Rotation of flows on generalized solenoids
by Yurii Borisovich Shvetsov

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences
Montana State University
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Abstract:
Rotation sets have been defined and studied for maps and flows on various spaces, including the circle, an annulus, and a torus. They have proven useful in the analysis of the dynamics of such maps and flows.

In this dissertation, we analyze ergodicity and rotation properties of flows on one dimensional generalized solenoids, which include true solenoids and substitution tiling spaces. First, covering projections and lifts are constructed for homeomorphisms and flows on solenoids. We argue that only lifts homotopic to the identity should be considered, and show that a lift homotopic to the identity is unique.

The concept of a rotation set is then defined for flows on solenoids. It is shown that a fixed-point free flow is uniquely ergodic, and that its rotation set contains exactly one point. It is also proved that a flow with fixed points may or may not be uniquely ergodic, and as a consequence, the rotation set of such a flow is either a point or an interval. We give a criterion for distinguishing between these two cases. We also construct an example of two fixed-point free flows that have the same rotation set but are not topologically conjugate.

Finally, all the above results are restated for flows on one-dimensional substitution tiling spaces.
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A dissertation submitted in partial fulfillment of the requirements for the degree
of
Doctor of Philosophy
in
Mathematical Sciences

MONTANA STATE UNIVERSITY
Bozeman, Montana

April 2003
APPROVAL

of a dissertation submitted by

Yurii Borisovich Shvetsov

This dissertation has been read by each member of the dissertation committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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ACKNOWLEDGEMENTS

I would like to thank my advisor, Dr. Marcy Barge, for his patient guidance and invaluable assistance, without which, this thesis would have never been possible. I would like to thank my committee members, Dr. Richard Gillette, Dr. Richard Swanson, Dr. Jack Dockery, and Dr. Bill Quimby, for their time, suggestions, and comments. I would like to thank Dr. Jarek Kwapisz for his many helpful suggestions, and Dr. John Lund for his support throughout my undergraduate and graduate career. I would like to thank my high school and undergraduate instructors, Valentina Saprykina, Vladimir Yurgelas, and Marina Sternina, for their special attention and for helping me choose a path in my life that I am now following. I would also like to thank my friends and family for their constant support and encouragement.
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ABSTRACT

Rotation sets have been defined and studied for maps and flows on various spaces, including the circle, an annulus, and a torus. They have proven useful in the analysis of the dynamics of such maps and flows.

In this dissertation, we analyze ergodicity and rotation properties of flows on one-dimensional generalized solenoids, which include true solenoids and substitution tiling spaces. First, covering projections and lifts are constructed for homeomorphisms and flows on solenoids. We argue that only lifts homotopic to the identity should be considered, and show that a lift homotopic to the identity is unique.

The concept of a rotation set is then defined for flows on solenoids. It is shown that a fixed-point free flow is uniquely ergodic, and that its rotation set contains exactly one point. It is also proved that a flow with fixed points may or may not be uniquely ergodic, and as a consequence, the rotation set of such a flow is either a point or an interval. We give a criterion for distinguishing between these two cases. We also construct an example of two fixed-point free flows that have the same rotation set but are not topologically conjugate.

Finally, all the above results are restated for flows on one-dimensional substitution tiling spaces.
CHAPTER 1

INTRODUCTION

Historical Overview

Rotation properties of both discrete and continuous dynamical systems have attracted a great deal of interest throughout last century as well as nowadays. In many cases, studying such properties allows one to make important conclusions about typical orbits of a dynamical system, and about long-term behavior of such orbits.

The idea of a rotation number was first introduced by Poincaré in Chapter 15 of the third of his memoirs [58]. The concept was inspired by his study of toral flows whose return maps are circle homeomorphisms. Rotation number is a measure of the asymptotic average rotation rate of a point under the homeomorphism. Poincaré proved that the rotation number of an orientation preserving circle homeomorphism exists and is independent of the point on the circle. He also gave a complete description of the possible behavior of orbits of circle homeomorphisms: if a homeomorphism has a rational rotation number, then it has a periodic orbit, and if a homeomorphism has an irrational rotation number, then the \( \omega \)-limit set of every point on the circle is either the entire circle or a Cantor subset of the circle. Poincaré also posed a question of what conditions a given homeomorphism or diffeomorphism should satisfy in order to be equivalent to a rotation. This question led him to the
result commonly known as Poincaré Classification (see, for instance, [38]), in which the concept of rotation number plays a central role.

Since this concept has proven so useful, many generalizations of the rotation number have been made. In 1979, Newhouse, Palis, and Takens [56] introduced the rotation interval for circle maps of degree 1, and Ito [37] proved that the rotation interval of a degree-one circle map is closed. [3] contains a comprehensive treatment of rotation intervals and a detailed historical overview on the subject.

The concept of a rotation vector is a generalization of the rotation number for homeomorphisms (or flows) on higher-dimensional tori. Given a homeomorphism \( f : T^m \rightarrow T^m \) and a lift \( F : \mathbb{R}^m \rightarrow \mathbb{R}^m \) of \( f \), the rotation set of \( x \in \mathbb{R}^m \) is

\[
\rho(x, F) = \left\{ \lim_n \frac{F^n(x) - x}{n} \right\},
\]

and the pointwise rotation set of \( F \) is

\[
\rho_p(F) = \bigcup_{x \in \mathbb{R}^m} \rho(x, F).
\]

An element \( v \in \rho_p(F) \) is called a rotation vector.

The pointwise rotation set describes the average rotation speed and direction of individual orbits of \( F \), but it lacks some important properties. It need not be connected or convex. Also, it is not known if \( \rho_p(F) \) is closed for every homeomorphism of \( T^2 \). Swanson and Walker [67] have constructed an example of an analytic diffeomorphism on \( T^3 \) whose pointwise rotation set is not closed. Barge and Walker [9] provided examples of \( C^\infty \) diffeomorphisms on \( T^m \) \((m \geq 3)\) that have rotation sets
with nonempty interior. Further, they showed that a rotation set with interior does
not guarantee the existence of periodic orbits. Dumonceaux [22] obtained results
pertaining to rotation sets on $\mathbb{T}^m$, where $m \geq 3$. A good survey of the main results
concerning rotation vectors for toral maps and flows is provided in [70].

Rather than defining a rotation set as the union of rotation sets of individual
orbits, Misiurewicz and Ziemian [52] proposed a more general definition. If $f : \mathbb{T}^m \to
\mathbb{T}^m$ is a continuous map of the $m$-torus, homotopic to the identity, and a lift $F : \mathbb{R}^m \to
\mathbb{R}^m$ of $f$, then the rotation set $\rho(F)$ is the set of all limits of convergent sequences
\[ \left( \frac{F^{n_i}(x_i) - x_i}{n_i} \right)_{i=1}^{\infty}, \]
where $x_i \in \mathbb{R}^m$ and $n_i \to \infty$. This generalized definition (sometimes referred to as the
MZ rotation set) has been adopted in subsequent works. Misiurewicz and Ziemian
[52] proved that the rotation set is compact and connected. Handel [35] showed
that a periodic-point free homeomorphism of $\mathbb{T}^2$, homotopic to the identity, cannot
have a rotation set with interior. Llibre and MacKay [45] analyzed the dynamics of
homeomorphisms of the torus that have rotation sets with nonempty two-dimensional
interior. Their results were subsequently extended in [53]. Franks [28] has shown that
every vector with rational coordinates, that lies in the interior of the rotation set, is
realized by some periodic orbit. Kwapisz [41] demonstrated that every convex polygon
with rational vertices is realized as a rotation set of some homeomorphism of $\mathbb{T}^2$.

Franks and Misiurewicz [29] gave a complete characterization of rotation sets for
the flows on $\mathbb{T}^2$. They proved that a rotation set of a toral flow is either a single point,
a segment of a line through 0 with rational slope, or a line segment with irrational slope and one end point equal to 0.

In 1957, Schwartzman [63] introduced the concept of asymptotic cycles, which is a generalization of the rotation set for $C^1$ flows on compact differentiable manifolds. The rotation vector for a fixed-point free flow on $\mathbb{T}^2$ is a coordinate representation of the asymptotic cycle with respect to the standard basis in the first cohomology group [38].

Because of the way it is defined, the concept of rotation set is closely related to ergodic properties of a map or a flow. We extensively use such interdependence in this dissertation.

At the same time, significant progress has been made in the study of hyperbolic attractors. While solenoids (also called the Vietoris - van Dantzig solenoids) have been known to topologists for quite some time, they first appeared in the dynamical systems setting in 1967, in Smale’s paper [64]. Smale considered two types of diffeomorphisms of the torus: the DE (Derived from Expanding) diffeomorphisms, which have solenoids as attractors, and the DA (Derived from Anosov) diffeomorphisms, whose attractors were later shown to be homeomorphic to substitution tiling spaces. Williams in a series of articles [72, 73, 74] introduced and developed the idea of representing such attractors as inverse limits. He defined generalized solenoids to be inverse limits of branched one-dimensional manifolds, with bonding maps that are generalizations of expanding circle maps. This inverse-limit viewpoint has been dominant in subsequent
studies. Plykin [57] reformulated Smale’s and Williams’ results using slightly different initial assumptions.

Following Williams’ work, there has been steady interest in solenoids during the last three decades. McCord [50] proved that solenoids are homogeneous. Aarts and Fokkink [1] obtained necessary and sufficient conditions for two solenoids to be homeomorphic. Hagopian [34] showed that a continuum is a solenoid if and only if it is homogeneous and all of its proper subcontinua are arcs. De Man [46] proved that any two composants of any two solenoids are homeomorphic. Various algebraic properties of solenoids were studied by Yi in a series of papers [75, 76, 77].

Multiple results have been obtained regarding maps and flows on solenoids. Some properties of continuous maps on solenoids were discovered by Ustinov [68]. Keesling [39] determined the precise topological structure of the group of homeomorphisms of a solenoid. Kwapisz [43] obtained an explicit decomposition of orientation preserving homeomorphisms on solenoids. Clark [17] investigated solenoidal homeomorphisms homotopic to the identity, and showed that a certain class of solenoids admits no expansive homeomorphisms.

As far as flows on solenoids are concerned, Aarts and Martens [2] obtained a relevant result by showing that a one-dimensional, separable and metrizable space is an orientable matchbox manifold if and only if it is the phase space of some fixed-point free flow. Since a solenoid is a matchbox manifold, fixed-point free flows on solenoids
coincide with the suspensions of homeomorphisms of the underlying zero-dimensional space $C$.

Clark investigated flows on solenoids of any finite dimension. In [14] he introduced linear flows on solenoids and addressed the question of when two linear flows are equivalent. He then extended his research, giving in [16] a method of constructing flows on $k$-dimensional solenoids from a given flow on the torus $T^k$, and showing in [15] that flows on solenoids are generically not almost periodic.

Many of the aforementioned contributions deal with generalized solenoids in Williams' setting. Substitution tiling spaces, which in fact form a subclass of Williams' generalized solenoids, have also been studied separately, since they exhibit many peculiar phenomena that are absent in true solenoids. The study of substitution tiling spaces has grown out of certain problems of symbolic dynamics, namely, the analysis of substitutions on finite alphabets. One of the well-known contributions in this area is a paper by Coven and Keane [19], who analyzed topological and measure-theoretic properties of substitutions of constant length on two symbols. Dekking [21] gave a complete spectral classification for substitutions of constant length. Gottschalk [30], Martin [49], and others considered the so called substitution minimal flows, which in essence are actions on a subshift of finitely many symbols by the infinite cyclic transformation group generated by the left shift homeomorphism. It was a natural step to replace this transformation group by the reals, which led to the introduction of substitution tiling spaces.
Substitution tilings of Euclidean spaces, obtained through the process of inflation and substitution of prototiles, have been studied by many researchers, including Kenyon [40], Mozes [54], and others. Radin and Sadun [60] developed an algebraic invariant, related to a subgroup of rotations, that helps determine when two substitution tiling systems are dynamically equivalent. Another noteworthy result is that of Solomyak [65], who looked into the spectral properties of tiling systems arising from self-affine tilings of $\mathbb{R}^d$, and proved that such systems are uniquely ergodic.

One of the first to look at substitution tiling spaces from the inverse limit viewpoint, was the paper by Anderson and Putnam [4]. They showed that substitution tiling spaces are a special case of expanding attractors, thereby providing a relation between substitution tiling theory and hyperbolic attractors. Using the inverse limit approach, they also computed cohomology for tiling spaces in one- and two-dimensional case. Sadun and Williams [62] later showed that a tiling space forms a bundle over a manifold whose fiber is a Cantor set. They also demonstrated that tiling spaces on $\mathbb{R}^d$ are suspensions of $\mathbb{Z}^d$ subshifts. A related result of local nature was obtained by Gutek [32]. This in turn establishes a connection between the topology of substitution tiling spaces and homeomorphisms of Cantor sets, which are studied in [26] and [23]. Such homeomorphisms break into two classes: the adding machines, whose suspensions are true solenoids, and subshifts generated by primitive substitutions, whose suspensions are substitution tiling spaces [7].
Among recent contributions, the work of Barge and Diamond [7] is of particular interest to us. In this article, they investigate the topological structure of one-dimensional substitution tiling spaces. First, they prove that all one-dimensional substitution tiling spaces have a finite and nonempty collection of 'asymptotic composants', thus providing a geometric insight into the structure of tiling spaces. The authors then proceed to establish a condition for two tiling spaces to be homeomorphic, using the concept of weak equivalence (see also [8]). The consequence of their results and [72, 73] is the following classification.

*Every orientable hyperbolic one-dimensional attractor is either homogeneous, and hence a true solenoid, or nonhomogeneous, and hence a one-dimensional substitution tiling space.*

In the latter case, the inhomogeneity exhibits itself in the asymptotic composants.

Among the results of geometric nature, we also mention Canterini and Siegel [13]. For Pisot substitution systems, they give an explicit continuous semi-conjugacy between the shift on the system and a translation on the torus. Queffélec [59], along with a comprehensive review of the theory, offers various new results and poses some interesting questions, including the one partially answered in [13]. Another, more recent compendium on the subject has been written by Arnoux *et al* [5].
Results

The goal of this thesis is to analyze rotation properties of flows on one-dimensional generalized solenoids, including true solenoids (which we refer to simply as solenoids), and substitution tiling spaces. Such analysis is carried out by means of investigating ergodicity and invariant measures of the flows. Fixed-point free flows and flows with fixed points are treated separately, because they exhibit substantially different ergodic properties and rotation behavior.

As we show in Chapter 3, a fixed-point free flow on a solenoid is uniquely ergodic, and its rotation set consists of just one point (which we can call the rotation number). We also give an explicit formula for computation of the rotation number, using return times of the flow to some cross-section of the solenoid. This formula proves handy in constructing an example of a flow that is not topologically conjugate to a linear flow.

The flows with fixed points, as we discover in Chapter 4, can manifest two types of behavior. Some of them are uniquely ergodic, and consequently have a trivial rotation set, while others admit more than one ergodic measure, and their rotation sets are intervals. We develop criteria for distinguishing between these two types of behavior, using the ‘speed’ of the flow.

We also address the problem of describing pointwise rotation sets for flows on solenoids. After a detailed discussion of Katok’s example in its original setting (on the torus), we adapt it for the flows on solenoids. Katok’s example provides an algorithm for the construction of flows whose pointwise rotation set contains points
other than the extremes of the MZ rotation set. The question remains whether it is possible to construct a flow with a unique interior point of $\rho(\phi)$ being contained in $\rho_p(\phi)$, that is, with $\rho_p(\phi) = \{0, r, \rho\}$, where $\rho(\phi) = [0, \rho]$, $r \in (0, \rho)$.

Finally, we discuss the question of whether two fixed-point free flows having the same rotation number are actually topologically conjugate. Our findings show that this is not always the case. We construct an explicit example of a flow that is not topologically conjugate to a linear flow with the same rotation number.

The following theorems summarize the main results of this thesis concerning rotation and ergodicity of flows on solenoids.

Let $\phi$ denote a continuous flow on the solenoid $S_p$, $P = (p_1, p_2, \ldots)$, and let $\tilde{\phi}$ be the lift of $\phi$ to $\mathbb{R} \times C$, so that $h \circ \tilde{\phi} = \phi \circ h$. Denote by $\tau(x)$ the return time of the flow to the cross-section of the identity $C_\beta$, that is, $\tau(x)$ is such that $\phi^{\tau(x)}(h(0, x)) \in C_\beta$. Let also $\beta$ denote the Bernoulli measure on the Cantor set $C$. $\Omega$ denotes the fundamental domain in $\mathbb{R} \times C$, and $B(S_p)$ the $\sigma$-algebra of Borel sets on $S_p$.

**Theorem A.** Let $\phi$ be a continuous fixed-point free flow on $S_p$, then it is uniquely ergodic, with the invariant measure given by

$$
\mu(B) = \frac{1}{\bar{r}_C} \int_C \tau(x) \, d\beta, \quad \forall B \in B(S_p)
$$

where $\bar{r}_C = \int_C \tau(x) \, d\beta$, $\tau(x)$ is the 'time-length' of the arc component of $B$ with the Cantor-set coordinate equal to $x$. 
THEOREM B. If \( \phi \) be a continuous fixed-point free flow on \( S_P \), then the rotation set of \( \phi \) is \( \rho(\phi) = \left\{ \frac{1}{\tau C} \right\} \).

Now let \( \phi \) be a differentiable flow on \( S_P \) with fixed points, such that \( \frac{d}{dt} \phi^t(\hat{z}) = \Phi(\hat{z}) \), and let \( \tilde{\Phi} \) be the lift of \( \Phi \) to the covering space \( \mathbb{R} \times C \). \( \Omega \) denotes the fundamental domain in \( \mathbb{R} \times C \). By \( L^1(\Omega) \) we mean \( L^1(\Omega) \) with respect to the Lebesgue measure \( \lambda \).

THEOREM C. Let \( \phi \) be a flow on \( S_P \) with one fixed point \( p_0 \), \( \tilde{\Phi} \) a lift of \( \Phi \), then:

1. If \( \frac{1}{\tilde{\Phi}} \) is \( L^1(\Omega) \), then \( \phi \) is not uniquely ergodic. It has a \( \phi \)-invariant ergodic measure \( \mu \), given by \( \mu(B) = \frac{\int_B (1/\tilde{\Phi}) \, d\lambda}{\int_\Omega (1/\tilde{\Phi}) \, d\lambda} \), such that \( \rho(z, \phi) = \rho > 0 \) for \( \mu \)-almost all \( z \in S_P \). Furthermore, such a measure is unique.

