapter 5) to exhibit the properties in the conclusion of Theorem 2.

We begin this chapter by introducing some preliminary definitions and making some initial adjustments to a C^r-diffeomorphism having a saddle fixed point and, later, add in the condition that it exhibits a homoclinic tangency.

Preliminaries

Let $f : U \in \mathbb{R} \to \mathbb{R}$ a C^r-diffeomorphism. We will use d_x^k to mean the k-th derivative with respect to x or sometimes simply d^k when the independent variable is understood. The C^r-norm of f will be taken to be $\sup\limits_{x \in U, 0 \leq k \leq r} \{|d^k f(x)|\}$. For $G : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, we take the C^r-norm of G to be $||G||_r = \sup\limits_{(x,y) \in U, 0 \leq k \leq r} \left\{ \max\limits_{i+j=k} \left| \frac{\partial^k}{\partial x^i \partial y^j} G(x,y) \right| \right\}$. Lastly for $F : U \in \mathbb{R}^2 \to \mathbb{R}^2$, we take the C^r-norm of $F = (F_1, F_2)$ to be $\max\{||F_1||_r, ||F_2||_r\}$.

Definition 4.1. (Definition 2.1 in [GG]) Let X and Y be smooth manifolds, and p in X. Suppose $f, g : X \to Y$ are smooth maps with $f(p) = q = g(p)$.

(i) f has *first order contact* with g at p if $(df)_p = (dg)_p$ as mappings of $T_pX \to T_qY$.

(ii) f has *k-th order contact* with g at p (for $k \in \mathbb{N}, k > 1$) if $(df) : TX \to TY$ has $(k-1)$-st order contact with (dg) at every point in T_pX. We write this as $f \sim_k g$ at p.

Definition 4.2. A C^r diffeomorphism F of a closed 2-dimensional manifold with saddle periodic point p of period n is said to *exhibit a homoclinic tangency* if a branch of the stable manifold, $W_+^s(p)$, meets a branch of the unstable manifold, $W_+^u(p)$, at a point q different from p and if there exists a neighborhood V of q and C^1-immersions i_s and i_u of (a,b) into V, such that

(i) $i_s((a,b)) = \Gamma_s(q)$ and $i_u((a,b)) = \Gamma_u(q)$,

(ii) $i_s(\tilde{q}) = q = i_u(\tilde{q})$ for some $\tilde{q} \in \mathbb{R}$ and

(iii) i_s has first order contact with i_u at $\tilde{q}$.

where Γ_s is the arc-component of q in $W_+^s(p) \cap V$ and Γ_u is the arc-component of q in $W_+^u(p) \cap V$.

If $i_s \sim_k i_u$ at $\tilde{q}$ for C^k-immersions i_s and i_u, $k \leq r$, then F is said to exhibit a *k-th order tangency*.

The following lemma will be useful in that it will allow us to view homoclinic tangencies in terms of normal forms (See the section on normal forms below).

LEMMA 4.3. (*Lemma 2.2 and Corollary 2.3 in [GG]*) *Let V be an open subset of $\mathbb{R}^n$ containing $\tilde{q}$. Let $f, g : V \to \mathbb{R}^m$ be smooth mappings. Then f and g have k-th order contact at $\tilde{q}$ if and only if the Taylor series expansions of f and g agree up to (and including) order k are identical at $\tilde{q}$.*

Initial Perturbations and Coordinate Changes

Let F be a C^r-diffeomorphism of a 2-dimensional manifold M and let p be a saddle fixed point with the eigenvalues of DF being σ and μ, $0 < |\sigma| < 1 < |\mu|$. Then, we have the following definitions.

DEFINITION 4.4. The *saddle exponent of p* is defined to be the number $\rho(p, F) = -\frac{\log|\sigma|}{\log|\mu|}$. We call p a *ρ-shrinking saddle*, where $\rho = \rho(p, F)$. If ρ is greater than some k, then p is also said to be *at least k-shrinking*. If $\rho > 1$ (i.e. $|\sigma\mu| < 1$), p is said to be a *locally dissipative saddle*.

DEFINITION 4.5. A saddle p is called *non-resonant* if ρ is irrational, that is, if for any pair of integers n and m both not equal to zero, the number $\sigma^n\mu^m$ is different from one.

