Rotation sets of flows on higher dimensional tori
by Doreen Norma Dumonceaux

A dissertation submitted in partial fullillment of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences
Montana State University

Abstract:
Rotation sets of points under a flow measure the average displacement of orbits as time goes to infinity. In lower dimensions, it has been shown that there is a strong link between the properties of the rotation sets of a flow and the dynamics of that flow.

In this dissertation flows on the 3-torus are constructed with rotation sets that have 3-dimensional interior and periodic-point-free flows on the n-torus ($n \geq 4$) are constructed with rotation sets that have n-dimensional interior. A natural question is which sets can be rotation sets of flows. It is shown that: every polyhedron in R^3 with rational vertices that does not contain the origin is the rotation set for a flow on the 3-torus; every C^r curve in R^n is the rotation set of a flow on (n + l)-torus; and every compact 2-manifold that can be embedded in is the rotation set for a flow on the (n + 2)-torus. In assessing the box dimension of rotation sets of flows it is shown that: for any α □ [0,2] □ {3} there is a continuous flow on the 3-torus such that the rotation set of the flow has box dimension equal to α; and for any α □ [0,1] □ {2} □ {3} there is a smooth flow on the 3-torus such that the rotation set of the flow has box dimension equal to α.

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APPROVAL

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Doreen Norma Dumonceaux

This dissertation has been read by each member of the dissertation committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the college of Graduate Studies.

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STATEMENT OF PERMISSION TO USE

To my father, Dr. Robert Dumonceaux,
your example is the one I choose to follow.

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ABSTRACT

Rotation sets of points under a flow measure the average displacement of orbits as time goes to infinity. In lower dimensions, it has been shown that there is a strong link between the properties of the rotation sets of a flow and the dynamics of that flow.

In this dissertation flows on the 3-torus are constructed with rotation sets that have 3-dimensional interior and periodic-point-free flows on the n-torus ($n \geq 4$) are constructed with rotation sets that have n-dimensional interior. A natural question is which sets can be rotation sets of flows. It is shown that: every polyhedron in $\mathbb{R}^3$ with rational vertices that does not contain the origin is the rotation set for a flow on the 3-torus; every C^r curve in $\mathbb{R}^n$ is the rotation set of a flow on $(n+1)$-torus; and every compact 2-manifold that can be embedded in $\mathbb{R}^n$ is the rotation set for a flow on the $(n+2)$-torus. In assessing the box dimension of rotation sets of flows it is shown that: for any $\alpha \in [0,2] \cup \{3\}$ there is a continuous flow on the 3-torus such that the rotation set of the flow has box dimension equal to α; and for any $\alpha \in [0,1] \cup \{2\} \cup \{3\}$ there is a smooth flow on the 3-torus such that the rotation set of the flow has box dimension equal to α.

CHAPTER 1

HISTORY AND INTRODUCTION

History

The idea of a *rotation number* was first introduced by Henri Poincaré in the late nineteenth century ([17]). The concept was inspired by his study of the qualitative nature of the orbits of those flows on the torus which generate return maps that are circle homeomorphisms. The rotation number (also referred to as the *winding number*) measures the asymptotic rotation rate of iterates of these circle return maps. In particular, Poincaré proved that, in the setting where the return map from a circular cross-section of the torus back to itself is an orientation preserving homeomorphism of a circle, the rotation number exists and is independent of the point on the circle. He also proved that if a rotation number is rational, then that rotation number is realized by a periodic orbit; that is, there exists a periodic orbit of the homeomorphism with that rational as its rotation number ([17]). The rotation number has proven to be a useful invariant in the case of circle maps and much effort has been given to extend this concept to higher dimensional settings. Some relevant properties of rotation sets of homeomorphisms, maps, and flows on the circle, annulus, and n-dimensional torus are summarized in Tables 1, 2, and 3, respectively.

Unlike the circle homeomorphism case, rotation sets of individual orbits of circle endomorphisms are dependent upon the orbit under consideration. In 1979, the concept of rotation number was extended by Sheldon Newhouse, Jacob Palis, and Floris Takens to the *rotation set* of circle endomorphisms homotopic to the identity ([19]). Such rotation sets are the union of rotation sets of individual orbits and are invariant under conjugacy. The rotation set of an orbit of a circle endomorphism can be a singleton or a closed interval. Therefore, rotation sets of circle endomorphisms may have interior. Every rational contained in the full rotation set of the endomorphism is realized as the rotation number of a periodic orbit ([19]). In 1989, Ryuichi Ito proved that the full rotation set of a degree-one circle endomorphism is closed ([10]).

The emphasis of this dissertation is on rotation sets of flows on tori of various dimensions. By continuity, the rotation set of a flow is equal to the rotation set of the time-one map of that flow. In the one-dimensional case, the rotation set of a flow on the circle is always a single number and is independent of the orbit. We now turn to the annulus case.