2. If \( \frac{1}{\tilde{\Phi}} \) is not \( L^1(\Omega) \), then \( \phi \) is uniquely ergodic, with the Dirac measure \( \delta_{p_0} \) being the only \( \phi \)-invariant probability measure on \( S_P \).

THEOREM D. Let \( \phi \) be a flow on \( S_P \) with fixed points, \( \tilde{\Phi} \) a lift of \( \Phi \), then the following is satisfied:

1. If \( \frac{1}{\tilde{\Phi}} \) is \( L^1(\Omega) \), then \( \rho(\phi) = [0, \rho] \), where \( \rho = \frac{1}{\int_\Omega (1/\tilde{\Phi}) \, d\lambda} \).

2. If \( \frac{1}{\tilde{\Phi}} \) is not \( L^1(\Omega) \), then \( \rho(\phi) = \{0\} \).

Essentially the same results for flows on substitution tiling spaces are stated in Chapter 5.
Structure of this Dissertation

This dissertation is organized into five chapters. In Chapter 1, we give an account of the history of development of the concepts of rotation sets, solenoids, and substitution tiling spaces, which we bring together in the remainder of the thesis. Here we also state the main results of the thesis.

Chapter 2 opens with basic definitions concerning solenoids, and a summary of the most important properties of solenoids. We also introduce the universal cover that can be used for both maps and flows on solenoids, and discuss typical covering projections associated with this universal cover. Then the focus of the discussion turns to homeomorphisms of solenoids, as well as their lifts. We devote a separate section to homeomorphisms homotopic to the identity. Such homeomorphisms include time-one maps of flows, therefore, their properties are extensively used in subsequent chapters. In the last section of the chapter, a known result ([43]) is stated about the decomposition of a homeomorphism of a solenoid into a composition of three maps: a homeomorphism homotopic to the identity, a group multiplication by a fixed element of the solenoid, and a shift on the inverse limit space.

In the subsequent chapters, we adopt the following method of presenting the results. Some proofs are given for the case of a flow on the dyadic solenoid, $S_2$, in order to maximize clarity of presentation. The majority of such proofs remain essentially the same, with obvious replacements in notation, for general solenoids $S_p$. This being the case, the corresponding results for $S_p$ are stated without proof, or
with comments that indicate the changes that need to be made in the proofs for $S_2$. Whenever a proof in the general case differs substantially from the dyadic case, it is given in its most general form.

In Chapter 3, we analyze fixed-point free flows on solenoids. After a brief discussion of basic characteristics of flows and their lifts, the definitions of rotation sets are given, and a few propositions are proved, establishing some facts about rotation sets of solenoidal flows. These facts, although redundant for fixed-point free flows, prove useful for flows with fixed points, discussed in Chapter 4. The next two sections of Chapter 3 are intended for the proof of the main results of the chapter, Theorems A and B. With the unique ergodicity and triviality of the rotation set thus established, we discuss some desirable geometric properties of fixed-point free flows, and show that they are not always present. This discussion leads us to an example that illustrates a breakdown in the link between the equality of rotation sets and topological conjugacy of the corresponding flows.

Chapter 4 deals with solenoidal flows with fixed points, and some related material. The results of this chapter are developed for the case of a flow with one fixed point. Here, we show that such flows may be either uniquely or non-uniquely ergodic, obtain criteria for that to happen, and, applying these facts to rotation sets, prove the other two main results of the thesis, Theorem C and Theorem D. Due to a certain similarity between solenoidal flows and flows on the torus $T^2$ that follow a foliation of the torus by vectors of irrational slope, we treat corresponding results on the torus first,
translating them later to the solenoid. Also in Chapter 4, we state a theorem on the measure-theoretic realization of points in the rotation set, and discuss possibilities for the pointwise rotation set, illustrating the discussion with a famous Katok’s example, adapted to the solenoid.

Finally, in Chapter 5 we restate the results of the previous two chapters for flows on substitution tiling spaces, indicating the changes that need to be made in the proofs, and giving new proofs whenever necessary. The chapter opens with definitions and background information on substitution tiling spaces, followed by the discussion of the universal cover and lifts. Next, we outline two different approaches to defining a rotation set for flows on substitution tiling spaces, and show their equivalence for primitive substitutions. Then, unique ergodicity of the underlying subshift is proved (see [59]), and Theorems A, B, C, and D are reformulated for substitution tiling spaces.
CHAPTER 2

HOMEOMORPHISMS OF SOLENOIDS AND THEIR LIFTS

Solenoids and Their Properties

Definitions and Notation

We shall now define a solenoid and state the most important properties of solenoids. To begin with, we give an intuitive explanation of these spaces.

A solenoid can be visualized in the following fashion. Let us imagine that we start with a rubber doughnut, then stretch it, making it thinner, and fold it, so it wraps around its original shape twice. The resulting double doughnut is also stretched and folded in the same way. This process is continued infinitely many times. In the limit, we obtain a solenoid.

The process described above lies at the core of one of the first definitions of a solenoid, proposed by van Dantzig [20].

**Definition 2.1 (van Dantzig).** Let $\mathbb{T}^2 = \mathbb{D}^2 \times S^1$ be a solid torus, and let $\mathcal{P} = (p_1, p_2, \ldots)$ be a sequence of natural numbers, such that $p_i \geq 2 \forall i$. Suppose that $\forall k \geq 2 \ F_k : \mathbb{T}^2 \to F_k (\mathbb{T}^2)$ is a homeomorphism given by

$$F_k (u, z) = (f_k (u, z), z^k), \quad u \in \mathbb{D}^2, \ z \in S^1,$$
where \( f_k(u, z) \) is a contraction in the first argument. Define a solenoid as

\[
S_p = \bigcap_{k \geq 1} F_{p_k} \circ F_{p_{k-1}} \circ \cdots \circ F_{p_1} \quad (T^2).
\]  

(2.1)

Even though this definition is geometric and intuitive, it is not very practical. Another, more recent and convenient approach is to define solenoids through inverse limits.

**Definition 2.2 (Williams).** Let \( S^1 \) be a unit circle, \( \mathcal{P} = (p_1, p_2, \ldots) \) a sequence of natural numbers, such that \( p_i \geq 2 \forall i \), and let \( f_j : S^1 \to S^1 \) be given by \( f_j(z) = z^{p_j} \). Define the \( \mathcal{P} \)-adic solenoid as the following inverse limit:

\[
\Sigma_{\mathcal{P}} = \lim \left\{ S^1, f_j \right\} = \left\{ (z_1, z_2, \ldots) \in \prod_{j=1}^{\infty} S^1 : z_j = f_j(z_{j+1}) \right\}.
\]  

(2.2)

These two definitions are in fact equivalent, as the following proposition shows.

**Proposition 2.3.** \( S_p \) and \( \Sigma_{\mathcal{P}} \) are homeomorphic.

**Proof.** Consider the diagram in Figure 1. Here, \( p(u, z) = z \) is a projection map, it induces the map \( \hat{p} : \lim \left\{ T^2, F_{p_j} \right\} \to \Sigma_{\mathcal{P}} \), given by

\[
\hat{p} \left( (u_1, z_1), (u_2, z_2), \ldots \right) = (z_1, z_2, \ldots).
\]

By a straightforward argument, one shows that \( \pi_1 : \lim \left\{ T^2, F_{p_j} \right\} \to S_p \), given by \( \pi_1 ((u_1, z_1), (u_2, z_2), \ldots) = (u_1, z_1) \) (projection onto the first coordinate), is a homeomorphism. It can also be shown that \( \hat{p} \) is a bijection. Clearly, \( \hat{p} \) is continuous,
Figure 1. Homeomorphism between $S_p$ and $\Sigma_p$.

since the projections on every coordinate are continuous. Then $\hat{\rho}^{-1}$ is also continuous, and hence $\hat{\rho}$ is a homeomorphism. It follows that $\hat{\rho} \circ \pi_1^{-1} : S_p \to \Sigma_p$ is a homeomorphism. □

The second definition is easily generalized to include inverse limits of bouquets of circles [72, 73], which we treat in Chapter 5, and inverse limits of tori that yield solenoids of higher dimensions. Clark [14] defines solenoids as inverse limits of maps of $T^n$:

$$
\Sigma_{\tilde{M}} = \lim \left\{ T^n, f_j \right\},
$$

(2.3)

where $\tilde{M} = \{M_1, M_2, \ldots \}$ is a sequence of $n \times n$ matrices with integer entries and nonzero determinants, $f_j : T^n \to T^n$ are automorphisms represented by the matrices $M_j$. In this setting, $\Sigma_{\tilde{M}} \subset \prod_{j=1}^{\infty} T^n$.

Such solenoids occur as attractors of hyperbolic dynamical systems. Examples of such dynamical systems are given in [24], [38], and [61].
The fact that solenoids occur as attractors justifies the importance of studying them, since properties of an attractor can give an insight into the behavior of the dynamical system. In this dissertation, we concentrate on one-dimensional solenoids.

**NOTATION.** Throughout the rest of this dissertation, we denote a $p$-adic solenoid by $S_p$, and use Definition 2.2. Furthermore, by $C_w$ we denote the cross-section of $S_p$ through the element $w = (w_1, w_2, \ldots) \in S_p$:

$$C_w = \{ (z_1, z_2, \ldots) \in S_p : z_1 = w_1 \},$$

and by $S_w$ we denote the arc component that contains $w \in S_p$. We equip $S_p$ with the subspace topology from the product topology on $\prod_{i=1}^{\infty} S^1$.

**Proposition 2.4.** $S_p$ is a compact abelian topological group, with the group operation (denoted ‘$*$’) given by factor-wise multiplication, and the identity element $e = (1, 1, \ldots)$.

**Proof.** Compactness follows from $S_p$ being a continuum. The fact that $S_p$ is a topological group follows because $f_j$ is a topological group homomorphism for all $j$, and the inverse limit of topological group homomorphisms is a topological group. □

In the next subsection and throughout this thesis, we use the concept of an adding machine, which we now introduce. Roughly, the adding machine is 'addition with carry': add 1 to the first coordinate and carry to the right. Our more precise definition follows that of [10].
DEFINITION 2.5. Consider the Cantor set $C_{\mathcal{P}} = \prod_{i=1}^{\infty} (\mathbb{Z} \mod p_i)$, where $\mathcal{P} = (p_1, p_2, \ldots)$. The adding machine operator $A : C_{\mathcal{P}} \to C_{\mathcal{P}}$ is defined by

$$A : (\ldots, \alpha_k, \ldots) \mapsto (\ldots, \gamma_k, \ldots),$$

so that $\gamma_k = \begin{cases} 0 : & 1 \leq k < r \\ \alpha_r + 1 : & k = r \\ \alpha_k : & k > r \end{cases}$, where $\alpha_k = \begin{cases} p_k - 1 : & 1 \leq k < r \\ \alpha_r : & k = r \\ \alpha_k : & k > r \end{cases}$, $\alpha_r < p_r - 1$, $1 \leq r \leq \infty$.

Let $z = (z_1, z_2, \ldots, z_j, \ldots) \in S_{\mathcal{P}}$, where $\mathcal{P} = (p_1, p_2, \ldots)$, then

$$z_1 = z_2^{p_1}, \ z_2 = z_3^{p_2}, \ldots, \ z_j = z_{j+1}^{p_j}, \ldots$$

For every value of $z_j \in S^1$ there are $p_j$ points $z_{j+1}$ in $S^1$ with $z_{j+1}^{p_j} = z_j$. In other words, if the value of $z_j$ is known, we can order the $p_j$-roots of it and use the number of the root to describe the value of $z_{j+1}$.

Define a function $\chi : C_{\mathcal{P}} \to C_{\mathcal{P}}$ by

$$\chi(z) = (s_1, s_2, \ldots, s_j, \ldots) \text{ if and only if } z = (1, z_2^{(s_1)}, \ldots, z_{j+1}^{(s_j)}, \ldots) \quad (2.4)$$

where $z_{j+1}^{(s_j)}$ is the $(s_j + 1)$-st of the $p_j$-th roots of $z_j$, counting counterclockwise, starting with the root that has the smallest argument. More precisely,

$$s_j = \frac{\arg z_{j+1} p_j - \arg z_j}{2\pi}.$$

We can call $\chi$ the $\mathcal{P}$-adic representation of $C_\mathcal{P}$.

Then the adding machine $\hat{A}$ on $C_\mathcal{P}$, induced by $A$, is given by $\hat{A} = \chi^{-1} A \chi$.

Abusing notation, we shall denote this adding machine also by $A$. 
In the following discussion, we shall also use the concept of a natural flow on a solenoid, as defined in [16].

**Definition 2.6.** The natural flow on a solenoid $S_\mathcal{P}$ is defined as follows:

$$\varphi_t^+ (z) = z * \pi_\mathcal{P} (t), \quad (2.5)$$

where $\pi_\mathcal{P} : \mathbb{R} \to S_\mathcal{S}$ is the continuous isomorphism onto the arc component of the identity $S_\mathcal{S}$, given by

$$\pi_\mathcal{P} (t) = \left( \exp (2\pi it), \exp \left( \frac{2\pi it}{p_1} \right), \exp \left( \frac{2\pi it}{p_1 p_2} \right), \ldots, \exp \left( \frac{2\pi it}{p_1 p_2 \cdots p_j} \right), \ldots \right). \quad (2.6)$$

**Remark 2.7.** In the case of the dyadic solenoid $S_2$, we have

$$\pi_2 (t) = \left( \exp (2\pi it), \exp (\pi it), \exp \left( \frac{\pi}{2} it \right), \ldots, \exp \left( \frac{\pi}{2^{j-1}} it \right), \ldots \right). \quad (2.7)$$

**Remark 2.8.** The Poincaré map of the natural flow is conjugate to the adding machine on the cross-section $C_\mathcal{S} = \{ z \in S_\mathcal{P} : z_1 = 1 \}$.

**Remark 2.9.** The topology on $S_\mathcal{P}$ is induced by the metric

$$d(z, w) = \sum_{j=1}^{\infty} \frac{1}{2^j} d(z_j, w_j), \quad (2.8)$$

where $d(z_j, w_j)$ is a metric on $S^1$.  

Properties of Solenoids

We now list some properties of solenoids.

**Proposition 2.10.** $S_P$ is a matchbox manifold, i.e. locally a product of a Cantor set with an arc.

**Proof.** Any segment of an arc component can be defined as

$$S = \{ \varphi^t(z) : 0 \leq t \leq T, z \in S_P \}.$$

It immediately follows that $S$ is homeomorphic to an arc $[0,T]$. Thus we only need to show that a cross-section $C_w = \{ z \in S_P : z_1 = w_1 =: w \}$ of $S_P$ is a Cantor set.

To that end, consider the spaces

$$X_1 = \{ w \}, \quad X_2 = f_1^{-1}(X_1), \quad X_3 = f_2^{-1}(X_2), \quad \ldots$$

Every $X_i$ (Figure 2) is contained in $S^1$, is finite, and the subspace topology on it is discrete.

Then we represent $C_w$ as an inverse limit, $C_w = \varprojlim \{ X_i, g_i \}$, where $g_i = f_i |_{X_{i+1}}$ for all $i$. Note that $g_i(z) = z^{w_i}$ and $\forall x_i \in X_i, \ k \geq i$,

$$\lim_{k \to \infty} \text{card} \left( g_k^{-1} \circ \cdots \circ g_i^{-1}(\{ x_i \}) \right) = \lim_{k \to \infty} \prod_{j=i}^{k} p_j$$

$$\geq \lim_{k \to \infty} 2^{k-i-1} = \infty$$

and apply the well-known Theorem 2.11. $\square$
THEOREM 2.11. Let \( \{X_i, f_i\} \) be an inverse sequence, where every \( X_i \) is a finite space with the discrete topology, every \( f_i \) is onto, and for each \( i \), if \( x_i \in X_i \), then
\[
\lim_{k \to \infty} \text{card} \left( f_{n-1} \circ \cdots \circ f_{i-1} (\{x_i\}) \right) = \infty
\]
(i.e. the number of preimages of any \( x_i \) tends to \( \infty \)). Then \( X = \lim \{X_i, f_i\} \) is a Cantor set.

A discussion of further properties of solenoids that are consequences of Proposition 2.10 can be found in [25].

PROPOSITION 2.12. \( S_P \) is a homogeneous continuum, i.e. for any two points in \( S_P \), there is a homeomorphism of \( S_P \) that swaps these two points.
PROOF. This statement follows from Proposition 2.4 and the fact that all topological groups are homogeneous. □

**PROPOSITION 2.13.** Every nondegenerate proper subcontinuum of $S_p$ is an arc.

**PROOF.** The proof is easy and well-known. See [55]. □

We note that the last two propositions actually characterize solenoids. Hagopian [34] proved the following theorem:

**THEOREM 2.14 (HAGOPIAN).** $M$ is a homogeneous continuum and every proper subcontinuum of $M$ is an arc if and only if $M$ is a solenoid.

Mardesic and Segal [48] showed that a solenoid is a circle-like continuum. Moreover, every homogeneous circle-like continuum that contains an arc is a solenoid [11].

**NOTATION.** By $S(z)$ we denote the arc component of $S_p$ containing $z$. Set

$$S^+(z_0) = \{ \varphi^t(z_0) : t \geq 0 \},$$

$$S^-(z_0) = \{ \varphi^t(z_0) : t \leq 0 \}.$$

$d_{arc}(x, y)$ denotes the signed distance between $x$ and $y$ along the arc component of $S_p$:

$$d_{arc}(x, y) = l \quad \text{if and only if} \quad y = \varphi^l(x). \quad (2.9)$$
Universal Cover and Covering Projections

As we mentioned in the previous section, a solenoid $S_P$ is locally a product of an arc with a Cantor set. So the natural choice for a covering space would be the product of the real line and a Cantor set, $\mathbb{R} \times C$, where $C = C_e$. Let $h : \mathbb{R} \times C \to S_P$ denote the desired covering projection. We would like $h$ to satisfy the following properties:

1. $h$ maps $[0, 1) \times C$ one-to-one onto $S_P$.