For $F \in \mathrm{Diff}^r(M)$ as above, we apply a C^r-perturbation to make F a C^∞ diffeomorphism. If $|\sigma\mu| > 1$, we replace F by F^{-1}. Theorem 2 will then hold for the stable manifold instead of the unstable. Also, we may C^r-perturb if F is resonant. Thus we are left with a C^∞, locally dissipative, non-resonant diffeomorphism. By the Sternberg linearization theorem ([St]), the new F is C^r-linearizable in a neighborhood U of

p and we may choose coordinates so that inside U, the stable and unstable manifolds coincide with the x-axis and y-axis, respectively. That is,

$$F|_U(x,y) = (\sigma x, \mu y).$$

In this dissertation, we assume that $0 < \sigma < 1 < \mu$, though the condition that the eigenvalues are positive is certainly not necessary.

Normal Forms at a Point of Tangency

For an F as above with p exhibiting homoclinic tangency, we intend to C^r-perturb F so as to obtain a C^r-diffeomorphism having the desired properties of Theorem 2. Here we describe normal forms for F^N in a neighborhood of a point of k-th order tangency in the linearized neighborhood U. Since all of our modifications will take place in the linearized neighborhood U, we will assume we are working in $\mathbb{R}^2$.

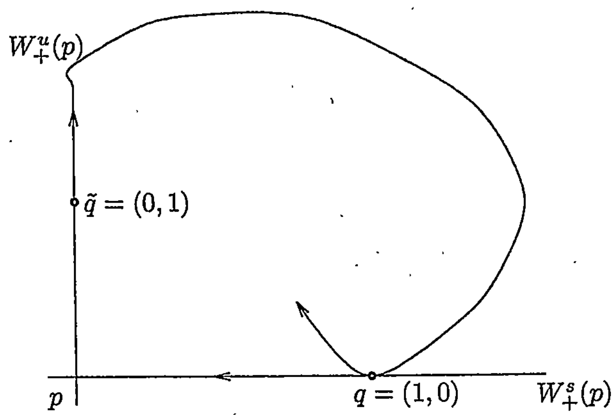


Figure 6. The linearized neighborhood U.

In order to make calculations easier, we will assume that a point of k-th order tangency between $W^s_+(p)$ and $W^u_+(p)$ is at $q=(1,0)\in U$ and for some N, $F^{-N}(q)=\tilde{q}=(0,1)\in U$ (see Figure 6). We will also assume without loss of generality that, as in the figure, the directions of $W^u(p)$ and $W^s(p)$ agree at the point of tangency. Let $V,\tilde{V}\in U$ be neighborhoods of q and $\tilde{q}$, respectively, so that $\Gamma=F^N(\tilde{V}\cap\{y-axis\})$ is the first arc-connected component of $W^u(p))$ in V. Define new coordinates inside V as $(\bar{x},\bar{y})=(1-x,y)$. Then Γ is the graph of $y=C\bar{x}^{k+1}+o(\bar{x}^{k+1}), C\neq 0$ (See Lemma 4.3).

Define new coordinates inside $\tilde{V}$ as $(\tilde{x},\tilde{y})=(x,y-1)$. Then $\tilde{V}$ and V can be rescaled so that the map $F^N|_{\tilde{V}}:(\tilde{x},\tilde{y})\mapsto(\bar{x},\bar{y})$ can be written as:

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}\overset{F^N}{\longmapsto}\begin{pmatrix}\tau\tilde{y}+\tilde{H}_1(\tilde{x},\tilde{y})\\ C\tilde{y}^{k+1}+\gamma\tilde{x}+\tilde{H}_2(\tilde{x},\tilde{y})\end{pmatrix},$$

where $C,\tau\neq 0$ and at $\tilde{x}=\tilde{y}=0$ we have $\tilde{H}_1=\partial_y\tilde{H}_1=0$ and $\tilde{H}_2=\partial_x\tilde{H}_2=\partial^j_y\tilde{H}_2=0$ for $1\leq j\leq k$.