The study of rotation sets has been further extended to homeomorphisms, maps, and flows on the annulus, $\mathbb{A} = \mathbb{S}^1 \times [0,1]$. In 1990, Michael Handel proved that if $f : \mathbb{A} \to \mathbb{A}$ is an orientation preserving, boundary component preserving homeomorphism, then the rotation set of f is closed ([8]). Furthermore, if f is also area-preserving, then the rotation set is a closed interval ([8]). John Franks showed that for each rational in the interior of the rotation set of f, there exists a periodic orbit with rotation number equal to that rational ([4]). It follows that, if the rotation set has interior, then the map must have periodic points. If $f : \mathbb{A} \to \mathbb{A}$ is map of the annulus, then it is an open question as to whether its rotation set is closed.

Again, because the rotation sets for flows of the annulus are the same as the time-one map, the rotation set of an annulus flow must be closed. A flow on the annulus which is the union of periodic orbits with varying periods has full rotation set with interior.

Now we consider the rotation sets of homeomorphisms, maps, and flows on n-dimensional tori, $\mathbb{T}^n = \overbrace{\mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1}^{n\text{- times}}$, $(n \geq 2)$. We will see that as n increases the link between rotation set structure and dynamical properties weakens. Michal Misiurewicz and Krystyna Ziemian developed an alternative definition of rotation set for homeomorphisms and maps on the n-dimensional torus and have established various properties of that rotation set ([15]). Consider a homeomorphism, $f : \mathbb{T}^2 \to \mathbb{T}^2$, which is homotopic to the identity. It follows from the work of Franks, as well of that of Misiurewicz and Ziemian, that the rotation set of f must contain its extreme points as well as the 2-dimensional interior of its closed convex hull ([5])([15]). Franks has also shown that every rational vector in the interior of the rotation set is the rotation vector for some periodic orbit ([5]). This followed the work of Handel who proved that periodic-point-free homeomorphisms of the 2-torus cannot have rotation sets with interior ([7]). Jaume Llibre and Robert MacKay proved related results ([14]). But, it is still unknown whether or not the rotation set of a homeomorphism, $f : \mathbb{T}^2 \to \mathbb{T}^2$, is closed.

A natural question is the following: Which subsets of the plane may be realized as rotation sets of n-dimensional toral homeomorphisms? Jaroslaw Kwapisz has shown that every convex polygon with rational vertices is realized as the rotation set for some homeomorphism of the 2-torus ([12]). Since rotation sets depend continuously on the map, this implies that some polygons with other than rational vertices can be rotation sets of some toral homeomorphisms ([16]). Kwapisz later constructed a smooth diffeomorphism on the 2-torus with a non-polygonal rotation set with interior ([13]).

Marcy Barge and Russell Walker provide examples of periodic-point-free endomorphisms on $\mathbb{T}^n$ which are C^∞, and C^∞ diffeomorphisms on the n-torus, ($n \geq 3$), with rotation sets which have interior ([1]). These examples illustrate that the link between rotation sets of maps with interior and the periodicity of orbits breaks down on n-tori when $n \geq 3$.

If $\varphi^t : \mathbb{T}^2 \to \mathbb{T}^2$ is a flow on the torus with lift, $\widetilde{\varphi}^t : \mathbb{R}^2 \to \mathbb{R}^2$, then again the nature of the rotation set of φ implies the existence of certain dynamics. Franks and Misiurewicz consider the more general Misiurewicz-Ziemian (M-Z) rotation set,

$\rho_{\text{M-Z}}(\varphi)$ ([15]). This is defined by $v \in \rho_{\text{M-Z}}(\varphi)$ if and only if there are sequences $x_i \in \mathbb{R}^2$ and $t_i \in \mathbb{R}^+$ with $\lim_{i\to\infty} t_i = \infty$ such that $\lim_{i\to\infty} \frac{\widetilde{\varphi}^{t_i}(x_i) - (x_i)}{t_i} = v$. Franks and Misiurewicz show that there are only three possibilities for the M-Z rotation set of a 2-torus flow.

1. The rotation set may be a single point, $v \in \mathbb{R}^2$.
2. The rotation set may be a closed segment contained in a line passing through 0 and another rational point in $\mathbb{R}^2$ (the segment need not contain 0).
3. The rotation set may be a closed line segment with one end at 0 and having irrational slope.

In particular, M-Z rotation sets of flows on the 2-torus must be closed and cannot have 2-dimensional interior ([6]). The M-Z rotation set contains the point rotation set which is the case of the M-Z rotation set when a fixed point in the domain is considered rather than a sequence of points. We use the point rotation set throughout this dissertation. In Chapter 6, we show that there is a flow on the 2-torus with (point) rotation set equal to a Cantor set in $\mathbb{R}^2$.

Now consider the case of homeomorphisms of $\mathbb{T}^n$ ($n \geq 3$). These rotation sets may have interior, but as previously discussed, Barge and Walker demonstrate that a rotation set with non-empty interior does not guarantee the existence of periodic points ([1]). Richard Swanson and Walker have constructed homeomorphisms on $\mathbb{T}^n$ with rotation sets that are not closed ([21]). Since the homeomorphisms constructed by Swanson and Walker are flowable (time-one maps of a flow), rotation sets for flows on the n-torus, ($n \geq 3$) need not be closed.