2. $h$ is a covering map (in particular, it is continuous and a local homeomorphism.)

3. $h$ is orientation preserving.

4. The lift of the natural flow $\varphi$ by $h$ is a linear flow on $\mathbb{R} \times C$, that is, a flow with a constant speed across all arc components of $\mathbb{R} \times C$.

**Remark 2.15.** A cylinder set of $C$ is a set of the form:

$$U_{q_1q_2...q_k} = \chi^{-1}\{x \in \prod_i (\mathbb{Z} \mod p_i) : x_1 = q_1, x_2 = q_2, \ldots, x_k = q_k \text{ fixed}\}$$

We now turn to the construction of the covering projection. Define $h$ as follows:

$$h(t, x) = \varphi^t(x).$$

By definition, $h$ maps $[0, 1) \times C$ (or, for that matter, any set $[t, t+1) \times C, t \in \mathbb{R}$) one-to-one onto $S_P$. Also by definition of $h$, the lift of $\varphi^t$ to $\mathbb{R} \times C$ is $\tilde{\varphi}^t : (\tau, x) \mapsto (\tau + t, x)$. Since $\tilde{\varphi} \mid_{\mathbb{R} \times \{x\}}$ and $h \mid_{[0] \times C}$ are continuous, and since $\mathbb{R} \times C$ has the product topology,
we know \( h \) is continuous. Also, if \( |t_2 - t_1| < 1 \), then \( h \mid_{[t_1,t_2] \times C} \) is one-to-one onto its image, \((h \mid_{[t_1,t_2] \times C})^{-1}\) is continuous, since \( \varphi^{-1} \), \((h \mid_{\{0\} \times C})^{-1}\) are continuous. Thus \( h \) is a local homeomorphism. The following propositions will assist us in showing that \( h \) is a covering map.

The map \( h \) defined above possesses another intrinsic property, which we now describe. For simplicity, we shall denote a point \((0, x) \in \{0\} \times C\) by \( x \). Now by Remark 2.8,

\[
  h(1, x) = \varphi^{-1} h(0, x) = h(0, Ax).
\]  

(2.12)

This implies the following two propositions.

**Proposition 2.16.** If \( h \) is the map defined by (2.11), then

\[
  h(t + 1, x) = h(t, Ax) \quad \forall (t, x) \in \mathbb{R} \times C.
\]  

(2.13)

**Proposition 2.17.** If \( h \) is the map defined by (2.11), then

\[
  h(t + n, x) = h(t, A^n x) \quad \forall (t, x) \in \mathbb{R} \times C, \forall n \in \mathbb{Z}.
\]  

(2.14)

Conversely, if \((t_1, x_1), (t_2, x_2) \in h^{-1}\{z\}\), then

\[
  (t_2, x_2) = (t_1 - n, A^n x_1), \quad \text{for some } n \in \mathbb{Z}.
\]  

(2.15)

For convenience, we define a map

\[
  T(t, x) = (t + 1, A^{-1} x).
\]  

(2.16)
If \( p \in S_p \), take an open neighborhood \( U \) of \( p \), of diameter less than \( \frac{1}{2} \). Let \( \bar{U} = h^{-1}[0,1) \times C \subset [0,1) \times C \). Then the collection \( \{T^n(\bar{U})\}_{n=-\infty}^{\infty} \) evenly covers \( U \).

Each \( T^n(\bar{U}) \subset [t,t+1) \times C \) for some \( t \in \mathbb{R} \). Also, \( h \mid_{[t,t+1) \times C} : [t,t+1) \times C \to S_p \) is a homeomorphism, so \( h \) maps every \( T^n(\bar{U}) \) homeomorphically onto \( U \). It follows that \( h \) is a covering map.

Note also that \( h^{-1}(C_p) = \mathbb{Z} \times C \).

### Lifts of Solenoidal Maps

**Definition 2.18.** A homeomorphism \( f : S_p \to S_p \) is called orientation preserving if whenever \( y = x * \pi_p(t) \), \( t > 0 \), we have \( f(y) = f(x) * \pi_p(\tau), \tau > 0 \). It is called orientation reversing, if \( t > 0, \tau < 0 \), respectively.

In other words, \( \forall t > 0 \forall x \in S_p \) \( f(\varphi^t(x)) = \varphi^\tau(f(x)), \tau > 0 \).

**Definition 2.19.** Let \( f : S_p \to S_p \) be an orientation preserving homeomorphism on \( S_p \). A lift of \( f \) is a map \( F : \mathbb{R} \times C \to \mathbb{R} \times C \), such that \( F \) is continuous and

\[
h \circ F = f \circ h, \tag{2.17}
\]

where \( h : \mathbb{R} \times C \to S_p \) is the covering projection.

Note that in general, a lift of \( f \) is not unique, however, as we show in the next section, a lift of a homeomorphism homotopic to the identity, that is also homotopic
to the identity, is in fact unique. We also note that a lift of a homeomorphism is not necessarily a homeomorphism, as the following example demonstrates.

**Example 2.1.** Consider the identity homeomorphism $f(z) = z$ of $S_2$. Decompose $C$ into a union of two clopen sets, $C_0$, $C_1$, that permute under the adding machine. Define a lift $F$, so that

$$[0,1) \times C_0 \mapsto [0,1) \times C_0, \quad [0,1) \times C_1 \mapsto [1,2) \times C_0.$$ 

Then $F$ is a lift of $f$, but it is not a homeomorphism.

**Proposition 2.20.** Let $F : \mathbb{R} \times C \to \mathbb{R} \times C$ be a lift of $f : S_p \to S_p$, then

$$F(t + n, A^{-n}x) = (s + k_n(x), A^{-k_n(x)}y),$$

where $(s, y) = F(t, x)$, and $k_n(x)$ is an integer that depends on $x \in C$, $n \in \mathbb{Z}$. In particular,

$$F(t + 1, A^{-1}x) = (s + k(x), A^{-k(x)}y).$$

**Proof.** By properties of the covering projection $h$, we have $h(t + n, A^{-n}x) = h(t, x)$. So $f \circ h(t + n, A^{-n}x) = f \circ h(t, x)$. By Equation (2.17),

$$h \circ F(t + n, A^{-n}x) = f \circ h(t + n, A^{-n}x) = f \circ h(t, x) = h \circ F(t, x).$$

Now again by properties of $h$, $\exists k = k_n(x)$, such that

$$F(t + n, A^{-n}x) = (s + k_n(x), A^{-k_n(x)}y).$$
Remark 2.21. Using the function $T$, defined in (2.16), Equation (2.19) can be rewritten as

$$F \circ T(t, x) = T^{k(x)} \circ F(t, x).$$

(2.20)

Proposition 2.22. A composition of lifts is a lift, i.e. if $F$ is a lift of $f$, and $G$ is a lift of $g$, then $G \circ F$ is a lift of $g \circ f$.

Proof. $h \circ G \circ F = g \circ h \circ F = g \circ f \circ h$. □

For our purposes, lifts of homeomorphisms are more important than lifts of continuous maps, therefore, we establish several properties of such lifts. In the following lemmas, it is assumed that $f: S_P \rightarrow S_P$ is an orientation preserving homeomorphism, and $F$ a lift of $f$.

Lemma 2.23. $F$ maps arc components to arc components, i.e. if $t_1, t_2 \in \mathbb{R}, x \in C$, then $F(t_1, x) = (s_1, y), \quad F(t_2, x) = (s_2, y)$ for some $y \in C$.

Proof. The set $[t_1, t_2] \times \{x\}$ is arc connected, so its image under a continuous map $F$ must also be arc connected. Thus $F(t_1, x)$ and $F(t_2, x)$ belong to the same arc component. □

Lemma 2.24. $F$ is 'strictly' orientation preserving on the arc components $\mathbb{R} \times \{x\}, x \in C$. That is, if $(s_1, y) = F(t_1, x), (s_2, y) = F(t_2, x)$, and $t_1 < t_2$, then $s_1 < s_2$. 
Proof. The homeomorphism \( f \) is orientation preserving, and so is \( h \), so \( F \) must be orientation preserving, i.e. if \( (s_1, y) = F(t_1, x) \), \( (s_2, y) = F(t_2, x) \), and \( t_1 \leq t_2 \), then \( s_1 \leq s_2 \). To show strict inequality, suppose that \( F(t_1, x) = F(t_2, x) \) and \( t_1 < t_2 \), then \( F \) is constant on the set \([t_1, t_2] \times \{x\}\), so \( h \circ F = y \forall (t, x) \in [t_1, t_2] \times \{x\}\). But \( h \circ F = f \circ h \), so \( h(t, x) = y \forall (t, x) \in [t_1, t_2] \times \{x\}\), since \( f \) is a homeomorphism. This is impossible, because \( h \) is a local homeomorphism. So the lemma is proved. \( \square \)

Corollary 2.25. The lift of a homeomorphism \( f : S_p \to S_P \) has the form

\[
F(t, x) = (F_1(t, x), F_2(x)), \tag{2.21}
\]

where \( F_2 \) does not depend on \( t \).

Lemma 2.26. Suppose that \( F(t, x) = (s, y) \), for some \( t, s \in \mathbb{R}, x, y \in C \). Then \( F \mid_{\mathbb{R} \times \{x\}} : \mathbb{R} \times \{x\} \to \mathbb{R} \times \{y\} \) is surjective.

Proof. Denote \( F(t, x) = (F_1(t, x), F_2(t, x)) \). \( F \mid_{\mathbb{R} \times \{x\}} \) is continuous and strictly orientation preserving by Lemma 2.24, so for surjectivity we just need to show that \( F_1(t, x) \) is unbounded in both directions. Suppose it is bounded, i.e. \( \bar{t} \leq F_1(t, x) \leq \bar{t}, \ \forall t \in \mathbb{R} \), for some finite \( \bar{t}, \bar{t} \in \mathbb{R} \). Then we can take \( \bar{t} = \inf_t F_1(t, x), \bar{t} = \sup_t F_1(t, x) \).

Consider now the set \( \{(k, x) : k \in \mathbb{Z}, x \in C\} \). On this set, \( F_1 \) is strictly monotone increasing by Lemma 2.24, and \( F_1 \leq \bar{t} \). Thus \( F_1(k, x) \to \bar{t} \) as \( k \to \infty \). Then \( \exists K \forall k > K \bar{t} - F_1(k, x) \leq \varepsilon \). Take \( k_1, k_2 > K, k_1 < k_2 \), then \( F_1(k_1, x), F_1(k_2, x) \in \)
[\bar{t} - \varepsilon, \bar{t}], that is, the images of \((k_i, x), i = 1, 2\) under \(F\) are within a distance \(\varepsilon\) of each other.

Suppose that \(k_2 - k_1 = n\), and let \(y = h(k_1, x) \in C_{\bar{t}}\). Then \(h(k_2, x) = h(k_1 + n, A^{-n}A^n x) = A^n y\) by Proposition 2.17. Now \(h\) is an isometry by definition, and

\[ |F_1(k_1, x) - F_1(k_2, x)| < \varepsilon, \]

so

\[ d(h \circ F_1(k_1, x), h \circ F_1(k_2, x)) = d(f \circ h(k_1, x), f \circ h(k_2, x)) = d(f(y), f(A^n y)) < \varepsilon. \]

By choosing different \(k_2\) and thus varying \(n\), we get \(d(f(y_1), f(y_2)) < \varepsilon \quad \forall y_1, y_2 \in C_{\bar{t}}\). Now \(\varepsilon\) is arbitrary, so \(f(y) \equiv \text{const on } C_{\bar{t}}\). This contradicts the fact that \(f\) is a homeomorphism. Therefore, \(\bar{t} = +\infty\). Similarly it can be shown that \(\bar{t} = -\infty\), and the lemma is proved. \(\square\)

In order to use lifts of solenoidal maps, it is important to establish their existence. This is done in the next two theorems, where we first prove existence of lifts for homeomorphisms homotopic to the identity, and then for any continuous map of a solenoid.

**Definition 2.27.** A homeomorphism \(f : S_p \to S_p\) is called homotopic to the identity (denoted \(f^h \text{id}\)) provided there exists an \(H : S_p \times [0, 1] \to S_p\) such that:

a) \(H\) is continuous;

b) \(H(z, 0) = f(z) \quad \forall z \in S_p\);

c) \(H(z, 1) = z \quad \forall z \in S_p\).
Similarly we define a lift $F : \mathbb{R} \times C \to \mathbb{R} \times C$ homotopic to the identity.

**Remark 2.28.** A homeomorphism $f$ of $S_P$, homotopic to the identity, fixes arc components of $S_P$, in the sense that

$$\forall z \in S_P \quad \exists \tau_z \in \mathbb{R} \quad f(z) = \varphi^\tau_z(z) = z \pi_P(\tau_z). \quad (2.22)$$

**Theorem 2.29.** Let $f$ be an orientation preserving homeomorphism of $S_P$, homotopic to the identity. Then there exists a lift $F : \mathbb{R} \times C \to \mathbb{R} \times C$ of $f$.

**Proof.** First, denote by $f_\mathbb{R}$ the identity map: $f_\mathbb{R}(z) = z \quad \forall z \in S_P$. Then it can be checked directly that $F_\mathbb{R}(t, x) = (t, x)$ is a lift of $f_\mathbb{R}$. Let $f : S_P \to S_P$ be a homeomorphism homotopic to the identity. We will show that it has a lift $F$.

Denote $f' = f \circ h$, $f'_\mathbb{R} = f_\mathbb{R} \circ h$. Let $G : \mathbb{R} \times C \times [0, 1] \to S_P$ be the homotopy between them (see Figure 3):

$$G(t, x, 0) = f'_\mathbb{R}(t, x)$$

$$G(t, x, 1) = f'(t, x)$$
Note that \( G(t, x, 0) = f'_e(t, x) = f_e \circ h = h \circ F_e \). So we are in a position to use the homotopy lifting property (see [66]). \( h \) is a covering projection, so it has the homotopy lifting property, that is, there is a \( \tilde{G} : \mathbb{R} \times C \times [0, 1] \rightarrow \mathbb{R} \times C \), such that \( \tilde{G}(t, x, 0) = F_e \) and \( h \circ \tilde{G} = G \). Denote \( F(t, x) := \tilde{G}(t, x, 1) \), then

\[
  h \circ F(t, x) = h \circ \tilde{G}(t, x, 1) = G(t, x, 1) = f'(t, x),
\]

so \( h \circ F = f' = f \circ h \), therefore, \( F \) is a lift of \( f \). □

**Theorem 2.30.** Let \( f \) be a continuous map of \( S_p \). Then it has a lift \( F : \mathbb{R} \times C \rightarrow \mathbb{R} \times C \).

**Proof.** Let \( f : S_p \rightarrow S_p \) be a map. Denote \( f' = f \circ h \), and consider \( f'_0 : \mathbb{R} \times C \rightarrow S_p \), given by \( f'_0(t, x) = f'(0, x) \).

**Claim 2.31.** \( f'_0(t, x) \) has a lift \( F_0(t, x) \) such that \( h \circ F_0 = f'_0 \).

**Proof.** In fact, \( f'_0 = f \circ h |_{\{0\} \times C} \), \( f \) is continuous, so \( f |_{C_x} \) is also continuous. We construct \( F_0 \) as follows. Consider open sets \( U_0, U_1 \subset S_p \) (see Figure 4) defined as follows:

\[
  U_0 = \{ z : z_1 = e^{\theta_i}, -\frac{\pi}{5} < \theta < \frac{6\pi}{5}, \quad U_1 = \{ z : z_1 = e^{\theta_i}, \frac{4\pi}{5} < \theta < \frac{11\pi}{5}. \}
\]

Then the following is satisfied:

- a) \( U_0 \cup U_1 = S_p \) and thus \( U_0 \cap U_1 \neq \emptyset \);
Figure 4. Construction of the lift $F_0$ of $f'_0$.

b) if $z \in U_i$ then $C_z \subset U_i$ for $i = 0, 1$ (that is, entire cross-sections belong to these sets).

Without loss of generality, we assume that $f'_0(0,0) \in U_0$.

The collections $\{\tilde{U}_{0,k}\}_{k=-\infty}^{\infty}$, $\{\tilde{U}_{1,k}\}_{k=-\infty}^{\infty}$, where $\tilde{U}_{0,k} = (-0.1 + k, 0.6 + k) \times C$, $\tilde{U}_{1,k} = (0.4 + k, 1.1 + k) \times C$, are the even covers of $U_0, U_1$, respectively. Also, $h |_{\tilde{U}_{i,k}} : \tilde{U}_{i,k} \to U_i$ is a homeomorphism $\forall k, i = 0, 1$.

Choose $F_0(0,0)$ from the even cover of $f'_0(0,0)$. Then there is a set $\tilde{U}_0 \in \{\tilde{U}_{0,k}\}$ such that $F_0(0,0) \in \tilde{U}_0$. Now take $\tilde{U}_1 \in \{\tilde{U}_{1,k}\}$ so that $\tilde{U}_0 \cap \tilde{U}_1 \neq \emptyset$. That is, if $\tilde{U}_0 = \tilde{U}_{0,t}$, take $\tilde{U}_1 = \tilde{U}_{1,t-1}$ or $\tilde{U}_{1,t}$. Thus we have $h |_{\tilde{U}_0} : \tilde{U}_0 \to U_0$, $h |_{\tilde{U}_1} : \tilde{U}_1 \to U_1$, which are homeomorphisms; $U_0 \cup U_1 = S_P$, so in addition, $U_0 \cup U_1 \supset f'_0[C_0]$, where
\[ C_0 = \{0\} \times C. \] Therefore, \( V_0 = (f_0')^{-1}(U_0), \ V_1 = (f_0')^{-1}(U_1) \) cover \( C_0, \) and \( V_0, V_1 \)

are open since \( f_0' \) is continuous.