We can again take coordinates $(\hat{x},\hat{y})=\left(\frac{\bar{x}}{\tau},\frac{\bar{y}}{C}\right)$ and then $F^N|_{\tilde{V}}:(\tilde{x},\tilde{y})\mapsto(\hat{x},\hat{y})$ has the normal form:

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}\overset{F^N}{\longmapsto}\begin{pmatrix}\tilde{y}+H_1(\tilde{x},\tilde{y})\\ \tilde{y}^{k+1}+B\tilde{x}+H_2(\tilde{x},\tilde{y})\end{pmatrix},$$

where $B=\gamma/C$ and $H_1=\dfrac{\tilde{H}_1}{\tau}, H_2=\dfrac{\tilde{H}_2}{C}$.

Remark 4.6: Lastly, we note that we may C^k-perturb so that $C,\tau>0$ and $H_1(0,\tilde{y})=H_2(0,\tilde{y})=0$ in $\tilde{V}$. It will be assumed that this perturbation has been performed whenever $k=r$, but not otherwise. When r is an odd number, the condition that

C be positive forces the tangency to be same-sided. Such a perturbation of C would look much the same as the perturbation in Figure 7 below.

Creating a Same-Sided Quadratic Tangency

It is necessary for our calculations to make the homoclinic tangency of the above $F \in \mathrm{Diff}^r(M)$ into one of order exactly r. The next two sections perform the perturbations necessary to achieve this end. The creation of a r-th order tangency from a k-th order tangency for $k < r$ has been done by Kaloshin ([K]). That technique will be discussed in the next section. In order to apply it, however, we must first C^r-perturb our k-th order tangency above to a same-sided quadratic tangency. The ability to do this falls into two cases:

Case 1: $k > 1$. Assume our diffeomorphism has a k-th order tangency for $k > 1$. Then, the local component of the unstable manifold near q, Γ, is the graph of $y = C\bar{x}^{k+1} + o(\bar{x}^{k+1}), C \neq 0$. We add a small $\epsilon \bar{x}^2$ term with $\epsilon > 0$, so as to make this tangency same-sided (see Figure 7).

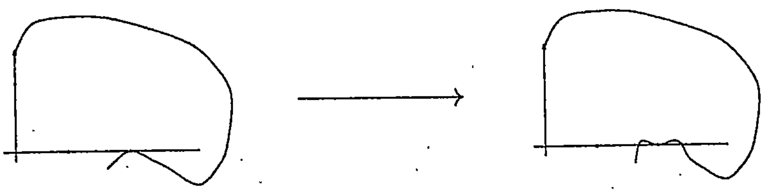


Figure 7. Changing higher order $C < 0$ to lower order $\epsilon > 0$.

Case 2: $k = 1$ and $r > 1$. A similar argument to what follows appears in [PT], Chapter 3, §1, to show that generic unfoldings of tangencies produce more tangencies.

There, however, the authors are not concerned with the question of the side on which the tangency occurs. Here we argue that it can be chosen to occur on either side.

Suppose again that Γ is the graph of $y = C\bar{x}^2 + o(\bar{x}^2)$ for $C < 0$. Consider a generic unfolding of this tangency by adding a $\epsilon > 0$ term to the second coordinate of the normal form small enough so that the local component of the unstable manifold, Γ^u_ϵ, crosses the stable manifold twice and is of the "parabolic" form $y = \epsilon + C\bar{x}^2 + o(\bar{x}^2)$. Then there are two topological pictures (depending on the sign of γ in the normal form of the last section) for how the local component of the stable manifold, Γ^s_ϵ must cross the unstable near $\tilde{q}$. First, we will assume it is as in Figure 8(a), which corresponds to $\gamma > 0$.

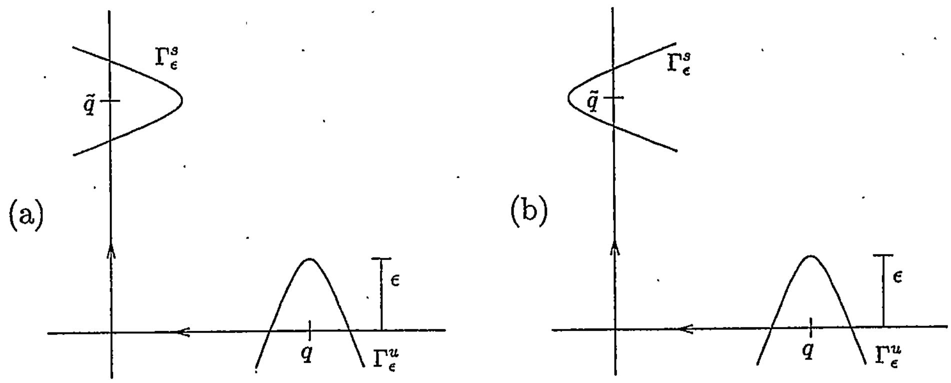


Figure 8. Two scenarios for unfolding a different-sided tangency.