Table 1. Rotation Sets of Homeomorphisms

	Must be Singleton	Must be Closed	Can have Interior	Can Have Interior w/o Periodic Points
$\mathbb{S}^1$	Yes [17]	Yes [17]	No [17]	No [17]
$\mathbb{A}^2$	No	Yes [8]	Yes	No [4]
$\mathbb{T}^2$	No	Open	Yes [5][16]	No [5][14]
$\mathbb{T}^3$	No	No [21]	Yes	Yes [1]
$\mathbb{T}^n$	No	No [21]	Yes	Yes

Table 2. Rotation Sets of Maps

	Must be Singleton	Must be Closed	Can have Interior	Can have Interior w/o Periodic Points
$\mathbb{S}^1$	No [19]	Yes [10]	Yes [19]	No [19]
$\mathbb{A}^2$	No	Yes	Yes [2]	No
$\mathbb{T}^2$	No	Open	Yes	Yes [1]
$\mathbb{T}^3$	No	No [21]	Yes	Yes [1]
$\mathbb{T}^n$	No	No [21]	Yes	Yes [1]

Table 3. Rotation Sets of Flows

	Must be Singleton	Must be Closed	Can have Interior	Can Have Interior w/o Periodic Points
$\mathbb{S}^1$	Yes	Yes	No	No
$\mathbb{A}^2$	No	Yes	Yes	No
$\mathbb{T}^2$	No	Yes	No [6]	No
$\mathbb{T}^3$	No	No [21]	Yes Thm 2.1	Open
$\mathbb{T}^4$	No	No	Yes Thm 3.4	Yes Thm 3.4
$\mathbb{T}^n$	No	No	Yes Thm 3.4	Yes Thm 3.4

Results

In this dissertation, we prove several theorems about the rotation sets of flows on higher dimensional tori. These results further demonstrate the breakdown in the link between rotation set topology and flow dynamics on n-tori, as n increases. Several of these results also address the question as to which sets can be the rotation set of a flow on the n-torus.

We first use direct techniques to prove two results about flows with "thick" rotation sets:

THEOREM 2.1. There exists a C^∞ flow, $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$, that has a rotation set with 3-dimensional interior.

THEOREM 3.4. For each $n \geq 4$, there exists a C^∞ flow, $\varphi^t : \mathbb{T}^n \to \mathbb{T}^n$, such that

(i) the rotation set ρ_φ, has n-dimensional interior, and

(ii) φ^t has no periodic points.

Next, we use more sophisticated but indirect techniques to extend Theorem 2.1.

THEOREM 4.2. Let $K \subset \mathbb{R}^3$ be a convex polyhedron with vertices at rational points of $\mathbb{R}^3$ such that $(0,0,0) \notin K$. Then there exists a $\mathbb{C}^\infty$ flow, $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$, such that the rotation set $\rho_\varphi = K$.

Since the rotation set of a flow is equal to the rotation set of its time-one map, we have the following corollary to Theorem 4.2.

COROLLARY 4.5. Let $K \subset \mathbb{R}^3$ be a convex polyhedron with vertices at rational points of $\mathbb{R}^3$ such that $(0,0,0) \notin K$. Then there exists a $\mathbb{C}^\infty$ homeomorphism, $f : \mathbb{T}^3 \to \mathbb{T}^3$ with lift $\widetilde{f} : \mathbb{R}^3 \to \mathbb{R}^3$, such that the rotation set $\rho_f = K$.

Focusing on the question of which sets can be rotation sets, we prove the following:

THEOREM 5.1. For any C^r curve, $\gamma : [0,1] \to \mathbb{R}^n$, there exists a C^r flow, $\varphi^t : \mathbb{T}^{n+1} \to \mathbb{T}^{n+1}$, such that the rotation set $\rho_\varphi = \text{Image}(\gamma) \times \{0\} \subset \mathbb{R}^{n+1}$.

Using a Theorem due to Hans Hahn and Stefan Mazurkiewicz ([9]), we have the following corollary:

COROLLARY 5.2. For any $K \subset \mathbb{R}^n$, that is compact, connected, and locally connected, there exists a continuous flow, $\varphi^t : \mathbb{T}^{n+1} \to \mathbb{T}^{n+1}$, such that the rotation set $\rho_\varphi = K \times \{0\} \subset \mathbb{R}^{n+1}$.

For Theorem 5.1 and Corollary 5.2 the smoothness of the flow is restricted by the smoothness of γ. If the dimension of the image of γ is greater than 1, γ can be only continuous. But, if the dimension of the domain of the map is higher, more smoothness can be realized:

THEOREM 5.3. Let $H : [0,1] \times [0,1] \to \mathbb{R}^n$ be a C^∞ map such that $H(r,0) = H(r,1)$, for all $r \in [0,1]$, and $H(0,s) = H(1,s)$, for all $s \in [0,1]$. Then there exists a C^∞ flow, $\varphi^t : \mathbb{T}^{n+2} \to \mathbb{T}^{n+2}$, such that the rotation set $\rho_\varphi = \text{Image(H)} \times \{0\} \times \{0\} \subset \mathbb{R}^{n+2}$.