The set \( C_0 \) is compact, so there is a Lebesgue number \( \delta \) for the cover \( V_0, V_1 \) of \( C_0. \) Let \( W_1, W_2, \ldots, W_n \) be a cover of \( C_0 \) by disjoint relatively clopen (i.e. closed and open) sets of diameter less than \( \delta. \) Then

\[ \forall k = 1, 2, \ldots, n \quad W_k \subset V_0 \text{ or } W_k \subset V_1. \]

It follows that \( f_0'[W_k] \subset U_i \) for the corresponding \( i = 0, 1. \) For every \((0, x) \in W_k\)

define

\[ F_0(0, x) = h \big|_{U_i}^{-1} \circ f_0'(0, x). \]

For every \( k, F_0 \) is continuous on \( W_k, \) therefore, it is continuous on \( C_0 \) by the Pasting Lemma. Also, \( F_0 \) is well defined, since \( \forall i, j W_i \cap W_j = \emptyset. \) Furthermore, on every \( W_k \)

(and therefore \( \forall (0, x) \in C_0)\),

\[ h \circ F_0 = h \circ h \big|_{U_i}^{-1} \circ f_0' = f_0', \]

so \( F_0 \) is the desired 'lift' of \( f_0'. \)

Now back to the proof of Theorem 2.30, construct a homotopy \( G : \mathbb{R} \times C \times [0, 1] \to S^p, \) by taking \( G(t, x, s) = f'(st, x). \) Then \( G \) is continuous as a composition of continuous functions, and

\[ G(t, x, 0) = f_0'(t, x), \quad G(t, x, 1) = f'(t, x). \]

Furthermore,

\[ G(t, x, 0) = f_0' = h \circ F_0. \]
The covering projection \( h \) has the homotopy lifting property, so there exists \( \tilde{G} : \mathbb{R} \times C \times [0,1] \to \mathbb{R} \times C \), such that \( \tilde{G}(t,x,0) = F_0 \) and \( h \circ \tilde{G} = G \). Denote \( F(t,x) = \tilde{G}(t,x,1) \). Then

\[
    h \circ F(t,x) = h \circ \tilde{G}(t,x,1) = G(t,x,1) = f'(t,x) = f \circ h(t,x),
\]

so \( F \) is a lift of \( f \). \( \square \)

We now analyze the relationship between two lifts of the same map of the solenoid. Let \( F, G : \mathbb{R} \times C \to \mathbb{R} \times C \) be two lifts of \( f : S_p \to S_p \). Denote \( (s_1,y_1) = F(t,x), (s_2,y_2) = G(t,x), (t,x) \in \mathbb{R} \times C \). Then by Proposition 2.17 these two lifts are related as follows:

\[
    (s_1 + k(t,x), y_1) = (s_2, A^{k(t,x)}y_2), \quad \text{where } k(t,x) \in \mathbb{Z}.
\]

Since everything is continuous along the arc components of \( \mathbb{R} \times C \), and the integers are discrete, \( k \) does not depend on \( t \). Thus we have

\[
    (s_1 + k(x), y_1) = (s_2, A^{k(x)}y_2), \quad k(x) \in \mathbb{Z}, (t,x) \in \mathbb{R} \times C, \tag{2.23}
\]

or, using the function \( T \) defined in (2.16),

\[
    G(t,x) = T^{k(x)} \circ F(t,x). \tag{2.24}
\]

Also, if \( k : C \to \mathbb{Z} \) is continuous and \( F \) is a lift then so is \( G \) defined by (2.24). That is, given a lift \( F \), the map \( G \) is a lift if and only if there exists continuous \( k : C \to \mathbb{Z} \) such that (2.24) holds.
Example 2.2 illustrates that $k_n(x)$ in Equation (2.18) actually does depend on $n$, $x$, and $k(x)$ in Equation (2.23) depends on $x$.

**Example 2.2.** Consider the identity homeomorphism on the dyadic solenoid $S_2$, $f(z) = z$, and its lift $F(t, x) = (t, x)$. Construct another lift $G$ as follows. Decompose $C$ into four cylinder sets: $C_{00}$, $C_{01}$, $C_{10}$, $C_{11}$. Define the lift $G$ so that

\[
G \{[0,1] \times C_{00}\} = [2,3] \times C_{01}, \quad G \{[0,1] \times C_{01}\} = [1,2] \times C_{10},
\]

\[
G \{[0,1] \times C_{10}\} = [1,2] \times C_{00}, \quad G \{[0,1] \times C_{11}\} = [0,1] \times C_{11}.
\]

(see Figure 5). Clearly this map satisfies the definition of a lift, i.e. equation (2.17). Now if $(t, x) \in [0,1] \times C_{00}$, then $T(t, x) = (t + 1, A^{-1}x) \in [1,2] \times C_{11}$ and $F \circ T(t, x) = T^{-1} \circ F(t, x)$. Thus $k_1(x) = -1$ for $x \in C_{00}$.

At the same time, $T^2(t, x) = (t + 2, A^{-2}x) \in [2,3] \times C_{10}$ and $F \circ T^2(t, x) = T \circ F(t, x)$. Thus $k_2(x) = 1$ for $x \in C_{00}$.
We also note that $G(t, x) = T^{k(x)} \circ F(t, x)$, where

$$k(x) = \begin{cases} 
2 & : x \in C_{00} \\
3 & : x \in C_{01} \\
3 & : x \in C_{10} \\
0 & : x \in C_{11} 
\end{cases}$$

Thus we see that $k_n(x)$ and $k(x)$ depend on $(n, x)$ and $x$, respectively.

### Homeomorphisms of Solenoids Homotopic to the Identity and Their Lifts

In our discussion of flows on solenoids in Chapters 3, 4, we heavily employ the concept of homeomorphisms homotopic to the identity, both in the definitions of flows and in proofs of their properties. In this section, we analyze homeomorphisms of $S_P$ homotopic to the identity and their lifts.

**Theorem 2.32.** Every homeomorphism $f^h \text{id}$ of $S_P$ has a unique lift $F^h \text{id}$.

**Proof.** We construct such a lift. Observe first that $\text{Id}(t, x) = (t, x)$ is a lift of $\text{id} : S_P \rightarrow S_P$. Now $\forall (t, x) \in [0, 1) \times C$ let $\tau(t, x) = \tau_z$ be as in Equation (2.22), where $z = h(t, x)$. Define

$$F(t, x) = (t + \tau(t, x), x).$$

(2.25)

Then $F(t, x)$ is continuous because $\tau(t, x)$ is. Also, by properties of $h$,

$$h \circ F(t, x) = z * \pi_P(\tau) = f(z) = f \circ h(t, x).$$

Furthermore, since $h$ is a homeomorphism on $[0, 1) \times C$, $F = h^{-1} \circ f \circ h$ is a homeomorphism from $[0, 1) \times C$ to a fundamental domain in $\mathbb{R} \times C$. Extend $F$ to all of...
$\mathbb{R} \times C$ as follows:

$$
\text{if } (t, x) \in T^n ([0, 1] \times C) \quad \text{then } F(t, x) = T^n \circ F \circ T^{-n}(t, x),
$$

where $T$ is the 'coil translation' map defined by (2.16). It is easy to see that $F$ is continuous on the 'joints' $(n(x), x)$ (since $h$ is a homeomorphism on $\{(n(x) - \varepsilon, n(x) + \varepsilon) \times \{x\}\}$) and that $h \circ F = f \circ h$. Thus $F$ is a lift of $f$.

$G(t, x, \alpha) = (t + (1 - \alpha) \tau(t, x), x)$ is a homotopy between $F$ and $\text{Id}$.

We now prove uniqueness of $F$. Let $F, G$ be two lifts of $f$, both homotopic to the identity, and let $(s_1, y_1) = F(t, x)$, $(s_2, y_2) = G(t, x)$. By (2.23), $(s_1 + k(x), y_1) = (s_2, A^{k(x)}y_2)$, and in particular,

$$
y_1 = A^{k(x)}y_2. \tag{2.26}
$$

By the first part of this proof and equation (2.25),

$$
(s_1, y_1) = (t + \tau_1, x), \quad (s_2, y_2) = (t + \tau_2, x);
$$

so

$$
y_1 = y_2. \tag{2.27}
$$

It follows from (2.26), (2.27) that $A^{k(x)} = I$, so $k(x) \equiv 0$, and thus $F = G$. □

**Corollary 2.33.** If $F^h \text{ id}$ is the lift of $f^h \text{ id}$, then $F \circ T = T \circ F$.

**Proposition 2.34.** $\tau(t, x)$ is bounded on $\mathbb{R} \times C$. 
Proof. \( \forall (t, x) \in \mathbb{R} \times C \quad (t, x) = T^n(t_0, x_0) \), where \( t_0 \in [0, 1], x \in C \). Then

\[
F(t, x) = F \circ T^n(t_0, x_0) = T^n \circ F(t_0, x_0)
\]

\[
= (t_0 + \tau(t_0, x_0) + n, A^{-n}x_0) = (t + \tau(t_0, x_0), x),
\]

so \( \tau(t, x) = \tau(t_0, x_0) \). Thus,

\[
\text{range}_{\mathbb{R} \times C} \tau = \text{range}_{[0,1) \times C} \tau = \text{range}_{[0,1] \times C} \tau.
\]

Since \([0, 1] \times C\) is compact and \( \tau \) is continuous, it follows that \( \tau \) is bounded. \( \square \)

We conclude this section by proving a criterion for a homeomorphism homotopic to the identity to have a fixed point. We will use the following notation.

**Notation.** We write \( z_1 < z_2 \) if and only if \( z_2 = \varphi^t(z_1) \) for some \( t > 0 \).

**Proposition 2.35.** Let \( f \) be a homeomorphism of \( S_\mathcal{P} \) homotopic to the identity. If there are two points \( z_1, z_2 \in S_\mathcal{P} \) such that \( f(z_1) < z_1 \) and \( z_2 < f(z_2) \), then \( f \) has a fixed point.

**Proof.** Let \( U = \{ z : f(z) < z \} \) and \( V = \{ z : z < f(z) \} \). Note that

\[
h^{-1}(U) = \{ (t, x) : F_1(t, x) < t \}\)

and

\[
h^{-1}(V) = \{ (t, x) : F_1(t, x) > t \}\)

are open. It follows that \( U, V \) are open. Also, \( U, V \) are disjoint by construction.

Now if \( U \cup V = S_\mathcal{P} \), then \( U, V \) would form a separation of \( S_\mathcal{P} \), which is impossible, since \( S_\mathcal{P} \) is connected. Thus the union of \( U \) and \( V \) is not all of \( S_\mathcal{P} \), i.e. \( \exists z_0 \in S_\mathcal{P} \setminus (U \cup V) \), which implies that \( f(z_0) = z_0 \). \( \square \)
Decomposition of Homeomorphisms on Solenoids

Main Classes of Homeomorphisms

In this section we state a result of Kwapisz [43] about decomposition of homeomorphisms on solenoids into a composition of maps that belong to a few main classes of homeomorphisms. First, we list these main classes of homeomorphisms.

1. Homeomorphism $\tau(z)$ homotopic to the identity has a lift

$$F(t, x) = (t + \delta(t, x), x),$$

where $\delta : \mathbb{R} \times C \to \mathbb{R}$ is a continuous function invariant under the coil map $T$.

2. Translation by $w \in S_p$:

$$t_w(z) = z \ast w.$$  \hspace{1cm} (2.29)

3. Power homeomorphism:

$$g_{a/b} = g_a \circ g_{1/b}, \quad \text{where } g_a : z \mapsto z^a, \ g_{1/b} = (g_b)^{-1}. \hspace{1cm} (2.30)$$

**Remark 2.36.** If $\mathcal{P} = (p, p, \ldots, p, \ldots)$, we have a special case of a power homeomorphism, a right shift homeomorphism:

$$s_R(z_1, z_2, \ldots, z_n, \ldots) = (z_1^p, z_2, \ldots, z_n^p, \ldots), \quad \text{or } s_R(z) = z^p. \hspace{1cm} (2.31)$$

Clearly, $s_R = g_p$. We note that if the sequence $\mathcal{P}$ is not constant, then the shift map is not a homeomorphism of $S_p$. 
To determine for what values of $a, b$ the map $g_{a/b}$ is a homeomorphism, we need the following definition.

**Definition 2.37.** Let $\mathcal{P} = (p_1, p_2, \ldots)$. An integer $l$ is called $\mathcal{P}$-recurrent provided for every prime factor $l_i$ of $l$ and for every $n \in \mathbb{N}$ $\exists k \geq n$ such that $l_i$ is a factor of $p_k$. In other words, every prime factor $l_i$ of $l$ occurs infinitely many times in the sequence $\mathcal{P}$.

**Remark 2.38.** If $\mathcal{P} = (p_1, p_2, \ldots, p_n, p_1, p_2, \ldots, p_n, \ldots)$ is a periodic sequence, then $l$ is $\mathcal{P}$-recurrent if and only if every factor of $l$ is a factor of $p_1 p_2 \cdots p_n$.

The following proposition is stated and proved by Kwapisz in [43], who in turn attributes it to Fokkink [25], Keesling [39], McCord [50], and others.

**Proposition 2.39.** The map $g_l(z) = z^l$ is a homeomorphism if and only if $l$ is $\mathcal{P}$-recurrent.

**Remark 2.40.** If $a, b$ are $\mathcal{P}$-recurrent, then so is $ab$. Thus if $g_a, g_b$ are homeomorphisms, then $g_{ab} = g_a \circ g_b$ is also a homeomorphism, and so is $g_{a/b} = g_a \circ g_b^{-1}$.

**Example 2.3.** Consider the dyadic solenoid $S_2$. The only 2-recurrent integers are powers of 2, so the only possible power homeomorphisms are $g_1, g_{2^n}, g_{1/2^n}$. Note that $g_{2^n} = s_R^n$ (the right shift), and $g_{1/2^n} = s_L^n$ (the left shift).
Theorem 2.41. If \( f : S_2 \to S_2 \) is a homeomorphism, then
\[
f = \tau \circ t_w \circ s,
\]
where \( \tau \) is a homeomorphism homotopic to the identity, \( t_w \) is a translation by \( w \in C_\varepsilon \), \( s = s_k \) for some \( k \in \mathbb{Z} \).

Note that it is sufficient to take \( w \in C_\varepsilon \) in (2.32), since otherwise \( w = w_1 \ast w_2 \), where \( w_1 \in C_\varepsilon \), \( w_2 \in S(\varepsilon) \), so \( t_w = t_{w_2} \circ t_{w_1} \). But \( t_{w_2} \) is homotopic to the identity, so it can be included in \( \tau \).

Theorem 2.42. Every homeomorphism \( f : S_\mathcal{P} \to S_\mathcal{P} \) can be represented as
\[
f = \tau \circ t_w \circ g_{a/b},
\]
where \( \tau \) is a homeomorphism homotopic to the identity, \( t_w \) is a translation by \( w \in C_\varepsilon \), \( g_{a/b} \) is a power homeomorphism, \( a \) and \( b \) are \( \mathcal{P} \)-recurrent.
CHAPTER 3

ERGODICITY AND ROTATION OF FIXED-POINT FREE FLOWS ON SOLENOIDS

Flows on Solenoids and Their Lifts

DEFINITION 3.1. A flow \( \phi : S_P \times \mathbb{R} \to S_P \) is a continuous group action of the reals on \( S_P \), or equivalently, a one-parameter group of homeomorphisms \( \phi^t : S_P \to S_P \) that satisfies the group property:

\[
\phi^{t+s}(z) = \phi^t(\phi^s(z)) \quad \forall t, s \in \mathbb{R}, \forall z \in S_P.
\] (3.1)

DEFINITION 3.2. We say that a flow is positively oriented if its direction coincides with the direction of the natural flow \( \varphi \), that is, if \( \forall z \in S_P, \ t > 0 \quad z < \phi^t(z) \). A flow with the opposite direction is called negatively oriented.

Most of the proofs in this dissertation are given for positively oriented flows, unless stated otherwise. The reason for this is that only trivial adjustments are necessary in order to reformulate these proofs for negatively oriented flows.

PROPOSITION 3.3. If \( \phi \) is a flow on \( S_P \), then the arcs are not blown up under \( \phi \), or, more precisely, \( \exists M \) such that for any arc \( A = \{ \phi^t(z_0) : 0 \leq t \leq t_0 \} \) and for all \( T \in \mathbb{R}, \phi^T(A) \) has length not more than \( M \).
Figure 6. Definition of a lift of a flow.

PROOF. Let \( M = \max \{ d_{\text{arc}}(\phi^t(z), z) : z \in S_P \} \). Then

\[
\text{length} \phi^T(A) = d_{\text{arc}}(\phi^T(\phi^t(z_0)), \phi^T(z_0)) = d_{\text{arc}}(\phi^t(\phi^T(z_0)), \phi^T(z_0)) \leq M
\]

for any \( T \in \mathbb{R} \). \( \square \)

**Definition 3.4.** Let \( \phi : S_P \times \mathbb{R} \to S_P \) be a flow on the solenoid \( S_P \). A flow \( \tilde{\phi} : (\mathbb{R} \times C) \times \mathbb{R} \to \mathbb{R} \times C \) is called a lift of \( \phi \) to \( \mathbb{R} \times C \) if \( \phi \circ h(x, t) = h \circ \tilde{\phi}(x, t) \), that is, if the diagram in Figure 6 commutes.

As we argue in Remark 3.12, in order to obtain meaningful results pertaining to rotation sets of flows, one has to restrict attention to the lifts homotopic to the identity. Any flow has a unique lift homotopic to the identity. This follows from the fact that \( \forall t \in \mathbb{R} \) \( \tilde{\phi}^t \) is a lift of \( \phi^t : S_P \to S_P \), which is a homeomorphism of \( S_P \) homotopic to the identity. By Theorem 2.32 such a lift exists and is unique.
Rotation Sets of Flows on Solenoids

Pointwise Rotation Set

In this section, we define the pointwise rotation set of a flow and state several properties of this rotation set. We start by giving two definitions of a rotation set at a point of $S_{\mathcal{P}}$.

**Definition 3.5.** Let $\phi$ be a flow on $S_{\mathcal{P}}$, and $\bar{\phi}$ its lift to $\mathbb{R} \times C$. The rotation set of $\phi$ at the point $z \in S_{\mathcal{P}}$ is

$$
\rho(z, \phi) = \left\{ \text{limit points of } \frac{\tau \bar{\phi}^t(r, x) - \tau}{t} \text{, } \forall t \in \mathbb{R}, \ t \to \infty \right\}, \quad (3.2)
$$

where $h(r, x) = z$.