For a fixed ϵ_0, let J be large enough so that that $F^{-J}_{\epsilon_0}(\Gamma^s_{\epsilon_0})$ and $\Gamma^u_{\epsilon_0}$ intersect at four points. As ϵ approaches zero, the shape of $F^{-J}(\Gamma^s_\epsilon)$ persists since this "parabola" depends C^r on ϵ. For that same J, there exist two values, ϵ_1 and ϵ_2, with $F^{-J}_{\epsilon_i}(\Gamma^s_{\epsilon_i})$

tangent to $\Gamma^u_{\epsilon_i}$ (for $i = 1, 2$) such that the tangencies occur on different sides of $\Gamma^u_{\epsilon_i}$, respectively. Two examples of these tangencies are given in Figure 9. Figure 9(a) and Figure 9(b) show these tangencies when $F^{-J}(\Gamma^s_\epsilon)$ is "shrinking" at a faster rate than Γ^u_ϵ. Figure 9(c) and Figure 9(d) show these tangencies when the opposite occurs.Other scenarios are possible, including entire intervals of tangency in the parameter space, but eventually a tangency must present itself on the other side due to the C^r-continuity of the transition. We, of course, choose among ϵ_1 and ϵ_2, the one which causes a same-sided tangency of the unstable manifold with the $x-axis$ in the linearized neighborhood.

This argument can be modified for the case where the relative positions of Γ^s_ϵ and Γ^u_ϵ are as in Figure 8(b), which corresponds to $\gamma < 0$ in the normal form of the last section. Only, in this case, it is even simpler since as ϵ approaches zero, Γ^u_ϵ must pull through $F_\epsilon^{-J}(\Gamma^s_\epsilon)$ as in Figure 10. This is because $F_\epsilon^{-J}(\Gamma^s_\epsilon)$ is not affected on the positive side of the stable manifold by changes in ϵ. That is, as ϵ approaches zero, $F_\epsilon^{-J}(\Gamma^s_\epsilon)$ persists in its crossing of the original $\Gamma^u_{\epsilon_0}$.

Constructing an r-th Order Tangency

Here we outline the technique of Kaloshin in the creation of a r-th order tangency from a k-th order tangency for $k < r$. This process is quite involved, so below we merely give a brief outline of the steps involved. These steps are carried out in detail, however, in [K]. We simply note that the initial conditions of his process are met by