Whether there exist endomorphisms of the 2-torus with circular rotation sets is an open question. We have the following corollary for a smooth flow on the 4-torus:

COROLLARY 5.4. Let D^2 be the unit disk contained in $\mathbb{R}^2$. Then there exists a C^∞ flow, $\varphi^t : \mathbb{T}^4 \to \mathbb{T}^4$, such that the rotation set $\rho_\varphi = D^2 \times \{0\} \times \{0\} \subset \mathbb{R}^4$.

Next we show that any compact 2-manifold, M, imbedded into $\mathbb{R}^n$ is the rotation set for a flow on the $(n+2)$-torus.

COROLLARY 5.6. Let M be a compact 2-manifold imbedded in $\mathbb{R}^n$. Then there exists a C^∞ flow, $\varphi^t : \mathbb{T}^{n+2} \to \mathbb{T}^{n+2}$, such that the rotation set $\rho_\varphi = M \times \{0\} \times \{0\} \subset \mathbb{R}^{n+1}$.

In the last chapter we turn to the question of the box-counting dimension, $\dim_B$, of rotation sets of toral flows. We prove the following:

THEOREM 6.3. For any $0 \leq \alpha \leq 1$, there exists a C^∞ flow, $\varphi^t : \mathbb{T}^2 \to \mathbb{T}^2$, such that the box-counting dimension of the rotation set, $\dim_B(\rho_\varphi) = \alpha$.

The following corollary to Theorem 5.3 concludes that rotation sets for flows on the 3-torus may have fractional box dimension between 1 and 2.

COROLLARY 6.4. For any $1 < \alpha < 2$, there exists a continuous flow, $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$, such that the box-counting dimension of the rotation set, $\dim_B(\rho_\varphi) = \alpha$.

To summarize the results of this dissertation concerning the dimension of rotation sets, we present the following two theorems.

THEOREM 6.5. For any $\alpha \in [0, 2] \cup \{3\}$, there exists a continuous flow, $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$, such that the the box-counting dimension of the rotation set, $\dim_B(\rho_\varphi) = \alpha$.

THEOREM 6.6. For any $\alpha \in [0, 1] \cup \{2\} \cup \{3\}$, there exists a C^∞ flow, $\varphi^t : \mathbb{T}^3 \to \mathbb{T}^3$, such that the the box-counting dimension of the rotation set, $\dim_B(\rho_\varphi) = \alpha$.

Definitions

We now establish some notation and definitions that will be used throughout this dissertation.

Let $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ denote the unit circle and let $\mathbb{T}^n = \overbrace{\mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1}^{n \text{ times}}$ be the n-dimensional torus. As our universal cover we use the map, $\Pi : \mathbb{R}^n \to \mathbb{T}^n$, given by:

$$\Pi(x_1, x_2, \ldots, x_n) = (\exp(2\pi i x_1), \exp(2\pi i x_2), \cdots, \exp(2\pi i x_n)) \tag{1.1}$$

where $x_j \in \mathbb{R}$ for each $j = 1, \ldots, n$.

We denote by $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ the *lift of the map*, $f : \mathbb{T}^n \to \mathbb{T}^n$, if $\widetilde{f}$ is continuous and $\Pi \circ \widetilde{f} = f \circ \Pi$. Note that for any given f, which is isotopic to the identity, with lift $\widetilde{f}$, $\widetilde{f}(\widetilde{p} + k) = \widetilde{f}(\widetilde{p}) + k$ for all $\widetilde{p} \in \mathbb{R}^n$ and $k \in \mathbb{Z}^n$.

Definition 1.1. *The* rotation set of $\widetilde{p}$ under the lift $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ of $f : \mathbb{T}^n \to \mathbb{T}^n$ *is defined by*

$$\rho(\widetilde{f}, \widetilde{p}) = \mathrm{LIM}\left\{ \frac{\widetilde{f}^n(\widetilde{p}) - \widetilde{p}}{n} \middle| n \in \mathbb{N} \right\}$$

where $\widetilde{p} \in \Pi^{-1}(p)$ *for* $p \in \mathbb{T}^n$.

Here, and throughout this dissertation, we use "LIM" to denote "the set of all limit points." Note that since the set of all limit points of any set is closed, then by definition the rotation set of $\widetilde{p}$ is closed.

Definition 1.2. *The* rotation set of $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}^n$, *denoted by* $\rho(\widetilde{f})$, *is defined by*

$$\rho(\widetilde{f}) = \bigcup_{\widetilde{p} \in \mathbb{R}^n} (\rho(\widetilde{f}, \widetilde{p}))$$

where $\widetilde{p} \in \Pi^{-1}(p)$ *for* $p \in \mathbb{T}^n$.

Note that the definition of rotation set used here is called the "point-wise rotation set" by Misiurewicz and Ziemian ([15]). Rotation sets of different lifts of a map, f, differ only by integer translates. We will refer to the rotation set of a map, $f : \mathbb{T}^n \to \mathbb{T}^n$, computed using a fixed lift for by ρ_f.