Since this does not cause ambiguity, we shall simply write

$$
\rho(z, \phi) = \left\{ \text{limit points of } \frac{\tau \bar{\phi}^t(r, x) - \tau}{t} \text{, as } t \to \infty \right\}. \quad (3.3)
$$

**Remark 3.6.** Since $d_{\text{arc}}(z, \phi^t(z)) = \pi \bar{\phi}^t(r, x) - \tau$, we can rewrite (5.8) as

$$
\rho(z, \phi) = \left\{ \text{limit points of } \frac{1}{t} d_{\text{arc}}(z, \phi^t(z)) \text{, } \forall t \in \mathbb{R}, \ t \to \infty \right\}. \quad (3.4)
$$

**Definition 3.7.** The pointwise rotation set of flow $\phi$ is

$$
\rho_p(\phi) = \bigcup_{z \in S_{\mathcal{P}}} \rho(z, \phi). \quad (3.5)
$$

Similarly, the pointwise rotation set of a homeomorphism can be defined:
DEFINITION 3.8. Let \( f \) be a homeomorphism of \( S_\varphi \), and \( F \) its lift to \( \mathbb{R} \times \mathbb{C} \). The rotation set of \( F \) at the point \( z = h(\tau, x) \) is

\[
\rho(z, F) = \left\{ \text{limit points of } \frac{\pi_1F^{k_i}(\tau, x) - \tau}{k_i}, \quad \forall k_i \in \mathbb{Z}, \ k_i \to \infty \right\},
\]

and the pointwise rotation set of \( F \) is

\[
\rho_p(F) = \bigcup_{z \in S_\varphi} \rho(z, F).
\]

We now describe several basic properties of the pointwise rotation set.

PROPOSITION 3.9. If two points \( z_1, z_2 \in S_\varphi \) lie on the same orbit, then \( \rho(z_1, \phi) = \rho(z_2, \phi) \).

PROOF. Let \((\tau_1, x_1)\) and \((\tau_2, x_2)\) be points from the cover of \( z_1 \) and \( z_2 \), respectively. By assumption, \( \exists t_0 \) such that \( z_2 = \phi^{t_0}(z_1) \). This implies that \( x_2 = x_1 \) and \( (\tau_2, x_1) = \phi^{t_0}(\tau_1, x_1) \). Thus we have

\[
\rho(z_2, \phi) = \left\{ \text{limit points of } \frac{\pi_1\phi^{\tau_1}(\tau_2, x_1) - \tau_2}{T_i} \right\}
= \left\{ \text{limit points of } \frac{\pi_1\phi^{\tau_1+t_0}(\tau_1, x_1) - \tau_1 + \tau_1 - \tau_2}{T_i} \right\} = \rho(z_1, \phi),
\]

since \( \frac{\tau_1 - \tau_2}{T_i} \to 0 \) as \( T_i \to \infty \) and \( \left\{ \text{limit points of } \frac{\pi_1\phi^{\tau_1+t_0}(\tau_1, x_1) - \tau_1}{T_i} \right\} = \left\{ \text{limit points of } \frac{\pi_1\phi^{\tau_1+t_0}(\tau_1, x_1) - \tau_1}{T_i + t_0} \right\} \).

\]

COROLLARY 3.10. If the flow \( \phi \) has no fixed points on some arc component of \( S_\varphi \), then \( \rho(z, \phi) \) is constant on that arc component.
Proposition 3.11. For every $z \in S_p$, $\rho(z, \phi)$ is a closed interval.

Proof. First, we note that for every $z \in S_p$, the rotation set $\rho(z, \phi)$ is closed. This follows from the fact that the set of limit points is always closed. We now show that for every $z \in S_p$, $\rho(z, \phi)$ is connected, i.e. if $a, b \in \rho(z, \phi)$, then $\forall \alpha \in [0, 1] \quad c = \alpha a + (1 - \alpha) b \in \rho(z, \phi)$.

From the fact that $\phi$ is continuous and $h$ is a local homeomorphism, it follows that $\tilde{\phi}$ is continuous. In particular, $\tilde{\phi}^t(\tau, x)$ is continuous in $t$ for any $(\tau, x) \in \mathbb{R} \times C$.

Then $\pi_1\tilde{\phi}^t(\tau, x) - \tau$ is continuous, and so $r(t) = \frac{\pi_1\tilde{\phi}^t(\tau, x) - \tau}{t}$ is continuous for all $t \neq 0$.

Let $t_i^{(1)}, t_i^{(2)}$ be two sequences, such that $r(t_i^{(1)}) \rightarrow a, \quad r(t_i^{(2)}) \rightarrow b$ as $i \rightarrow \infty$.

Without loss of generality, we assume that $a \leq b$ and that $t_i^{(1)}, t_i^{(2)}$ are such that

$$\forall i. \quad t_{i+1}^{(k)} > \max \{t_i^{(1)}, t_i^{(2)}\} \quad (3.8)$$

(Otherwise, choose appropriate subsequences.) Take an $\varepsilon < \frac{1}{2} \min \{c - a, b - c\}$ and $N$ such that $\forall i \geq N \quad |r(t_i^{(1)}) - a| < \varepsilon$ and $|r(t_i^{(2)}) - b| < \varepsilon$. Then $\forall i \geq N$ we have $r(t_i^{(1)}) < c < r(t_i^{(2)})$. $r(t)$ is continuous for $t \neq 0$, so $\forall i \exists t_i^{(3)} \in [t_i^{(1)}, t_i^{(2)}]$ such that $r(t_i^{(3)}) = c$. Now $\forall i \quad t_{i+1}^{(3)} > t_i^{(3)}$ and $t_i^{(3)} \rightarrow \infty$ by (3.8). In addition, $\lim_{i \rightarrow \infty} r(t_i^{(3)}) = c$, so $c \in \rho(z, \phi)$, where $z = h(\tau, x)$.

We note that the pointwise rotation set $\rho(\phi)$ need not be connected, as Katok's Example in Chapter 4 demonstrates.
Remark 3.12. It is well-known that rotation sets of homeomorphisms or flows on the circle and on the torus do not depend on the choice of the lift, in the sense that if we choose a different lift, the rotation set would differ by an integer translate. However, that is not the case on solenoids. Due to a lack of connectedness in the Cantor set coordinate on the universal cover, we have more freedom in choosing lifts. As a consequence, we can construct lifts of the same homeomorphism, whose rotation sets differ by an arbitrary constant, thereby rendering the concept of a rotation set meaningless.

Example 3.1. In Example 2.2 we constructed a lift $G$ of the identify homeomorphism, whose rotation set is $\rho_p(G) = \{4/3\}$, while the rotation set of the standard (identity) lift is $\rho_p(F) = \{0\}$.

We would like to avoid this phenomenon for flows as well as homeomorphisms, since many homeomorphisms appear as time-one maps of continuous flows. The most logical way to rectify the situation is to restrict attention to the lifts homotopic to the identity, and that is what we do in this dissertation.

MZ Rotation Set

Definition 3.13. Let $\phi$ be a flow on $S_p$, and $\bar{\phi}$ its lift to $\mathbb{R} \times C$. The MZ rotation set (or simply the rotation set) of $\phi$ is

$$\rho(\phi) = \left\{ \text{limit points of } \frac{\tau_i \bar{\phi}^{t_i}(\tau_i, x_i) - \tau_i}{t_i}, \forall t_i \in \mathbb{R}, t_i \to \infty, (\tau_i, x_i) \in \mathbb{R} \times C \right\}$$
Similarly we define rotation sets for homeomorphisms:

**Definition 3.14.** Let \( f \) be a homeomorphism of \( S_p \), and \( F \) its lift to \( \mathbb{R} \times C \). The MZ rotation set of \( F \) is

\[
\rho(F) = \left\{ \text{limit points of} \frac{\pi_1 F^{k_i} (\tau_i, x_i) - \tau_i}{k_i}, \ \forall k_i \in \mathbb{Z}, \ k_i \to \infty, \ (\tau_i, x_i) \in \mathbb{R} \times C \right\}
\]

Misiurewicz and Ziemian [52] prove that the rotation set of a toral map or flow is compact. A nearly identical proof applies to the solenoidal flows. In addition, it follows directly from the definitions that

\[
\rho_p(\phi) \subset \rho(\phi). \quad (3.9)
\]

Because of the nice properties that the MZ rotation set possesses, including compactness, it has been extensively studied in the last decade: In this dissertation, the most important results are obtained for the MZ rotation set, but in some cases certain conclusions about the pointwise rotation set are also derived.

We conclude this section by stating a theorem proved by Misiurewicz and Ziemian [53]. Although originally this theorem was proved for toral maps and flows, a nearly identical proof works for solenoids as well.

**Theorem 3.15 (Misiurewicz – Ziemian).** For every extreme point \( v \) of \( \rho(\phi) \), there exists an ergodic probability measure \( \mu \), invariant under flow \( \phi \), such that \( \phi \) has rotation number \( v \) with respect to \( \mu \).
In this section, we introduce and study the concept of return time to a cross-section, that will be used later to construct and analyze ergodic measures for solenoidal flows.

**Definition 3.16.** Let $X \subset C$ be a clopen set, and let $x \in X$. The *return time* of $x$ to $X$ is

$$r(x) = \min \{ \tau > 0 : \phi^\tau(0, x) = (1, x), \text{ so that } h(1, x) = h(0, y), y \in X \} \quad (3.10)$$

or, equivalently (see Figure 7),

$$r(x) = \min \{ \tau > 0 : \phi^\tau(x) \in X \}. \quad (3.11)$$

![Figure 7. Return time to $X \subset C$.](image)

**Definition 3.17.** The *average return time* to $X$ is

$$\bar{r}_X = \frac{1}{\beta(X)} \int_X r(x) \, d\beta, \quad (3.12)$$
where $\beta$ is the Bernoulli measure on $C$ (see Theorem 3.18 below). For our purposes, the most important is the average return time to $C$:

$$r_C = \int_C r(x) \, d\beta.$$  \hfill (3.13)

Clearly, for a fixed-point free flow, all $r(x) < \infty$. It is sometimes convenient to define a fixed-point free flow by specifying $r(x)$, $x \in C$, since return times capture most of the dynamics of such a flow.

**Unique Ergodicity and Rotation Number**

Using Theorem 3.15 of Misiurewicz and Ziemian, we will prove that the rotation set of a fixed-point free flow on a solenoid is trivial, by first showing that such a flow is uniquely ergodic. We start by stating and proving a well-known theorem on unique ergodicity of the adding machine.

**Theorem 3.18.** The adding machine on $C$ is uniquely ergodic, and the only invariant measure is the Bernoulli measure:

$$\beta(U) = \frac{1}{p_1 p_2 \ldots p_k},$$

for any cylinder set $U_{q_1 q_2 \ldots q_k}$ defined in (2.10).

**Proof.** Since cylinder sets form a basis for the Borel $\sigma$-algebra on $C$, it is sufficient to restrict attention to such sets. Denote by

$$\Omega_f = \{U_{q_1 q_2 \ldots q_j} : q_i \in \mathbb{Z} \mod p_i, \ i = 1, 2, \ldots, j\},$$
the collection of cylinder sets of the same diameter. Then

\[
\text{card } \mathcal{U}_1 = p_1, \quad \text{card } \mathcal{U}_2 = p_1p_2, \quad \ldots \quad \text{card } \mathcal{U}_j = p_1p_2\ldots p_j.
\]

By definition, the adding machine \( A \) permutes cylinder sets in \( \mathcal{U}_j \) \( \forall j \), so if \( \mu \) is an invariant measure for \( A \), then \( \mu(\mathcal{U}_1) = \mu(\mathcal{U}_2) \) \( \forall U_1, U_2 \in \mathcal{U}_j \). So we must have:

\[
\mu(U) = \frac{1}{p_1} \quad \forall U \in \mathcal{U}_1,
\]

\[
\mu(U) = \frac{1}{p_1p_2} \quad \forall U \in \mathcal{U}_2,
\]

\[
\ldots
\]

\[
\mu(U) = \frac{1}{p_1p_2\ldots p_j} \quad \forall U \in \mathcal{U}_j.
\]

These formulas completely determine the measure \( \mu \) on the collection of cylinder sets, and therefore on the entire \( \sigma \)-algebra of Borel sets on \( C \). It follows that such an \( A \)-invariant measure on \( C \) is unique. The measure defined by (3.15) is called the Bernoulli measure and is denoted by \( \beta(U) \).

Let \( \phi \) be a flow on \( S^1 \) without fixed points. We shall define an invariant probability measure for \( \phi \). \( \Omega = [0, 1] \times C \) denotes the fundamental domain in \( \mathbb{R} \times C \).

**Definition 3.19.** Define a measure \( \mu \) as follows:

\[
\mu(B) = \frac{1}{\bar{\tau}_C} \int_C \tau_B(x) \, d\beta,
\]

(3.16)

where \( B \subset \Omega \subset \mathbb{R} \times C \) and \( \tau_B(x) = \lambda \{ t \in \mathbb{R} : \phi^t(x) \in B \} \), i.e. the 'time-length' of the arc component of \( B \) with Cantor-set coordinate equal to \( x \), \( \bar{\tau}_C \) as defined in (3.13).
REMARK 3.20. 1. In formula (3.16), since $B \subset \Omega$, then
\[ \tau_B(x) \leq \tau(x) \quad \forall x. \]

2. $\mu$ is a Borel probability measure on $S_p$. To see that $\mu$ is a measure, we note that clearly $\mu(B) \geq 0$, $\mu(\emptyset) = 0$ since $\tau_\emptyset(x) = 0$, and $\mu(S_p) = 1$ since $\tau_{S_p}(x) = 1$. For countable additivity, we use the fact that if $\{B_i\}_{i=1}^\infty$ is a disjoint collection of sets in $S_p$, then
\[ \tau_{\bigcup B_i}(x) = \sum_{i=1}^\infty \tau_{B_i}(x). \]
Countable additivity of $\mu$ then follows by Fubini’s Theorem.

PROPOSITION 3.21. $\mu$ is invariant with respect to the flow $\phi$.

PROOF. Let $B \in \mathcal{B}(S_p)$, $\tilde{B} = \phi^t(B)$ for some $t \in \mathbb{R}$. For simplicity, suppose $B$ is connected along arc components, i.e. $B = \{(t, x) : x \in X \subset C, C$ clopen, $t_0(x) \leq t \leq t_1(x)\}$ for some functions $t_0, t_1 : C \to \mathbb{R}$. (otherwise, it is a union of such sets).
Then $\forall x \in X$, $\tau_B(x) = t_1(x) - t_0(x)$, also set $\tau_B(x) = 0 \quad \forall x \in C \setminus X$. Since $\tilde{B} = \phi^t(B)$, then
\[ \tau_{\tilde{B}}(x) = \pi_1(\phi^t(t_1, x) - \phi^t(t_0, x)) = (t_1(x) + t) - (t_0(x) + t) = \tau_B(x), \]
so $\mu(\tilde{B}) = \frac{1}{\tau_C} \int_C \tau_{\tilde{B}}(x) \, d\beta = \frac{1}{\tau_C} \int_C \tau_B(x) \, d\beta = \mu(B)$. Thus $\mu$ is invariant. \qed

We now would like to show that measure $\mu$ defined above is the only probability measure invariant with respect to the flow $\phi$, that is, $\phi$ is uniquely ergodic.
THEOREM 3.22. A fixed-point free flow \( \phi \) on \( S_P \) is uniquely ergodic. Moreover, the unique invariant measure is defined by (3.16).

PROOF. The proof will consist of the following steps:

1. Show that every \( \phi \)-invariant Borel measure \( \mu \) on \( \Omega \) is given by

\[
\mu(B) = \int_X \tau(x) \, d\tilde{\beta},
\]

where \( X \) is the support of \( B \) in \( C \), \( \tilde{\beta} \) some measure on \( C \).

2. Show that measure \( \tilde{\beta} \) is invariant with respect to the adding machine \( A : C \to C \).

3. Such measure \( \tilde{\beta} \) is unique, \( \tilde{\beta} = \beta \), so measure \( \mu \) is also unique.

Step 1: Let \( B(\Omega) \) denote the \( \sigma \)-algebra of Borel sets in the fundamental domain \( \Omega \subset \mathbb{R} \times C \), and let \( \mu \) be any \( \phi \)-invariant measure on \( S_P \) (it exists by Proposition 3.21). It induces an invariant measure for \( \tilde{\phi} \) on \( \Omega \). Abusing notation, we shall denote this measure also by \( \mu \):

\[
\mu(B) = \mu(h(B)) \quad \forall B \in B(\mathbb{R} \times C), \ B \subset \Omega. \quad (3.17)
\]

Consider the collection of sets \( E(\Omega) \subset B(\Omega) \), such that every set \( B \in E(\Omega) \) is of the form

\[
B = \{(t, x) : x \in X, X \text{ clopen}, \ t \in [0, \pi_1 \phi^{\tau(x)}(0, x)]\} \quad (3.18)
\]

for some continuous function \( \tau : C \to \mathbb{R}_+ \) with \( \text{supp} \tau = X \). Then every such function \( \tau \) uniquely determines a set \( B \in E(\Omega) \). Denote by \( C_+(C) \) the set of nonnegative continuous functions on \( C \), and introduce \( \gamma : C_+(C) \to E(\Omega) \), defined by \( \gamma(\tau) = B \).
Since $\mathcal{E}(\Omega)$ contains cylinder sets of $\mathbb{R} \times C$, then it, together with its $\varphi^t$-translates, generates the $\sigma$-algebra of Borel sets on $\Omega$. As a consequence, if we show uniqueness of $\mu$ on $\mathcal{E}(\Omega)$, it would imply uniqueness on $\mathcal{B}(\Omega)$. So let us consider $\mu|_{\mathcal{E}(\Omega)}$. Construct a map $\hat{\mu} : C_+(C) \to \mathbb{R}_+$ by

$$\hat{\mu}(\tau) = \mu(\gamma(\tau)).$$

**Claim 3.23.** $\hat{\mu}$ is a linear functional.

**Proof.** a) First, we show that $\hat{\mu}(\tau_1 + \tau_2) = \hat{\mu}(\tau_1) + \hat{\mu}(\tau_2)$.

Let $B_1 = \gamma(\tau_1)$, $B_2 = \gamma(\tau_2)$. Then $\gamma(\tau_1 + \tau_2) = B_1 \cup \varphi^{\tau_1}(B_2)$, the union being disjoint. So

$$\hat{\mu}(\tau_1 + \tau_2) = \hat{\mu}(B_1 \cup \varphi^{\tau_1}(B_2)) = \hat{\mu}(B_1) + \hat{\mu}(\varphi^{\tau_1}(B_2)) = \hat{\mu}(\tau_1) + \hat{\mu}(\tau_2),$$

as desired.

b) Now we prove that $\forall c > 0 \ \hat{\mu}(c\tau) = c\hat{\mu}(\tau)$.