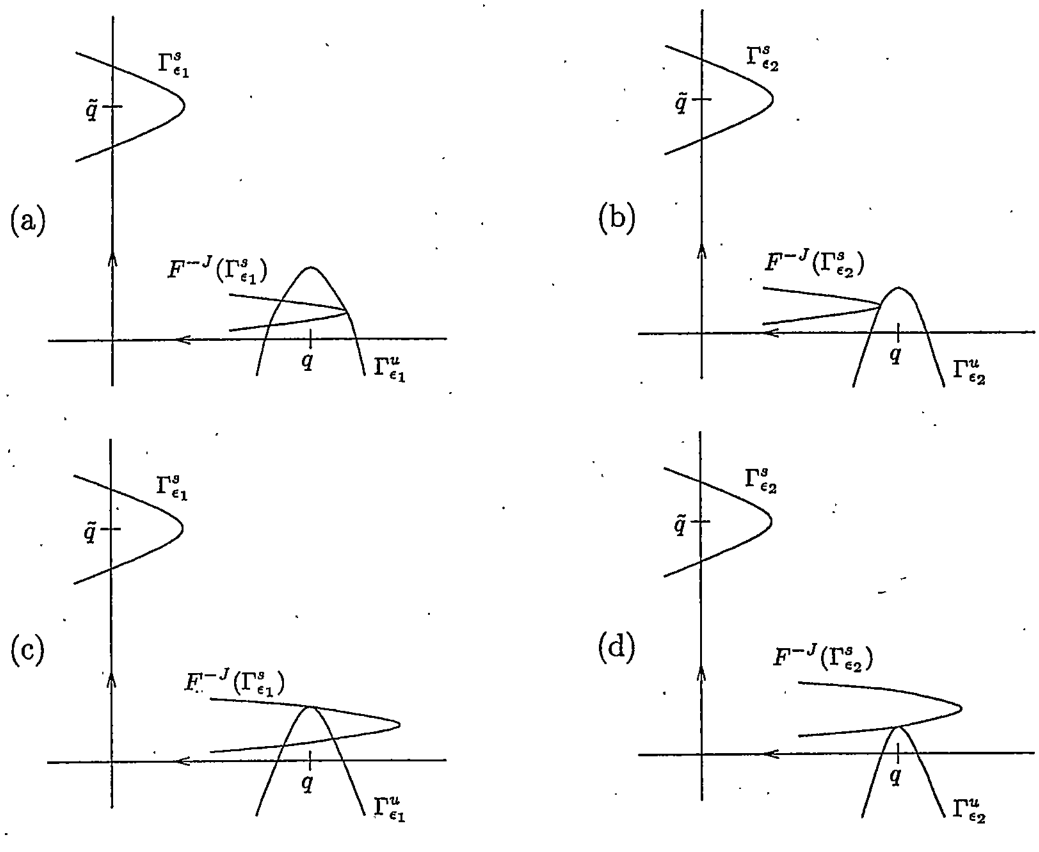


Figure 9. Creation of tangencies on either side (I).

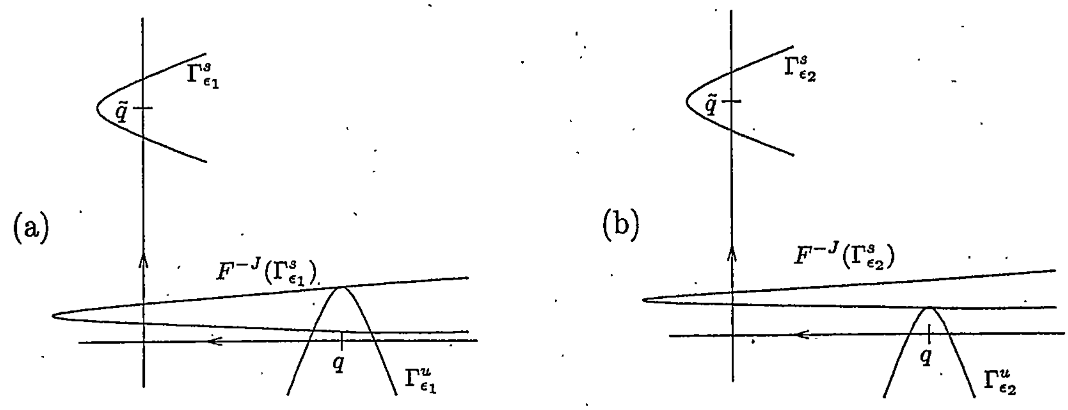


Figure 10. Creation of tangencies on either side (II).

the perturbations in the beginning of this chapter, the fact that our diffeomorphism is locally dissipative at p as well as the fact that our tangency is same-sided.

Remark 4.7: We note that we may skip this section for the case where $r = 1$ since the tangency is already of order C^1 by assumption. The condition that our diffeomorphism be locally dissipative at p will not be needed since we will not be applying the following technique in this case, which requires such a condition. This is of course true in general; that is, we may omit this step and the locally dissipative condition if the tangency is of order r already.

It should be noted that in actuality, the new tangency of order r, will not be a tangency of the branches of the unstable and stable manifolds for the original fixed point p above. This is the reason for the statement of Theorem 2, that a new periodic point exists with the property of the result. We therefore include this outline to give the reader some idea of where this new tangency occurs in relation to the old one. It will also be necessary in proving the main result of this dissertation in Chapter 6 to understand how this new tangency is formed.