A Toroidal Horseshoe Map on the 2-Torus

One type of map that will be utilized in the construction of several examples in this dissertation will be the 3-symbol toroidal horseshoe homeomorphism. For our purposes, we list the properties of such a horseshoe which we will need. See Barge-Walker for a detailed construction ([1]).

The 3-symbol toroidal horseshoe, $f : \mathbb{T}^2 \to \mathbb{T}^2$, contains a rectangle $R \subset \mathbb{T}^2$, which in turn contains three disjoint sub-rectangles, I_0, I_1, I_2. f contracts R in one direction while stretching R in the perpendicular direction about both generators of the 2-torus so that $f(I_0)$, $f(I_1)$, and $f(I_2)$ meet R as shown in Figure 1. On $I_0 \cup I_1 \cup I_2$, f linearly contracts in one direction and linearly expands in the perpendicular direction.

DEFINITION 1.3. Λ *is an* invariant set *of a map* $f : \mathbb{T}^n \to \mathbb{T}^n$, *if* $f(\Lambda) \subseteq \Lambda$.

Furthermore, Λ_f is the non-wandering set of f ([18]).

For our 3-symbol toroidal horseshoe f, $\Lambda_f = \bigcap_{k\in\mathbb{Z}} \{\{f^k(R)\} \cup \{p_1\} \cup \{p_2\}\}$ where $p_1,\ p_2 \in \mathbb{T}^2 \backslash R$ are fixed points of f ([1]). $\Lambda_f \cap R$ is a Cantor set and any point $p \in \Lambda_f \cap R$ can be represented by an infinite 3-symbol sequence $\underline{\eta}(p) \in \{0,1,2\}^{\mathbb{Z}}$. The sequence $\underline{\eta}(p)$ keeps track of the forward and backward itinerary of each $p \in \Lambda_f$ in the following manner: if $\underline{\eta}(p) = \{\eta_i\}_{i=-\infty}^{\infty}$, then

$$\eta_i = \begin{cases} 0 & \text{if } f^i(p) \in I_0 \\ 1 & \text{if } f^i(p) \in I_1 \\ 2 & \text{if } f^i(p) \in I_2. \end{cases}$$

The map $\underline{\eta} : \Lambda_f \cap R \to \{0,1,2\}^{\mathbb{Z}}$ is one-to-one and onto. For simplicity, we denote by $p : \{0,1,2\}^{\mathbb{Z}} \to \Lambda_f$, the inverse of $\underline{\eta}$. We will also make use of finite forward 3-symbol sequences $\underline{\eta} \in \{0,1,2\}^k$ for some fixed $k \in \mathbb{N}$. If $\underline{\alpha} \in \{0,1,2\}^k$ and $\underline{\beta} \in \{0,1,2\}^l$, then the juxtaposition of two sequences, $\underline{\alpha\beta} \in \{0,1,2\}^{k+l}$, is defined to be the terms of $\underline{\alpha}$ concatenated with the terms of $\underline{\beta}$. $\underline{\beta}^n$ denotes $\underline{\beta}$ repeated n-times; $\underline{\beta}^\infty$ denotes the element of $\{0,1,2\}^{\mathbb{N}}$ where β is concatenated with itself infinitely often. Let $\sigma : \{0,1,2\}^{\mathbb{Z}} \to \{0,1,2\}^{\mathbb{Z}}$ be the full shift map on 3-symbols such that $\sigma(\underline{\eta}) = \underline{\xi}$ where $\eta_i = \xi_{i+1}$. Notice that $\underline{\eta}(f(p)) = \sigma(\underline{\eta}(p))$ for all $p \in \Lambda_f$.

We now discuss the rotational properties of points, $p \in R$, which are such that their forward sequence representation is $\underline{\eta} \in \{0,1,2\}^{\mathbb{N}}$. We denote the set of all such points, p, by $p(\underline{\eta}) \in \Lambda_f$.