Let $\varepsilon > 0$ be arbitrarily small. Choose $m, n \in \mathbb{N}$ so that

$$c - \frac{\varepsilon}{\hat{\mu}(\tau)} < \frac{m}{2^n} < c \quad \text{and} \quad \hat{\mu}(\gamma((c - \frac{m}{2^n})\tau)) < \varepsilon.$$  

We note that $\gamma((c - \frac{m}{2^n})\tau)$ is a set in $\mathcal{E}(\Omega)$, whose measure is less than $\varepsilon$ provided the time-lengths of arc components, $(c - \frac{m}{2^n})\tau$, are sufficiently small. Then by part (a):

$$\hat{\mu}\left(\frac{m}{2^n}\tau\right) < \hat{\mu}(c\tau)$$

$$= \hat{\mu}\left(\frac{m}{2^n}\tau\right) + \hat{\mu}\left((c - \frac{m}{2^n})\tau\right)$$

$$< \hat{\mu}\left(\frac{m}{2^n}\tau\right) + \varepsilon.$$  

(3.19)
Now, \( \tilde{\mu}(\tau) = \mu \left( \frac{1}{2^n} \tau + \cdots + \frac{1}{2^n} \tau \right) = 2^n \mu \left( \frac{1}{2^n} \tau \right) \), so \( \mu \left( \frac{1}{2^n} \tau \right) = \frac{1}{2^n} \tilde{\mu}(\tau) \).

Then since \( m \in \mathbb{N} \),

\[
\hat{\mu} \left( \frac{m}{2^n} \tau \right) = m \mu \left( \frac{1}{2^n} \tau \right) = m \frac{1}{2^n} \tilde{\mu}(\tau).
\]

By (3.19):

\[
c\mu(\tau) - \varepsilon < \frac{m}{2^n} \hat{\mu}(\tau) < c\mu(\tau),
\]

so

\[
\mu(\tau c\tau) > \mu \left( \frac{m}{2^n} \tau \right) = \frac{m}{2^n} \hat{\mu}(\tau) > c\mu(\tau) + \varepsilon
\]

and

\[
\mu(\tau c\tau) < \mu \left( \frac{m}{2^n} \tau \right) + \varepsilon = \frac{m}{2^n} \hat{\mu}(\tau) + \varepsilon < c\mu(\tau) + \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, we have \( \mu(\tau c\tau) = c\mu(\tau) \). \( \square \)

We now return to the proof of Theorem 3.22. By the Riesz Representation Theorem, there exists a measure \( \tilde{\beta} \) on \( C \) such that

\[
\hat{\mu}(\tau) = \int_X \tau(x) \, d\tilde{\beta} = \int_C \tau(x) \, d\tilde{\beta},
\]

where \( X = \text{supp } \tau \). It follows that

\[
\mu(B) = \int_{X_B} \tau_B(x) \, d\tilde{\beta} \quad \forall B \in \mathcal{E}(\mathbb{R} \times C),
\]

where \( X_B = \text{supp } \tau_B \). Thus

\[
\mu(B) = \int_{X_B} \tau_B(x) \, d\tilde{\beta} \quad \forall B = h(B) \subset S, \tilde{B} \in \mathcal{E}(\mathbb{R} \times C), \tilde{B} \subset \Omega.
\]

Step 2: We now show that measure \( \tilde{\beta} \) from (3.21) is invariant with respect to the adding machine \( A : C \to C \), that is, \( \tilde{\beta}(AX) = \tilde{\beta}(X) \quad \forall X \in \mathcal{B}(C) \). It is sufficient to prove this for cylinder sets of \( C \), so let \( X \) be a cylinder set.
As before, let \( r(x) \) denote the return time of \( x \) to the cross-section of the identity, then by continuity of \( r(x) \) and compactness of \( X \), \( 0 < r \leq r(x) \leq \bar{r} < \infty \). \( (r > 0 \) since \( \phi \) is fixed-point free.) Without loss of generality, assume \( X \) is small enough so that \( \bar{r} - r < r \) (if not, represent \( X \) as a union of smaller cylinder sets with the required property).

Take \( r \) such that \( \bar{r} - r < r < \bar{r} \), and let \( \tilde{B} \) be a Borel set in \( \mathbb{R} \times C \), built on \( X \) by flowing each \( \{0\} \times X \) by \( \phi^r \):

\[
\tilde{B} = \{(t, x) : x \in X, \ t \in [0, \pi_1 \phi^r(0, x)]\}.
\]

Clearly, \( \{0\} \times C \supset \{0\} \times X \subset \tilde{B} \).

Now take \( T > 0 \) such that \( T < r \), \( T + r > \bar{r} \), and consider the set \( \tilde{B}' = \phi^T(\tilde{B}) \).

Then \( \forall x \in X \),

\[
\pi_1 \phi^T(0, x) < \pi_1 \phi^r(0, x) < \pi_1 \phi^{r(x)}(0, x)
\]

\[
< \pi_1 \phi^r(0, x) < \pi_1 \phi^{T + r}(0, x).
\]

So by connectedness of \( \tilde{B} \) along the arc components and by continuity of \( \phi \),

\[
\{\phi^{r(x)}(0, x)\} = \phi^T(\tilde{B}) \cap (\{1\} \times C),
\]

and consequently, \( \{1\} \times X = \tilde{B}' \cap (\{1\} \times C) \).

Technically, \( \tilde{B}' \notin \mathcal{E}(\Omega) \), so we now prove formula (3.21) for \( \tilde{B}' \). Denote

\[
\tilde{B}'_L = \{(t, x) : x \in X, \ t = \pi_1 \phi^T(0, x)\} - \text{‘left boundary’},
\]

\[
\tilde{B}'_R = \{(t, x) : x \in X, \ t = \pi_1 \phi^{T + r}(0, x)\} - \text{‘right boundary’}.
\]

Now \( \tilde{B}' = \tilde{E}'_1 \cup \tilde{E}'_2 \), where \( \tilde{E}'_1, \ \tilde{E}'_2 \) are constructed as follows. Take
\( \tau_1(x) \) such that \( \forall (t, x) \in \tilde{B}_L' \quad \tilde{\phi}^{\tau_1(x)}(t, x) = (1, x) \in \{1\} \times C, \)

\( \tau_2(x) \) such that \( \forall x \in X \quad \tilde{\phi}^{\tau_2(x)}(1, x) \in \tilde{B}_R' \),

then

\[
\tilde{E}_1' = \{(t, x) : \ x \in X, \ \pi_1 \tilde{\phi}^T(0, x) \leq t \leq 1\} ,
\]

\[
\tilde{E}_2' = \{(t, x) : \ x \in X, \ 1 \leq t \leq \pi_1 \tilde{\phi}^{T+T}(0, x)\} .
\]

It follows that \( \tilde{E}_2' \in \mathcal{E}(\Omega) \), so \( \tilde{\mu}(\tilde{E}_2') = \int_X \tau_2 \, d\tilde{\beta} \). Also, \( \tilde{\phi}^{-T}(\tilde{E}_1') \in \mathcal{E}(\mathbb{R} \times C) \), so by invariance of \( \tilde{\mu} \):

\[
\tilde{\mu}(\tilde{E}_1') = \int_X \tau_1 \, d\tilde{\beta} . \]

Therefore,

\[
\tilde{\mu}(\tilde{B}') = \tilde{\mu}(\tilde{E}_1') + \tilde{\mu}(\tilde{E}_2') = \int_X (\tau_1 + \tau_2) \, d\tilde{\beta} = \int_X \tau_{\tilde{B}'} \, d\tilde{\beta} .
\]

Set \( B = h(\tilde{B}) \), \( B' = h(\tilde{B}') \), then by properties of the covering projection \( h \):

\[
supp B = h(\{0\} \times X) = X ,
\]

\[
supp B' = h(\{1\} \times X) = AX .
\]

Also, \( B' = \phi^T(B) \), then by (3.22):

\[
\mu(B) = \int_X \tau \, d\tilde{\beta} = \tau \tilde{\beta}(X) ,
\]

\[
\mu(B') = \int_{AX} \tau \, d\tilde{\beta} = \tau \tilde{\beta}(AX) .
\]

By invariance of \( \mu \):

\[
\tau \tilde{\beta}(X) = \tau \tilde{\beta}(AX) , \quad \text{or} \quad \tilde{\beta}(X) = \tilde{\beta}(AX) ,
\]

so \( \tilde{\beta} \) is invariant with respect to \( A \).

Step 3: By Theorem 3.18, \( A \) is uniquely ergodic, so \( \tilde{\beta} \) coincides with the Bernoulli measure \( \beta \), and thus \( \mu \) is uniquely determined on \( \mathcal{E}(S_P) = \{B : B = h(\tilde{B}), \tilde{B} \in \mathcal{E}(\Omega)\} \). Since \( \mathcal{E}(S_P) \) generates the \( \sigma \)-algebra \( \mathcal{B}(S_P) \), it follows that \( \mu \) is unique on \( \mathcal{B}(S_P) \) (if we require it to be normalized), and

\[
\mu(B) = \frac{1}{\tau_C} \int_C \tau_B(x) \, d\beta .
\]
We have seen in Proposition 3.21 that such $\mu$ is indeed a probability measure invariant with respect to $\phi$. □

We are now in a position to state a theorem on rotation sets of fixed-point free flows.

**Theorem 3.24.** If $\phi$ is a fixed-point free flow on $S_P$, then its rotation set is

$$\rho(\phi) = \{\rho\}, \quad \text{where} \quad \rho = \frac{1}{\bar{r}_C} = \frac{1}{\int_C r(x) \, d\beta}. \quad (3.23)$$

**Proof.** It is proved in [52] that rotation set $\rho(\phi)$ is compact. If $\rho(\phi)$ contains more than one point, then it has more than one extreme point, thus by Theorem 3.15 it must have more than one ergodic measure. This contradicts Theorem 3.22. Hence, $\rho(\phi)$ contains exactly one point $\rho$. Then all the averages $\frac{\tilde{\phi}^{t_n}(\tau, x) - \tau}{t_n}$ converge to the same point for any $(\tau, x) \in \mathbb{R} \times C$ and for any sequence of times $\{t_n\}$. So let $\{t_n\}$ be the sequence of return times to the cross-section of the identity. Then

$$\frac{\tilde{\phi}^{t_n}(\tau, x) - \tau}{t_n} = \frac{n}{r(x) + r(Ax) + \cdots + r(A^{n-1}x)} \rightarrow \frac{1}{\bar{r}_C},$$

since by unique ergodicity of $A$

$$\frac{r(x) + r(Ax) + \cdots + r(A^{n-1}x)}{n} \rightarrow \int_C r(x) \, d\beta = \bar{r}_C.$$

So the formula (3.23) is now proved. □
We are interested in the question of how the rotation number relates to the
dynamics of a fixed-point free solenoidal flow.

It is well-known that for a homeomorphism on the circle $S^1$, if $\rho = \frac{p}{q} \in \mathbb{Q}$, then
there exists a periodic point of period at most $q$. (For flows on $S^1$, every point is periodic.) For toral flows, the existence of a periodic orbit also follows from the rotation vector being rational. We would like to find out if a similar result holds on a solenoid; that is, if $\rho = \frac{p}{q} \in \mathbb{Q}$, then maybe there is a Cantor subset of small diameter, such that after $q$ iterates it comes back to itself.

This turns out to be false, as the following example demonstrates.

**Example 3.2.** We define a flow $\phi$ by specifying return times for all $x \in C$. Here $C_{q_1, q_{12}, \ldots}$ denote the cylinder sets, as in (2.10). Define $r(x)$ as follows:

$$
r(x) = \begin{cases} 
\frac{1}{2} & : \ x \in C_0 \\
\frac{3}{4} & : \ x \in C_{10} \\
\frac{7}{8} & : \ x \in C_{110} \\
& \ldots \\
1 - \frac{1}{2^n} & : \ x \in C_{11\ldots10} \\
1 & : \ x = (\ldots111) 
\end{cases}
$$

(3.24)
Then \( \int_C r(x) \, d\beta = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{8} \cdot \frac{7}{8} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} (1 - \frac{1}{2^n}) = \frac{2}{3} \), so by Theorem 3.24, \( \rho(\phi) = \frac{3}{2} \). However, it is easy to see that no subset of \( C \) maps to itself under \( \phi^2 \).

A weaker statement, which claims the existence of a periodic transversal, rather than a periodic cross-section, if a rotation number is rational, is also false, as the following discussion illustrates.

Suppose that \( \rho(\phi) = \frac{p}{q} \). Let us assume that such an invariant transversal \( D \) exists. To construct it, we would take a point \( z \in S_p \) and let \( \phi^n(z) \in D \) for all \( n \in \mathbb{Z} \). However, these iterates of \( z \) might go around \( S_p \) faster and faster (as in Example 3.3 below), causing \( D \) to intersect itself. Thus, we get a contradiction.

To give a quantitative measurement of how fast the iterates of a point move around \( S_p \) under the flow \( \phi \), we define a function

\[
a(t, z) = d_{arc} (z \ast \pi_p (t\rho), \phi^t(z)),
\]

where \( \rho(\phi) = \rho \). In other words, \( a(t, z) \) is the difference between iterates of the same point under \( \phi \) and under a linear flow with speed \( \rho \).

One can easily show that if \( \rho(\phi) = \frac{p}{q} \) and \( \phi \) is such that \( a(t, z) \to 0 \) as \( t \to \infty \) for all \( z \in S_p \), then every point on the solenoid is uniformly recurrent, that is,

\[
\forall \varepsilon > 0 \exists n \in \mathbb{N} \text{ such that } \forall z \in S_p, k \in \mathbb{N} \quad d(\phi^{knq}(z), z) < \varepsilon.
\]
But \( a(t, z) \) does not necessarily tend to 0 as \( t \to \infty \). In fact, it may not even be bounded, as we shall see in Example 3.3. Therefore, we cannot establish any recurrence properties for flows with a rational rotation number.

To facilitate the discussion, we state a few facts about the function \( a(t, z) \).

1. \( a(t, z) \) is sublinear in \( t \), since otherwise, if \( a(t, z) \) is not sublinear in \( t \) for some \( z = h(\tau, x) \), then \( \frac{\pi_1 \varphi^k(\tau, x) - \tau - tp}{t} \) does not converge to 0 as \( t \to \infty \), so \( \rho(\phi) \neq \rho \), which is a contradiction.

2. If \( \phi \) is a differentiable flow, then \( a(t, z) \) is differentiable, and its derivative in \( t \), \( a'(t, z) \) is almost periodic. To see this, we note that if \( I_0 \subset S(z) \) is any interval on the arc component of \( z \), then there is a sequence of intervals \( I_n \subset S(z) \) when the iterates \( \varphi^k(z) \) come within distance \( \delta \) of \( I_0 \). By continuity, the change of \( a(t, z) \) on \( I_n \) is within \( \varepsilon \) of the change of \( a(t, z) \) on \( I_0 \). Say, \( I_0 = [t_0, t_0'] \), \( I_1 = [t_1, t_1'] \), etc. Then \( T_1 = t_1 - t_0 \), \( T_2 = t_2 - t_1 \), \( T_i = t_i - t_{i-1} \), \( \ldots \) are \( \varepsilon \)-almost periods (according to Bohr’s definition of almost periodicity). On each interval \([t_{i-1}, t_i] \) \( a'(t, z) \) almost repeats its behavior on \([t_0, t_1] \).

3. \( a'(t, z) \) cannot be bounded away from 0 (otherwise the rotation number is too small if \( a' > 0 \), or too large if \( a' < 0 \).) For the same reason, it also cannot asymptotically get closer to 0 without equaling it. So \( a' \) must alternate signs.

We now describe an example of a flow for which the function \( a(t, z) \) is unbounded.

**Example 3.3.** Here we construct a flow on the dyadic solenoid \( S_2 \).
If $0.q_1q_2\ldots q_n$ is a binary expansion of $x \in C$, that is, if $\chi^{-1}(q_1q_2\ldots q_n00\ldots) = x$, define $r(x) = \sum_{i=1}^{n} p_i$, where

$$p_1 = \frac{1}{4} q_1, \quad p_2 = \frac{3q_1}{4^2} q_2, \quad p_3 = \frac{3q_1+q_2}{4^3} q_3, \quad \ldots \quad p_n = \frac{3q_1+q_2+\ldots+q_{n-1}}{4^n} q_n.$$  \hspace{1cm} (3.26)

Then extend $r(x)$ continuously on all of $C$. The graph of $r(x)$ is given in Figure 8. There, along the horizontal axis are the values $y \in [0,1]$ whose binary expansion corresponds to points in $C$. In other words, the horizontal axis corresponds to the set $C$, where all the removed open intervals are collapsed. (In such a ‘coordinate system’, the Cantor ternary function (the ‘devil’s staircase’) would look like a straight line from $(0,0)$ to $(1,1)$.)
The function \( r(x) \) constructed above can also be viewed as a limit of a sequence of functions \( r_n(x) \), where \( r_0(x) \) is the devil’s staircase, and \( \forall n \ r_{n+1} \) is obtained by replacing every linear piece \((t_1, r_1) - (t_2, r_2)\) of the graph of \( r_n \) with two pieces, \((t_1, r_1) - (t_2, r_2)\) (see Figure 9).

Yet another way to view the function \( r(x) \) is through the following recursive formula, which will prove useful later. It can be easily checked using formulas (3.26).

**Proposition 3.25.** Let \( x_0 \in C \) have a binary expansion \( 0.q_1q_2\ldots q_n \), then:
a) if \( x = 0.0q_1q_2 \ldots q_n \), then \( r(x) = \frac{1}{4} r(x_0) \),

b) if \( x = 0.1q_1q_2 \ldots q_n \), then \( r(x) = \frac{1}{4} + \frac{3}{4} r(x_0) \).