Kaloshin's technique relies on the normal forms of the diffeomorphism near a point of tangency in the linearized neighborhood. Let q and $\tilde{q}$ be as above with $F^N(\tilde{q}) = q$ for some N. Also, let $\tilde{V}$ and V, the neighborhoods of $\tilde{q}$ and q, respectively, be as defined above. In V, one can place a sequence of rectangles T_n centered at $(1, \mu^{-n})$ so that $F^n(T_n) \subset \tilde{V}$ as in Figure 11. T_n can be chosen precisely (see [K]) so that $F^{n+N}(T_n)$ forms a curvilinear rectangle in V. Moreover, as long as p is locally dissipative, T_n and $F^{n+N}(T_n)$ form a horseshoe which has two periodic saddles of period $n + N$. Let p_n be the saddle with positive eigenvalues.

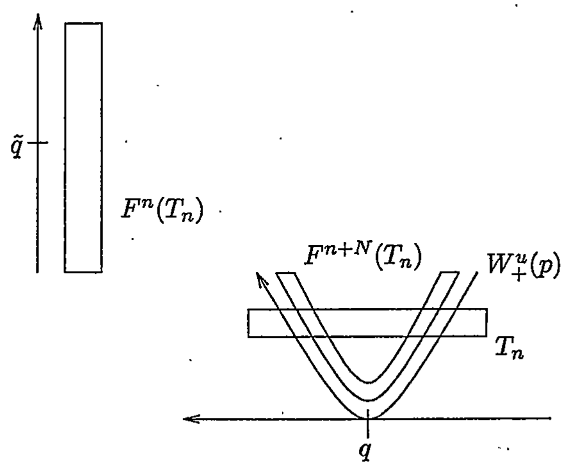


Figure 11. Neighborhood of tangency in U.

We are now ready to outline Kaloshin's technique in three steps. The first step is a technical lemma, though the last two afford some pictures which hint at the technique. Again, these calculations are quite involved, and the reader is referred to [K] for detailed analysis.

The first step. From the existence of a homoclinic tangency of a locally dissipative saddle, one deduces the existence (after a C^r-perturbation) of a homoclinic tangency of an at least k-shrinking saddle (see Definition 4.4 above), $k > r$. This result follows from the following lemma and corollary.

LEMMA 4.8. (*Lemma 1 in [K]*) *For $k \geq 1$, consider a generic $(k+1)$-parameter unfolding of a k-th order homoclinic tangency:*

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix} \overset{F^N_{\mu(n)}}{\longmapsto} \begin{pmatrix}\tau\tilde{y} + \tilde{H}_1(\tilde{x},\tilde{y})\\ C\tilde{y}^{k+1} + \sum_{i=0}^k \mu_i \tilde{y}^i + \gamma\tilde{x} + \tilde{H}_2(\tilde{x},\tilde{y})\end{pmatrix}.$$

For an arbitrary set of real numbers $\{M_i\}_{i=0}^k$, there exists a sequence of parameters $\{\mu(n) = (\mu_0(n), ..., \mu_k(n))\}_{n\in\mathbb{N}}$ such that $\mu(n)$ tends to 0 as $n \to \infty$ and a sequence of change of variables $R_n : T_n \to [-2,2] \times [-2,2]$ such that the sequence of maps: $\{R_n \circ F^{n+N}_{\mu(n)} \circ R_n^{-1}\}$ converges to the 1-dimensional map

$$\begin{pmatrix}x\\ y\end{pmatrix} \overset{\phi_M}{\longmapsto} \begin{pmatrix}y\\ y^{k+1} + \sum_{i=0}^k M_i y^i\end{pmatrix}$$

in the C^r-topology for every r.

COROLLARY 4.9. (*Corollary 2.4 in [K]*). *For $k = 1, M_0 = -2$, and $M_1 = 0$, by a C^r-perturbation of a C^∞-diffeomorphism F exhibiting a quadratic (C^1) tangency, one can create a C^∞-diffeomorphism F with a periodic saddle p exhibiting a homoclinic tangency and the eigenvalues of $DF^M|_p$ are close to 4 and +0, respectively.*

Here, 4 and 0 are the eigenvalues associated with the fixed point $(2,2)$ of $(x,y) \mapsto (y, y^2 - 2)$. The fact that we can perturb to a diffeomorphism which exhibits homoclinic tangency is shown in § 6.3 of [PT]. Thus the saddle exponent can be made as large as desired, so F is at least k-shrinking.