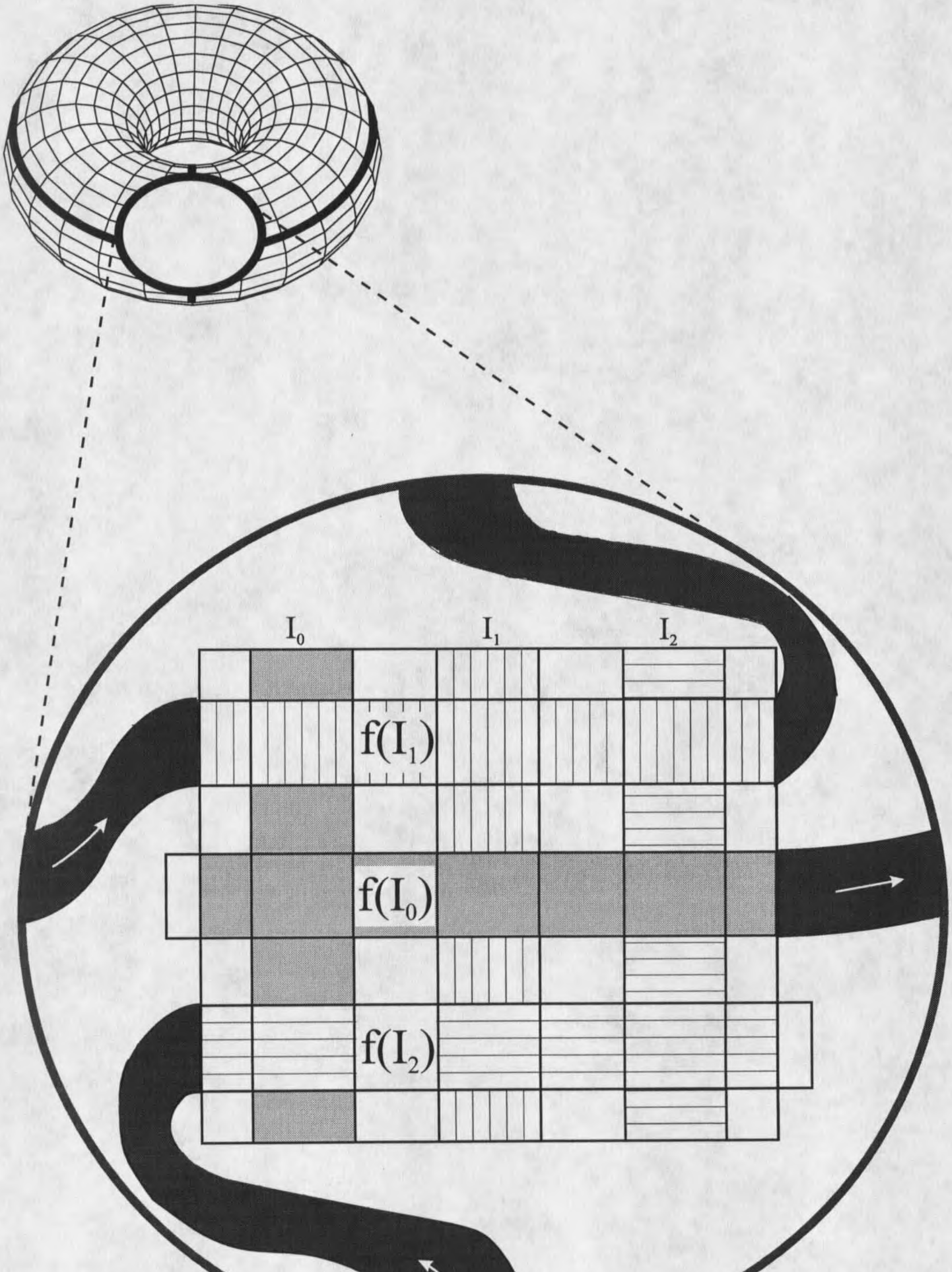

Figure 1. The toroidal horseshoe, f

For $k \in \{0,1,2\}$, define $\Gamma : \{0,1,2\} \to \mathbb{R}^2$ by

$$\Gamma(k) = \begin{cases} (0,0) & \text{if } k = 0 \\ (1,0) & \text{if } k = 1 \\ (1,1) & \text{if } k = 2. \end{cases} \tag{1.2}$$

PROPOSITION 1.4. *If f is the 3-symbol toroidal horseshoe as described above with lift $\widetilde{f}$ as in Figure 2, and $p \in \Lambda_f$ with forward 3-symbol sequence $\underline{\eta}(p)$, then*

$$\rho_f = \mathrm{LIM}\left\{ \frac{1}{n}\sum_{i=0}^{n-1} \Gamma(\eta_i) \;\middle|\; n \in \mathbb{N} \right\} = <(0,0),\ (0,1),\ (1,1)>$$

where $<x_1,\ldots,x_k>$ is the closed convex hull of $\{x_1,\ldots,x_k\}$.

Proof: Let $p \in \Lambda_f \cap R$ and $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be the fixed lift of f via the projection $\Pi : \mathbb{R}^2 \to \mathbb{T}^2$ as in Figure 2, $\widetilde{p} \in \Pi^{-1}(p)$, and $\widetilde{R}$ be the component of $\Pi^{-1}(R)$ which contains $\widetilde{p}$. The key observation in the construction of f which we make use of is that $\widetilde{f}(\widetilde{p}) \in \widetilde{R} + \Gamma(\eta_0)$, because $p \in I_{\eta_0}$, see Figure 2. Similarly, for $n \in \mathbb{N}$, $\widetilde{f}^n(\widetilde{p}) \in \widetilde{R} + \sum_{i=0}^{n-1}\Gamma(\eta_i)$ and thus, $\widetilde{f}^n(\widetilde{p}) - \sum_{i=0}^{n-1}\Gamma(\eta_i) \in \widetilde{R}$. Therefore,

$$\begin{aligned} \left\| (\widetilde{f}^n(\widetilde{p}) - \widetilde{p}) - \sum_{i=1}^{n-1}\Gamma(\eta_i) \right\| &= \left\| \left(\widetilde{f}^n(\widetilde{p}) - \sum_{i=1}^{n-1}\Gamma(\eta_i) \right) - \widetilde{p} \right\| \\ &\le \operatorname{diam}\widetilde{R}\,. \end{aligned}$$

Let $p = p_1$ or p_2. Then, p is fixed and $\rho_f(p) = (0,0)$ by our choice of lift, $\widetilde{f}$.