Let \( \phi \) be the flow defined by return times \( r(x) + 1 \). It can be easily checked that

\[
\int_C r(x) \, d\beta = \frac{1}{4}, \text{ so } \int_C (r(x) + 1) \, d\beta = \frac{5}{4}, \text{ and therefore}
\]

\[
\rho(\phi) = \frac{4}{5}.
\]

**Proposition 3.26.** The function \( a(t, \varepsilon) \) for the flow \( \phi \) constructed above is unbounded.

The graph of \( a(t, \varepsilon) \) is given in Figure 10, where the points on the horizontal axis are the numbers of consecutive returns to the cross-section of the identity \( C_\varepsilon \). It is evident from this graph that \( a(t, \varepsilon) \) evolves in ‘cycles’, where every \( n \)-th cycle lasts from return number \( 2^{n-1} \) to return number \( 2^n \), and at every \( 2^n \)-th return \( a(t, \varepsilon) \) takes the value 0.3125. The maximum values of \( a(t, \varepsilon) \) in the first 10 cycles, along with the numbers of returns that yield those maxima, and the points in \( C^* \) that correspond to such returns, are listed in Table 1. Figure 10 and Table 1 already suggest that \( a(t, \varepsilon) \) is unbounded. We shall now give a rigorous proof of Proposition 3.26.

The following notation will be used in the proof. Denote \( f(x) = \frac{1}{4} - r(x) \), \( y(n) = f(x) \) if \( x \in C \) corresponds to the \( n \)-th return of \( \varepsilon \) to \( C \) under the flow \( \phi \) (we
Figure 10. The graph of $a(t,e)$. 
Table 1. Maximum values of $a(t,e)$ in every cycle.

<table>
<thead>
<tr>
<th>Cycle</th>
<th>Maximum of $a$</th>
<th>Return #</th>
<th>Point in $[0,1]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3125</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.546875</td>
<td>3</td>
<td>0.25</td>
</tr>
<tr>
<td>3</td>
<td>0.722656</td>
<td>7</td>
<td>0.375</td>
</tr>
<tr>
<td>4</td>
<td>0.854492</td>
<td>15</td>
<td>0.4375</td>
</tr>
<tr>
<td>5</td>
<td>1.033936</td>
<td>27</td>
<td>0.34375</td>
</tr>
<tr>
<td>6</td>
<td>1.190491</td>
<td>55</td>
<td>0.421875</td>
</tr>
<tr>
<td>7</td>
<td>1.340866</td>
<td>119</td>
<td>0.429688</td>
</tr>
<tr>
<td>8</td>
<td>1.512585</td>
<td>219</td>
<td>0.355469</td>
</tr>
<tr>
<td>9</td>
<td>1.666436</td>
<td>439</td>
<td>0.427734</td>
</tr>
<tr>
<td></td>
<td></td>
<td>475</td>
<td>0.357422</td>
</tr>
<tr>
<td>10</td>
<td>1.819419</td>
<td>951</td>
<td>0.428711</td>
</tr>
</tbody>
</table>

denote this by $x(n))$. Also denote
\[
\bar{f}(x) = \bar{y}(n) = \sum_{k=0}^{n} y(k),
\]

(3.27)

the accumulated difference in return times between $\phi$ and the linear flow. Since returns occur for some $t_n$, $n = 1, 2, \ldots$, then the following is true:

**Lemma 3.27.** If the accumulated difference $\bar{y}(n)$ is unbounded as $n \to \infty$, then $a(t,e)$ is unbounded.

Thus we only need to show that $\bar{y}(n_k)$ is unbounded for some sequence of returns $n_k \to \infty$.

**Lemma 3.28.** For any two consecutive powers $2^{k-1}$, $2^k$, the following holds:
\[
\forall n \geq 0 \quad \frac{1}{4} + y(2^{k-1} + n) = y(2^k + 2n - 1) + y(2^k + 2n).
\]

(3.28)
PROOF. Assume that \( n < 2^{k-1}, m = k - 1 \), and let the binary expansion
\[
\text{bin}(2^{k-1} + n) = 0.q_1q_2\ldots q_m.
\]
Then \( \text{bin}(2^k + 2n) = 0.0q_1q_2\ldots q_m \). Let \( q_i \) be the first \( q_i = 1 \), then
\[
\begin{align*}
\text{bin}(2^{k-1} + n - 1) &= 0.111\ldots \underbrace{0q_{i+1} \ldots q_m}_{l-1}, \\
\text{so}
\text{bin}(2^k + 2n - 1) &= 0.111\ldots \underbrace{0q_{i+1} \ldots q_m}_{l-1}, \\
\text{bin}(2^k + 2n - 2) &= 0.0111\ldots \underbrace{0q_{i+1} \ldots q_m}_{l-1}.
\end{align*}
\]
Note that \( x(n) = \text{bin}(n - 1) \). So we need to show the following:
\[
\frac{1}{4} + r(0.\underbrace{11\ldots 1}_{l-1}0q_{i+1} \ldots q_m) = r(0.\underbrace{11\ldots 1}_{l}0q_{i+1} \ldots q_m) + r(0.0\underbrace{11\ldots 1}_{l-1}0q_{i+1} \ldots q_m).
\]
Let \( r_0 = r(0.q_{i+1} \ldots q_m) \), then by formulas (3.26):
\[
\begin{align*}
\frac{1}{4} + r(0.\underbrace{11\ldots 1}_{l-1}0q_{i+1} \ldots q_m) &= \frac{1}{4} + 3 \left( \frac{1}{4} + \frac{3}{4} \left( \frac{1}{4} + \ldots \left( \frac{1}{4} + \frac{3}{4} \left( \frac{1}{4} r_0 \right) \right) \right) \right), \\
\text{for } l-1 \text{ times,}
\frac{1}{4} + r(0.\underbrace{11\ldots 1}_{l}0q_{i+1} \ldots q_m) &= \frac{1}{4} + 3 \left( \frac{1}{4} + \frac{3}{4} \left( \frac{1}{4} + \ldots \left( \frac{1}{4} + \frac{3}{4} \left( \frac{1}{4} r_0 \right) \right) \right) \right), \\
\text{for } l \text{ times,}
\frac{1}{4} + r(0.0\underbrace{11\ldots 1}_{l-1}0q_{i+1} \ldots q_m) &= \frac{1}{4} \left( \frac{1}{4} + 3 \left( \frac{1}{4} + \ldots \left( \frac{1}{4} + \frac{3}{4} \left( \frac{1}{4} r_0 \right) \right) \right) \right), \\
\text{for } l-1 \text{ times.}
\end{align*}
\]
Therefore,
\[
\frac{1}{4} + r(0.\underbrace{11\ldots 1}_{l}0q_{i+1} \ldots q_m) + r(0.0\underbrace{11\ldots 1}_{l-1}0q_{i+1} \ldots q_m) = \frac{1}{4} + \frac{3}{4}(\ldots) + \frac{1}{4}(\ldots)
\]
\[
= \frac{1}{4} + (\ldots) = \frac{1}{4} + r(0.\underbrace{11\ldots 1}_{l-1}0q_{i+1} \ldots q_m).
\]
\[
\square
\]
Corollary 3.29. Since $\bar{y}(2) = \bar{y}(4) = 0.25$ and by Lemma 3.28, $\bar{y}(2^k) = 0.25 \ \forall k$.

Corollary 3.30. $\bar{y}(2^k + n) = \bar{y}(2^{k+1} + 2n) \ \forall k, n$.

Since $x(n) = \text{bin}(n - 1)$, then Corollary 3.30 states:

$$\bar{y}(0.q_1q_2 \ldots q_n) = \bar{y}(0.1q_1q_2 \ldots q_n). \quad (3.29)$$

Since $A(0.0q_1q_2 \ldots q_n) = 0.1q_1q_2 \ldots q_n$, i.e. $0.0q_1q_2 \ldots q_n$ is the point of the previous return of $\varepsilon$ to $C$, and by definition of $\bar{y}$, we have the following:

**Proposition 3.31.**

a) $\bar{y}(0.1q_1q_2 \ldots q_n) = \bar{y}(0.q_1q_2 \ldots q_n)$;

b) $\bar{y}(0.0q_1q_2 \ldots q_n) = \bar{y}(0.q_1q_2 \ldots q_n) - \bar{y}(0.1q_1q_2 \ldots q_n)$.

Table 1 suggests that the maximum values of $a(t, \varepsilon)$ in every cycle accumulate at two points in $C^*$. Further computations show that these points are $0.428571 \ldots = \frac{3}{7}$ and $0.357142 \ldots = \frac{5}{14}$. We show unboundedness of $a(t, \varepsilon)$ for a sequence of returns $r_n$ such that $x_n \rightarrow x \in C$ that corresponds to $0.428571 \ldots \in C^*$. This purpose is served by a sequence $r_n$ that corresponds to the following $x_n \in C$:

\[
\begin{align*}
x_0 &= .100000 \ldots \\
x_1 &= .0110000 \ldots \\
x_2 &= .011010000 \ldots \\
&\vdots
\end{align*}
\]
We shall now compute \( \bar{f}(x_n) \), \( n = 0, 1, 2, \ldots \)

\[
\bar{f}(x_0) = f(0.1) + f(0.0) = 0.25, \quad r_0 = 0,
\]

\[
\bar{f}(x_1) = \bar{f}(0.111) - f(0.111) = \bar{f}(0.1) - 0.25 + r(0.111)
\]

\[
= \bar{f}(x_0) - 0.25 + \left( \frac{1}{4} + \frac{3}{4} \left( \frac{1}{4} + \frac{3}{4} \left( \frac{1}{4} + \frac{3}{4} r_0 \right) \right) \right)
\]

\[
= \bar{f}(x_0) - 0.25 + \left( \frac{37}{64} + \frac{27}{64} r_0 \right);
\]

\[
\bar{f}(x_2) = \bar{f}(0.111011) - f(0.111011) = \bar{f}(0.011) - f(0.111011)
\]

\[
= \bar{f}(x_1) - 0.25 + \left( \frac{37}{64} + \frac{27}{64} \tilde{r}_1 \right),
\]

where \( \tilde{r}_1 = r(0.011) = \frac{1}{4} \left( \frac{1}{4} + \frac{3}{4} \left( \frac{1}{4} + \frac{3}{4} r_0 \right) \right) = \frac{7}{64} + \frac{9}{64} r_0 \). Continuing this process, for any \( n \) we obtain

\[
\bar{f}(x_n) = \bar{f}(x_{n-1}) - 0.25 + \left( \frac{37}{64} + \frac{27}{64} \tilde{r}_{n-1} \right) = \bar{f}(x_{n-1}) + \frac{21}{64} + \frac{27}{64} \tilde{r}_{n-1}.
\]

But \( \tilde{r}_2 = \frac{7}{64} + \frac{9}{64} \tilde{r}_1 = \frac{7}{64} + \frac{9 \cdot 7}{64^2} \); \( \tilde{r}_3 = \frac{7}{64} + \frac{9 \cdot 7}{64^2} + \frac{9^2 \cdot 7}{64^3} \); \ldots, so

\[
\bar{f}(x_n) = \bar{f}(x_{n-1}) + \frac{21}{64} + \frac{27}{64^2} \left( \frac{7}{64} + \frac{9 \cdot 7}{64^2} + \frac{9^2 \cdot 7}{64^3} + \cdots + \frac{9^{n-2} \cdot 7}{64^{n-1}} \right)
\]

\[
= \bar{f}(x_{n-1}) + \frac{21}{64} + \frac{27 \cdot 7}{64^2} \cdot \frac{(9^{n-1}/64^{n-1}) - 1}{(9/64) - 1},
\]

therefore,

\[
\bar{f}(x_n) = \frac{1}{4} + \sum_{i=1}^{n} \left( \frac{21}{64} + \frac{27 \cdot 7}{64^2} \cdot \frac{1 - (9/64)^{n-1}}{55} \right). \tag{3.30}
\]

The series (3.30) diverges, since its common term does not tend to zero, which means that \( \bar{f}(x_n) \rightarrow \infty \). So if \( t_n = r_n \quad \forall n \), then \( a(t_n, \varepsilon) \rightarrow \infty \).
In this section, we shall discuss what can be said about rotation numbers of fixed-point free flows on $S_2$ that are topologically conjugate, and conversely, whether two flows that have the same rotation number are topologically conjugate. First, we investigate what happens on the dyadic solenoid $S_2$.

**Theorem 3.32.** Let $\phi, \psi$ be two continuous positively oriented flows on $S_2$, that are topologically conjugate:

$$
\psi^t(z) = g \circ \phi^t \circ g^{-1}(z) \quad \forall z \in S_2, \ t \in \mathbb{R}.
$$

Then

$$
\rho(\psi) = 2^k \rho(\phi),
$$

for some integer $k$.

**Proof.** Step 1: Suppose first that $g : S_2 \to S_2$ is a homeomorphism isotopic to the identity, i.e. $g(z) = z \ast \pi_P(\xi(z))$. Since $g$ is continuous, then so is $\xi : S_2 \to \mathbb{R}$. $S_2$ is compact, so $\xi$ is bounded. Also, if $\theta = g(z)$, then $z = g^{-1}(\theta) = \theta \ast \pi_P(\xi(z))$.

For convenience, we shall denote $f_t^t(z) = d_{arc}(z, \phi_t^t(z))$, that is,

$$
\phi_t^t(z) = z \ast \pi_P(f_t^t(z)).
$$
Thus we have

\[ \psi^t(z) = g\phi^t g^{-1}(z) = g\phi^t (z * \pi_p(\zeta)) \quad \text{(some } \zeta) \]

\[ = g(z * \pi_p(\zeta) * \pi_p(f_t(z * \pi_p(\zeta)))) \]

\[ = z * \pi_p(\zeta) * \pi_p(f_t(z * \pi_p(\zeta))) * \pi_p(\tilde{\zeta}) \quad \text{(some } \tilde{\zeta}) \]

\[ = z * \pi_p \left( \zeta + f_t(z * \pi_p(\zeta)) + \tilde{\zeta} \right). \]

Now since both \( \rho(\phi), \rho(\psi) \) exist and do not depend on \( z \), and both \( \zeta, \tilde{\zeta} \) are bounded,

\[ \rho(\psi) = \lim_{t \to \infty} \frac{1}{d_{arc}(\psi^t(z), z)} = \lim_{t \to \infty} \frac{1}{t} \left( \zeta + f_t(z * \pi_p(\zeta)) + \tilde{\zeta} \right) \]

\[ = \lim_{t \to \infty} \frac{f_t(z * \pi_p(\zeta))}{t} = \lim_{t \to \infty} \frac{f_t(z)}{t} = \lim_{t \to \infty} \frac{1}{d_{arc}(\phi^t(z), z)} \]

\[ = \rho(\phi). \]

Thus if \( g \) is homotopic to the identity, then \( \rho(\psi) = \rho(\phi) \).

**Step 2:** Now suppose that \( g \) is a shift homeomorphism.

If \( g = s_R \) (the right shift by one coordinate), that is, \( g(z_1, z_2, \ldots) = (z_1^2, z_1, z_2, \ldots) \), then \( g^{-1} = s_L \) (the left shift by one coordinate), that is, \( g(z_1, z_2, \ldots) = (z_2, z_3, \ldots) \), and we have

\[ \psi^t(z) = g\phi^t(z_2, z_3, \ldots) \]

\[ = g \left( z_2 e^{2\pi i f_t(z_2, z_3, \ldots)}, z_3 e^{2\pi i f_t(z_2, z_3, \ldots)}, \ldots \right) \]

\[ = (z_2 e^{4\pi i f_t(z_2, z_3, \ldots)}, z_3 e^{2\pi i f_t(z_2, z_3, \ldots)}, \ldots) \]

\[ = (z_1 e^{4\pi i f_t(z_2, z_3, \ldots)}, z_2 e^{2\pi i f_t(z_2, z_3, \ldots)}, \ldots) \]

\[ = z * \pi_p \left( 2f_t(z_2, z_3, \ldots) \right). \]
So \( d_{arc}(\psi^t(z), z) = 2f_t(z_2, z_3, \ldots) \), and

\[
\rho(\psi) = \lim_{t \to \infty} \frac{1}{t} d_{arc}(\psi^t(z), z) = \lim_{t \to \infty} \frac{2f_t(z_2, z_3, \ldots)}{t} = 2 \lim_{t \to \infty} \frac{f_t(z)}{t} = 2 \rho(\phi),
\]

since the limit does not depend on \( z \).

Thus if \( g = s_R \), then

\[
\rho(\psi) = 2 \rho(\phi). \tag{3.32}
\]

Clearly, if \( g = s_L \), then \( g^{-1} = s_R \), and \( \rho(\psi) = \frac{1}{2} \rho(\phi) \).

Now if \( g = s_R^k \) is the right shift by \( k \) coordinates, then

\[
\psi = g \circ \phi \circ g^{-1} = s_R^k \circ \phi \circ (s_R^{-1})^k = s_R \circ \cdots \circ s_R \circ \phi \circ s_R^{-1} \circ \cdots \circ s_R^{-1}.
\]

Denote

\[
\psi(1) = s_R \circ \phi \circ s_R^{-1};
\]

\[
\psi(2) = s_R^2 \circ \phi \circ (s_R^{-1})^2 = s_R \circ \psi(1) \circ s_R^{-1};
\]

\[
\psi = \psi(k) = s_R^k \circ \phi \circ (s_R^{-1})^k = s_R \circ \psi(k-1) \circ s_R^{-1}.
\]

...
Then by (3.32):

\[ \rho(\psi(1)) = 2 \rho(\phi); \]
\[ \rho(\psi(1)) = 2 \rho(\psi(1)) = 4 \rho(\phi); \]
\[ \ldots \]
\[ \rho(\psi) \equiv \rho(\psi(k)) = 2 \rho(\psi(k-1)) = \cdots = 2^k \rho(\phi). \]

Thus if \( g = s^k_R \), then

\[ \rho(\psi) = 2^k \rho(\phi). \] (3.33)

Clearly, if \( g = s^k_T \), then \( \rho(\phi) = 2^k \rho(\psi) \), so \( \rho(\psi) = 2^{-k} \rho(\phi) \).