The second step. From the existence of a homoclinic tangency of an at least k-shrinking saddle, one creates a k-floor tower after a C^r-perturbation. Before proving this claim, we include the following definition.

DEFINITION 4.10. (Definition 4 in [K]) A *k-floor tower* consists of k saddle periodic points $p_1, ..., p_r$(of different periods) such that $W^u_{loc}(p_i)$ is tangent to $W^s_{loc}(p_{i+1})$ for $i = 1, ..., k-1$, and $W^u_{loc}(p_k)$ intersects $W^s_{loc}(p_1)$ transversally (Figure 12).

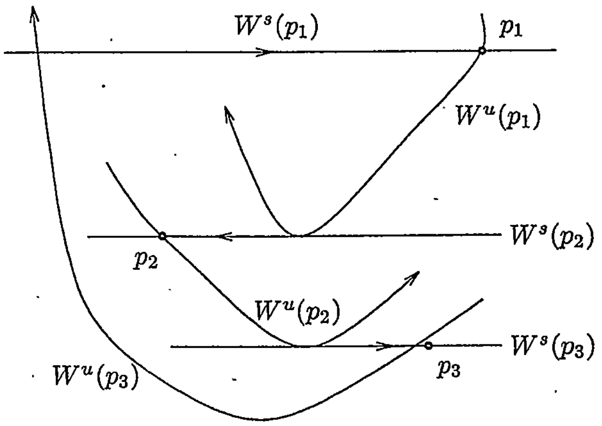


Figure 12. A 3-floor tower.

We describe the creation of the desired k-floor tower for the above saddle pictorially, so that we may give the reader an idea of where it occurs. Consider the rectangles T_n above. One can choose a subsequence T_{n_i} so that $T_{n_{i+1}}$ and $F^{n_i+N}(T_{n_i})$ intersect in a horseshoe-like way. Figure 13 shows this scenario. Also included in this figure are pieces of branches of the stable and unstable manifolds for the saddle point p_i of the horseshoes $\bigcap_{j=-\infty}^{\infty} F^{(n_i+N)j}(T_{n_i})$ as well as those for p_{i+1}. Furthermore, detailed analysis shows that as long as the diffeomorphism is at least k-shrinking, we can choose these T_{n_i} so that $F^{n_i+N}(T_{n_i})$ barely clears $T_{n_{i+1}}$ in its crossing for $i = 1, ..., k-1$. Thus, the branch of the unstable manifold of p_i crossing the branch of the stable manifold of p_{i+1} can be C^r-perturbed to obtain a tangency and so the k-floor tower can be formed.

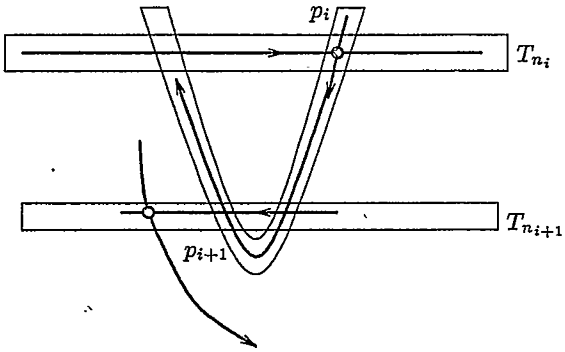


Figure 13. Forming of tower structure.

The third step. From the existence of a $(k+1)$-floor tower, one shows that a C^r-perturbation can make a k-th order homoclinic tangency. This step is done inductively and has two different topological pictures: one for k being odd, and another for k being even. Suppose one has a $(k+1)$-floor tower. It is desired to construct a 1-st order tangency of $W^u(p_k)$ and $W^s(p_1)$. One does this by perturbing $W^u(p_k)$ so that a piece of it, γ_0, "falls below" the stable manifold of p_{k+1} as in Figure 14. Then this piece will follow $W^u(p_{k+1})$ under iterations to its transversal crossing with $W^s(p_1)$. The perturbation can then be adjusted so that this piece is tangent to $W^s(p_1)$.

One can now ignore the $(k+1)$-st floor in the tower and consider the new k-floor structure. We will refer to this structure as a *k-floor quadratic tower* to emphasize that it differs from the normal towers in that $W^u(p_k)$ does not intersect $W^s(p_1)$ transversally, but rather in a quadratic heteroclinic tangency.