We can now easily compute the rotation set for all points $p \in \Lambda_f$ which have forward symbol sequence equal to $\underline{2}^\infty$.

$$\begin{aligned} \rho_f(p\ (\underline{2}^\infty)) &= \mathrm{LIM}\left\{ \frac{1}{n}\sum_{i=0}^{n-1}\Gamma(2) \;\middle|\; n \in \mathbb{N} \right\} \\ &= \mathrm{LIM}\left\{ \frac{n(1,1)}{n} \;\middle|\; n \in \mathbb{N} \right\} \\ &= (1,1). \end{aligned}$$

It is left to the reader to check that:

$$\rho_f(p\ (\underline{0}^\infty)) = (0,0); \qquad \rho_f(p\ (\underline{1}^\infty)) = (1,0)$$

and that for $\underline{\eta} = \{(\eta_1\eta_2\dots) \mid \eta_i \in \{0,1,2\} \text{ for all } i > 0\}$,

$$\rho_f = \bigcup_{\underline{\eta}} \rho_f(p\ (\underline{\eta})) = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; 0 \le y \le x \text{ and } 0 \le x \le 1\right\}.$$

or $<(0,0),(1,0),(1,1)>$. ∎

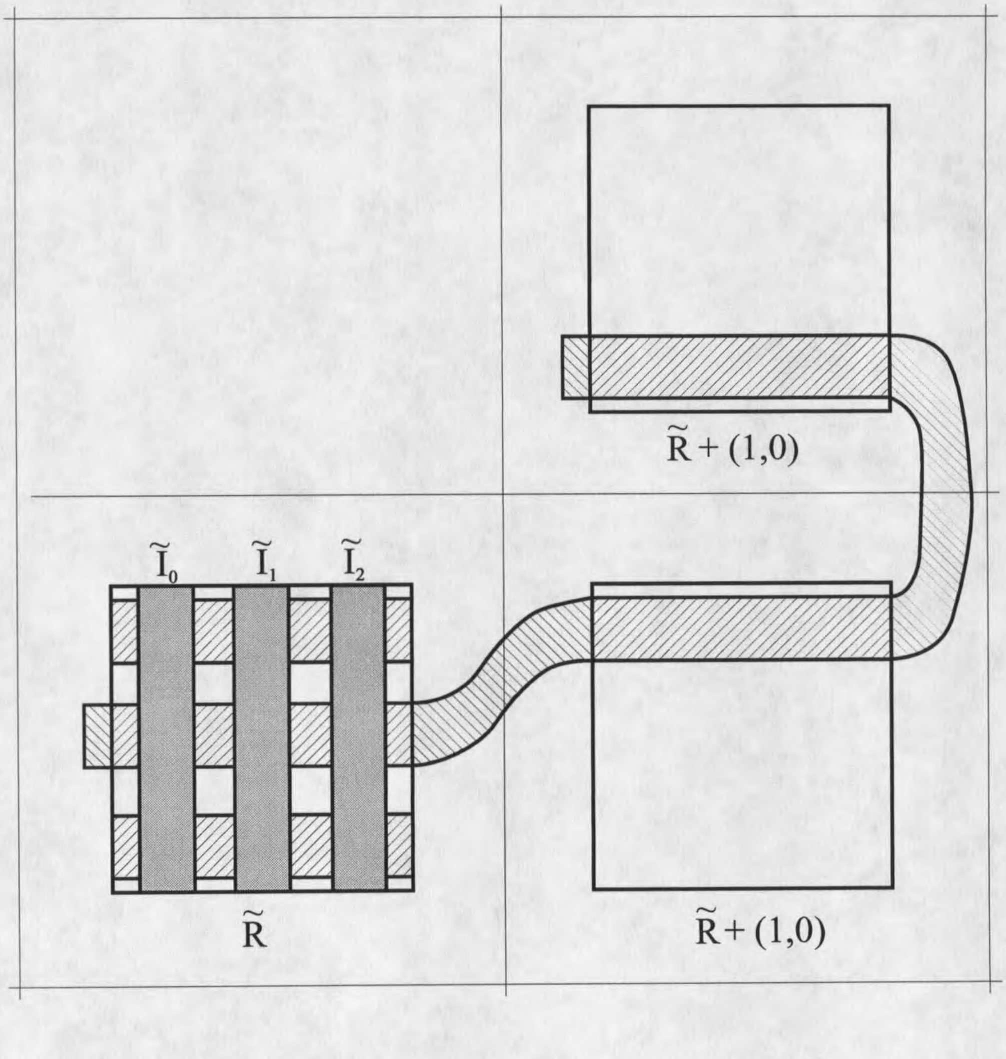

Figure 2. The lift of the toroidal horseshoe, $\widetilde{f}$

As mentioned previously, the focus of this dissertation is on the rotation sets of flows on the n-dimensional torus. In the transition from maps to flows, we shift our concern from the discrete-time dynamical system to the continuous-time dynamical system.