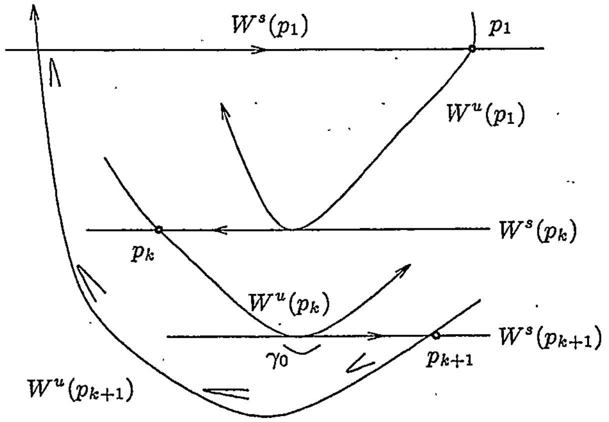
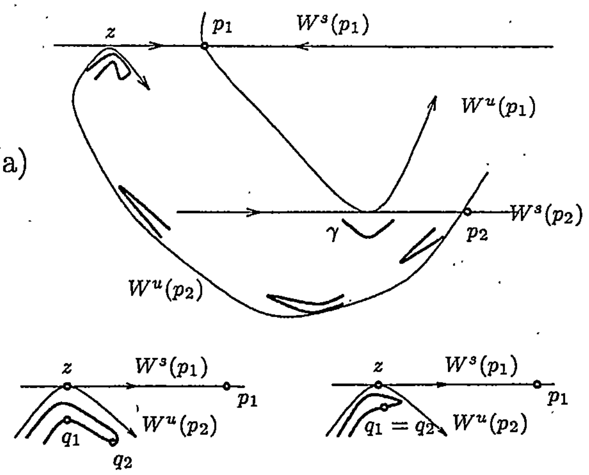


Figure 14. Creating tangency of $W^u(p_k)$ with $W^u(p_1)$.

One now proceeds inductively using the following result. Suppose there are three periodic points p_1, p_2 and p_3 (p_3 possibly the same point as p_1) such that $W^u(p_1)$ has a $(k-1)$-st order tangency with $W^s(p_2)$ and so that $W^u(p_2)$ has a tangency with $W^s(p_3)$. We will see that a small perturbation creates a k-th order tangency of $W^u(p_1)$ with $W^s(p_3)$. Figure 15(a) shows this for k even and the case in which p_3 is the same point as p_1. First, a perturbation is made so that a piece of $W^u(p_1)$ falls below $W^s(p_2)$. Call this piece γ as in the figure. Under iteration, γ follows the fate of $W^u(p_2)$ so that it ends up near the tangency z. Figure 15(b) shows that there then must be points q_1 and q_2 at which γ is tangent to the horizontal direction. Now, the perturbation can be adjusted so that q_1 and q_2 merge to the same point (see Figure 15(c)). Careful analysis shows that this point forms a k-th order tangency with the horizontal direction. A small perturbation pushes this tangency up so that it is a tangency with $W^s(p_1)$. A similar picture holds for k odd.

(a)



(b) (c)

Figure 15. Creating higher order tangencies for $p_1 = p_3$ and even k.

We are now ready to describe how to prove the claim. The original $(k+1)$-floor tower has been perturbed into a k-floor quadratic tower. Using the above result inductively, one may "push" $W^u(p_1)$ down the unstable manifolds of the periodic points so that $W^u(p_1)$ has a $(k-1)$-st order tangency with $W^s(p_k)$. Then, once again applying the result with p_1 playing the role of the first and last periodic point, $W^u(p_1)$ can be perturbed to have a k-th order tangency with $W^s(p_1)$. This completes our outline of Kaloshin's technique.

Remark 4.11: Though it is only necessary to form a $(r+1)$-floor tower in order to create a r-th order tangency, one can use a k-floor tower for any $k > r$ to form such a tangency. It will be convenient in Chapter 7 to choose k quite large.

Remark 4.12: The branches of the stable and unstable manifold which exhibit this new tangency are not those from the original fixed point, but rather from a new periodic point near the original fixed point. So once again, we replace some iterate