DEFINITION 1.5. *Let* $\varphi : \mathbb{T}^n \times \mathbb{R} \to \mathbb{T}^n$ *be a* C^r *function. Then* φ *is a* C^r flow $\mathbb{T}^n$ *if,*

(i) φ *satisfies the group property,* $\varphi^t \circ \varphi^s(p) = \varphi^{t+s}(p)$ *for each* $p \in \mathbb{T}^n$ *and* $t, s \in \mathbb{R}$ *and,*

(ii) for each fixed $t \in \mathbb{R}$*,* φ^t *is a homeomorphism on* $\mathbb{T}^n$.

For the rest of this dissertation $\varphi(p, t)$ will be denoted by $\varphi^t(p)$.

A flow can also be thought of as a family of homeomorphisms, $\varphi = \{\varphi^t : \mathbb{T}^n \to \mathbb{T}^n\}_{t\in\mathbb{R}}$ that satisfy the group property.

DEFINITION 1.6. $\widetilde{\varphi}^t : \mathbb{R}^n \to \mathbb{R}^n$ *is a* lift of the flow, $\varphi^t : \mathbb{T}^n \to \mathbb{T}^n$*, if the following diagram commutes for each* $t \in \mathbb{R}$.

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\widetilde{\varphi}^t} & \mathbb{R}^n \\ \downarrow \Pi & & \downarrow \Pi \\ \mathbb{T}^n & \xrightarrow{\varphi^t} & \mathbb{T}^n \end{array}$$

DEFINITION 1.7. *The* rotation set of $\widetilde{p}$, under the lift, $\widetilde{\varphi}^t : \mathbb{R}^n \to \mathbb{R}^n$*, of a flow,* $\varphi^t : \mathbb{T}^n \to \mathbb{T}^n$*, is defined by*

$$\rho(\widetilde{\varphi}, \widetilde{p}) = \underset{t\to\infty}{\mathrm{LIM}} \left(\frac{\widetilde{\varphi}^t(\widetilde{p}) - \widetilde{p}}{t} \right)$$

where $\widetilde{p} \in \Pi^{-1}(p)$ *for* $p \in \mathbb{T}^n$*. Here "* $\underset{t\to\infty}{\mathrm{LIM}}(f(t))$ *" means the set of all limits,* $\lim\limits_{j\to\infty}(f(t_j))$*, for all infinite sequences,* $\{t_1, t_2, \dots\}$ *such that* $\lim_{j\to\infty}(t_j) = \infty$.

Furthermore, the rotation set of $\widetilde{\varphi}$*, denoted by* $\rho(\widetilde{\varphi})$*, is defined by*

$$\rho(\widetilde{\varphi}) = \bigcup \rho(\widetilde{\varphi}, \widetilde{p})$$

where the union is taken over all $\widetilde{p} \in \mathbb{R}^n$.

Unlike the case of maps, there is only one lift of a flow φ. So, the rotation set of a flow is a uniquely defined set. We will refer to the rotation set of a flow by ρ_φ.

Suspension Flows

An interesting and useful flow construction that will be used throughout this dissertation is one which takes a C^r-diffeomorphism, $f : \mathbb{T}^{n-1} \to \mathbb{T}^{n-1}$, which is isotopic to the identity, and creates a C^r-*suspension flow of* f, $\varphi_f^t : \mathbb{T}^n \to \mathbb{T}^n$, which we now describe. For any given C^r-diffeomorphism, $f : \mathbb{T}^{n-1} \to \mathbb{T}^{n-1}$, that is isotopic to the identity, consider the space $X = \mathbb{T}^{n-1} \times \mathbb{R}$ under the equivalence relation, $(p, s+1) \sim (f(p), s)$ for all $p \in \mathbb{T}^{n-1}$ and $s \in \mathbb{R}$. Because f is isotopic to the identity, under this equivalence relation, the quotient space, $\hat{X} = X/_\sim$, is C^r-diffeomorphic to $\mathbb{T}^{n-1} \times \mathbb{S}^1$ ([18]). All points of $\hat{X}$, are represented by points (p, s) where $p \in \mathbb{T}^{n-1}$ and $0 \le s < 1$. Consider the "vertical" vector field on X given by:

$$\dot{p} = 0$$

$$\dot{s} = 1.$$

This vector field induces a C^r flow on X which passes to the C^r flow, $\varphi_f^t : \hat{X} \to \hat{X}$, under the equivalence relation $\sim$. See Figure 3. Then φ_f^t is a *suspension flow of* f ([18]).

PROPOSITION 1.8. *If* $f : \mathbb{T}^{n-1} \to \mathbb{T}^{n-1}$ *has rotation set,* $\rho_f \subset \mathbb{R}^{n-1}$, *then there is a suspension flow of* f, $\varphi_f^t : \mathbb{T}^n \to \mathbb{T}^n$, *such that* $\rho_{\varphi_f} = \rho_f \times \{1\} \subset \mathbb{R}^n$.

Proof: The proposition follows from the observation that $\varphi_f^1(p, 0) = (f(p), 0)$ for all $(p, 0) \in \hat{X}$ and that $\mathbb{T}^{n-1} \times \{0\}$ is a cross-section of the flow, φ_f^t. □