Step 3: Suppose now that \( g = t_w \) is a translation by some \( w \in C_\xi \), that is, \( g(z) = z * w \). Then \( g^{-1} = t_{w^{-1}} \), where \( w^{-1} \) is defined from \( w * w^{-1} = \xi \), that is, \( w^{-1} = (w_1^{-1}, w_2^{-1}, \ldots) \). So

\[ \psi^t(z) = g \phi^t g^{-1}(z) = g \phi^t(z_1 w_1^{-1}, z_2 w_2^{-1}, \ldots) \]
\[ = g \left( z_1 w_1^{-1} e^{2\pi i f_t(z_2 w_1^{-1})}, z_2 w_2^{-1} e^{2\pi i f_t(z_2 w_2^{-1})}, \ldots \right) \]
\[ = \left( z_1 w_1^{-1} w_1 e^{2\pi i f_t(z_2 w_1^{-1})}, z_2 w_2^{-1} w_2 e^{2\pi i f_t(z_2 w_2^{-1})}, \ldots \right) \]
\[ = z * \pi \left( f_t(z * w^{-1}) \right). \]

Thus \( d(\psi^t(z), \bar{z}) = f_t(z * w^{-1}) \), and we have:

\[ \rho(\psi) = \lim_{t \to \infty} \frac{1}{t} d_{arc}(\psi^t(z), \bar{z}) = \lim_{t \to \infty} \frac{f_t(z * w^{-1})}{t} \]
\[ = \lim_{t \to \infty} \frac{f_t(z)}{t} = \rho(\phi). \]
Step 4: Finally, if $g : S_2 \to S_2$ is any homeomorphism, then by Theorem 2.41 $g = r \circ t \circ s$, and by Steps 1-3 $\rho(\psi) = 2^k \rho(\phi)$ for some $k \in \mathbb{Z}$. 

Let us now consider flows on $S^p$. If $g = g_a : z \mapsto z^a$, where $a$ is $\mathcal{P}$-recurrent, then similar to Step 2 of Theorem 3.32 we show that $\rho(\psi) = a \rho(\phi)$. Also, if $g = g_{1/b} = (g_b)^{-1}$, then $\rho(\psi) = \frac{1}{b} \rho(\phi)$, and if $g = g_{a/b} = g_a \circ g_{1/b}$, then $\rho(\psi) = \frac{a}{b} \rho(\phi)$.

We note that by Remark 2.40, this formula is the most general for power homeomorphisms. Therefore, using Theorem 2.42, similarly to the above one proves the following statement:

**Theorem 3.33.** Let $\phi, \psi$ be two continuous positively oriented flows on $S^p$, and let $g : S^p \to S^p$ be a topological conjugacy between them, so that $\psi = g \circ \phi \circ g^{-1}$. Then

$$\rho(\psi) = \frac{a}{b} \rho(\phi),$$

(3.34)

where each of $a, b$ is either 1 or a $\mathcal{P}$-recurrent natural number.

**Remark 3.34.** The converse of Theorems 3.32, 3.33 is not true. If $\rho(\phi) = \rho(\psi)$, then $\phi, \psi$ are not necessarily topologically conjugate.

To see that, let $\phi$ be a linear flow on $S^2$ with 'speed' $\frac{4}{5}$:

$$\phi^t(z) = z \star \pi_\mathcal{P}\left(\frac{4}{5} t\right), \quad f_t^{(\phi)} = \frac{4}{5} t \quad \forall z,$$

(3.35)

and let $\psi$ be the flow from Example 3.3:

$$\psi^t(z) = z \star \pi_\mathcal{P}(f_t^{(\psi)}(z)).$$

(3.36)
We know that $\rho(\phi) = \rho(\psi) = \frac{4}{5}$. So if these flows are conjugate by $g : S_2 \to S_2$, then $g = \tau \circ t_{\omega}$. Then it follows from Steps 1, 3 of the proof of Theorem 3.32 that

$$\psi^t(z) = z * \pi_p \left( \zeta + f_t^{(\phi)}(z * W^{-1} * \pi_p(\zeta)) + \tilde{\zeta} \right), \quad (3.37)$$

where $\zeta, \tilde{\zeta}$ are bounded. We note that $f_t^{(\phi)}(z) = \frac{4}{5} t \quad \forall \ z$, so

$$\psi^t(z) = z * \pi_p \left( \zeta + \frac{4}{5} t + \tilde{\zeta} \right).$$

Now consider the difference between the iterates of the same point under these two flows:

$$d_{arc}(\psi^t(z), \phi^t(z)) \equiv a(t, z) = \left( \zeta + \frac{4}{5} t + \tilde{\zeta} \right) - \frac{4}{5} t = \zeta + \tilde{\zeta}.$$

It is bounded for every $z$ since $\zeta, \tilde{\zeta}$ are bounded. However, it has been shown in Proposition 3.26 that $a(t, z)$ is unbounded for some $z$, which is a contradiction. Hence, $\phi, \psi$ are not conjugate.

The question remains whether two flows with the same rotation number, for which the difference between iterates of the same point under those two flows is uniformly bounded, are necessarily topologically conjugate.
CHAPTER 4

ERGODICITY AND ROTATION OF FLOWS WITH FIXED POINTS

Flows with Fixed Points on $S_P$ versus Irrational Toral Flows

In this chapter, we study differentiable flows on solenoids, that have fixed points. Throughout the chapter, flow $\phi : \mathbb{R} \times S_P \to S_P$ is assumed to satisfy an initial-value problem

$$\frac{d}{dt} \phi^t(z) = \Phi(\phi^t(z)) \quad (4.1)$$

$$\phi^0(z) = z \quad (4.2)$$

where $\Phi : S_P \to \mathbb{R}_+ \cup \{0\}$ is a function that has a lift $\bar{\Phi}$.

**Remark 4.1.** For such $\phi$, formula (3.3) can be rewritten as

$$\rho(z, \phi) = \left\{ \text{limit points of } \frac{1}{T} \int_0^T \bar{\Phi}(\phi^t(\tau, x)) \, d\tau, \text{ as } t \to \infty \right\}. \quad (4.3)$$

While proving results about flows on $S_P$ with fixed points, we shall exploit certain similarity between solenoidal flows and irrational toral flows. Both $S_P$ and an irrational foliation of $T^2$ can be viewed as suspensions of some homeomorphism in the base space, which is an adding machine on the Cantor set $C$ in the first case, and an irrational rotation on the circle $S^1$ in the second case. As a consequence, the special
representations of flows (see [18]) on $S_p$ and irrational flows on $T^2$ possess similar characteristics.

For this reason, some proofs contained in this chapter follow the lines of corresponding proofs for $T^2$ that have already appeared in literature. Other statements, such as criteria for existence of non-unique ergodic measures for flows with one fixed point, are first proved for irrational flows on $T^2$, and then reformulated for flows on $S_p$, with an indication of the changes that should be made in the proofs.

We believe that this approach highlights the connection between our results and earlier work on toral flows, and reinforces the similarity of flows on $S_p$ and $T^2$.

On the other hand, there are various differences between solenoidal and toral flows, such as the uniqueness of the lift on $S_p$ and the lack of topological conjugacy of flows with the same rotation number. These phenomena have already been considered in Chapter 3.

Ergodicity and Rotation of Flows with One Fixed Point

Irrational Toral Flows

Let $\phi : \mathbb{R} \times S_p \to S_p$ be a flow that satisfies

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \Phi(x, y) \bar{X}$$

(4.4)

where $\Phi : T^2 \to \mathbb{R}_+ \cup \{0\}$ is an $L^1$ nonnegative function, $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is its lift to the universal cover, $\bar{X} = \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$ is a vector field on $T^2$ that has irrational slope, i.e. $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. 
Furthermore, we assume that $\Phi = 0$ at the unique point $p_0 = ((p_0)_x, (p_0)_y) \in \mathbb{T}^2$, that is, $\phi$ has only one fixed point, $p_0$. By $C_{p_0}$ we denote the cross-section of $\mathbb{T}^2$ through the point $p_0$: $C_{p_0} = h \{(x,y) \in [0,1]^2 : x = (p_0)_x, y \in [0,1]\}$. Here, as before, $h : \mathbb{R}^2 \to \mathbb{T}^2$ denotes the covering projection.

The MZ rotation set of $\phi$ is then defined as

$$
\rho(\tilde{\phi}) = \left\{ \text{limit points of } \frac{\tilde{\phi}^{t_i}(x_i) - x_i}{t_i}, \quad \forall t_i \in \mathbb{R}, \; t_i \to \infty, \; x_i \in \mathbb{R}^2 \right\},
$$

where $\tilde{\phi}$ is a lift of $\phi$ to $\mathbb{R}^2$. It is easy to see that $\rho(\tilde{\phi})$ is independent of the lift $\tilde{\phi}$ and is compact. Misiurewicz and Ziemian [52] have proved that $\rho(\tilde{\phi})$ is convex. Moreover, Franks and Misiurewicz obtained the following result:

**Theorem 4.2 (Franks, Misiurewicz).** If $\phi$ is a continuous flow on $\mathbb{T}^2$ and $\tilde{\phi}$ is its lift to $\mathbb{R}^2$, then the rotation set $\rho(\tilde{\phi})$ is one of the following:

(a) A single point $v \in \mathbb{R}^2$,

(b) A segment of a line passing through $(0,0)$ and some point of $\mathbb{Q}^2$, not necessarily containing $(0,0)$,

(c) A line segment from $(0,0)$ to some $v \in \mathbb{R}^2$, having irrational slope.

Clearly, for a flow that satisfies (4.4), only (a) and (c) are possible. Our goal in this section is to obtain conditions on the function $\Phi$, under which the situation (c) takes place.
Proposition 4.3 (Franks, Misiurewicz). If some $v \in \rho(\phi)$ has irrational slope, then there exists at most one $\phi$-invariant ergodic measure that has nonzero rotation vector.

With this proposition in mind, the question of what conditions $\Phi(x, y)$ should satisfy for $\phi$ to have a nontrivial rotation set, is equivalent to the question of when $\phi$ has more than one (in fact, exactly two) ergodic measures.

To that end, suppose that $\mu$ is an ergodic $\phi$-invariant measure on $\mathbb{T}^2$. For $p \in \mathbb{T}^2$, consider the rotation vector of $p$ under $\phi$:

$$\rho(p) = \lim_{T \to \infty} \frac{\phi^T(p) - p}{T} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{d}{dt} \tilde{\phi}^t(p) \, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T \tilde{\Phi}(\tilde{\phi}^t(p)) \, \vec{X} \, dt$$

$$= \left( \lim_{T \to \infty} \frac{1}{T} \int_0^T \tilde{\Phi}(\tilde{\phi}^t(p)) \, dt \right) \vec{X}$$

$$= \left( \int_{\mathbb{T}^2} \tilde{\Phi} \, d\mu \right) \vec{X},$$

for $\mu$-almost all $p \in \mathbb{T}^2$, by the Ergodic Theorem. Measure $\mu$ naturally induces a measure on the fundamental domain $\Omega = [0, 1]^2$ in $\mathbb{R}^2$, which we shall also denote by $\mu$: $\mu(B) = \mu(h(B))$, for $B \subset \Omega$, $B \in B(\mathbb{R}^2)$.

Theorem 4.4. If there exists a $\phi$-invariant ergodic measure $\mu$ that has a nonzero rotation vector, then it is absolutely continuous with respect to the Lebesgue measure.
Moreover, $\mu$ is given by

$$\mu(B) = \frac{\int_B (1/\Phi) \, d\lambda}{\int_\Omega (1/\Phi) \, d\lambda}$$  \hspace{1cm} (4.7)

**Proof.** Let $S$ denote the set of $x \in \mathbb{R}^2$ such that the limits

$$\lim_{t \to \infty} \frac{\phi^t(x) - x}{t} \quad \text{and} \quad \lim_{t \to -\infty} \frac{\phi^t(x) - x}{t}$$

exist, are equal and nonzero. Franks and Misiurewicz [29] show that flow $\phi$, restricted to the set $S$, has a special representation, i.e. is metrically isomorphic to a special flow $\phi_Y$ on

$$Y = \{(y, s) : y \in C_0 \cap S, \quad 0 \leq s \leq r(y)\},$$

where $r(y)$ is the return time to the cross-section $C_0$.

In the base space $C_0 \cap S$, $\phi_Y$ amounts to a map with irrational rotation number. In our assumptions, since $\phi$ has a constant irrational slope, the map in the base space is a rigid rotation.

Denote $\hat{Y} = \{(y, s) : y \in C_0, \quad 0 \leq s \leq r(y)\}$, the space for a special flow for $\mathbb{T}^2$. Then $Y = \hat{Y} \cap g(S)$, $g$ being the special representation isomorphism (see Figure 11).

Now $\phi_Y$ is a direct product of dynamical systems on the base space and on the fibers (see [18]), so the invariant measure $\mu_Y$ is a direct product of the invariant measure on the base and Lebesgue measure on the fibers. Since $g$ is a special representation, measure $\mu$ on $\mathbb{T}^2$ and $\mu_Y$ are related by the following equalities:

$$\mu_Y(U) = \mu(g^{-1}(U)), \quad \mu(B) = \mu_Y(g(B)) \hspace{1cm} (4.8)$$
We shall prove formula (4.7), from which the absolute continuity of $\mu$ with respect to $\lambda$ will immediately follow.

The Lebesgue measure on $\mathbb{T}^2$ is defined by means of the universal cover: if $B \in \mathcal{B}(\mathbb{T}^2)$ and $\tilde{B} \subset \Omega$ is its lift, then $\lambda(B) := \lambda(\tilde{B})$, and also, $\lambda$ on $\Omega$ is a direct product of the Lebesgue measures in the $x$- and $y$-direction. That is, if $\tilde{B} = \{(x, y) : y \in A \subset [0, 1], \ x \in I(y) \subset [0, 1]\}$, then

$$
\lambda(\tilde{B}) = \int_A \lambda(I(y)) \, dy. \quad (4.9)
$$

We want to represent $\lambda$ on $\Omega$ as a direct product of Lebesgue measures in the $y$-direction and in the direction of $\tilde{X}$.

Suppose that

$$
\tilde{B} = \{(y, s) : y \in A \subset [0, 1], \ s_1(y) \leq s \leq s_2(y)\},
$$
where $s_i(y) \in [0,1] \forall y, \ i = 1, 2. \ \sqrt{1+\alpha^2}$ is the length of the portion of $\bar{X}$ within $\Omega$. Then $s_2(y) - s_1(y) = \sqrt{1+\alpha^2} (x_2(y) - x_1(y))$, so

$$\lambda(\bar{B}) = \frac{1}{\sqrt{1+\alpha^2}} \int_A (s_2(y) - s_1(y)) \, dy.$$ 

In the general case, if (see Figure 11)

$$\tilde{B} = \{(y,s) : y \in A \subset [0,1], \ s \in J(y) \subset [0,1] \ \forall y\},$$

then

$$\lambda(B) = \lambda(\tilde{B}) = \frac{1}{\sqrt{1+\alpha^2}} \int_A \lambda(J(y)) \, dy, \quad (4.10)$$

setting $J(y) = \emptyset$ whenever $y \notin A$.

Now consider $g(B) \subset \tilde{Y}$. Every point $y \in A$ is mapped one-to-one to some point $\tilde{y} \in C_0 = [0,1]$, and $J(y)$ is mapped to some $\tilde{J}(y)$. In the simple case when $J(y) = [s_1(y), s_2(y)]$, we get $\tilde{J}(y) = [\tilde{s}_1(y), \tilde{s}_2(y)]$, where $\tilde{s}_1, \tilde{s}_2$ are determined from the following equations:

$$\phi^{\tilde{s}_1}(y,0) = (y, s_1(y)),$$
$$\phi^{\tilde{s}_2}(y,0) = (y, s_2(y)),$$

and in general, $\tilde{s} \in \tilde{J}(y)$ if and only if $\phi^{\tilde{s}}(y,0) = (y, s)$ for some $s \in J(y)$. Since in this case

$$\tilde{s} = \int_0^s \frac{1}{\Phi(\phi^s(y,0))} \, dt,$$

then

$$\tilde{s}_2(y) - \tilde{s}_1(y) = \int_{s_1(y)}^{s_2(y)} \frac{1}{\Phi(\phi^s(y,0))} \, ds.$$
in the simple case. In the more general case, we obtain

\[ \lambda(J(y)) = \int_{J(y)} \frac{1}{\Phi(\phi^t(y, 0))} \, dt. \]  \hspace{1cm} (4.11)

Also,

\[ \mu_X(g(B)) = \int_{\hat{A}} \lambda(J(y)) \, dy = \int_0^1 \int_{J(y)} \frac{1}{\Phi} \, dt \, dy = \int_B \frac{1}{\Phi} \, d\lambda, \]  \hspace{1cm} (4.12)

where again, \( \hat{J}(y) = \emptyset \) whenever \( \hat{y} \notin \hat{A} \). Therefore,

\[ \mu(\check{B}) = \frac{1}{M} \int_{\check{B}} \frac{1}{\Phi} \, d\lambda \quad \forall \check{B} \in B(\Omega), \]  \hspace{1cm} (4.13)

where \( M = \int_{\Omega} \frac{1}{\Phi} \, d\lambda \). Thus formula (4.7) is verified for \( \mu \) restricted to \( S \), and as a consequence, \( \mu \ll \lambda \).

Now we extend \( \mu \) to all of \( T^2 \) by setting \( \mu = 0 \) on all flow lines in \( T^2 \setminus S \), then we still have \( \mu \ll \lambda \). \( \square \)

**Remark 4.5.** In Theorem 4.4, after we extend \( \mu \) to all of \( T^2 \), equation (4.13) might not hold (even though \( \mu \) is still absolutely continuous with respect to \( \lambda \)), unless \( T^2 \setminus S \) consists of at most countably many \( \vec{X} \)-flow lines, in which case equation (4.13) will hold on all of \( T^2 \). As we shall see, this is the case if \( \phi \) has one fixed point, or at most countably many fixed points.

We now turn to the question of the conditions that \( \Phi \) should satisfy in order for \( \phi \) to have measure \( \mu \) given by (4.7).
Let \( r(y) \) denote the return time of \( y \in C_0 \) to \( C_0 \), \( R(y) = y + \theta \) the Poincaré map on \( C_0 (\theta \in \mathbb{R} \setminus \mathbb{Q}) \). Let also \( \gamma_y \) denote the trajectory of \( y \), for \( 0 \leq t \leq r(y) \). The time to travel one unit of length is \( \frac{1}{\Phi(y)} \), then

\[
\Phi(y) = \int_{\gamma_y} \frac{1}{\Phi} \, ds = \sqrt{1 + \alpha^2} \int_0^1 \frac{1}{\Phi} \, dx.
\] (4.14)

If \( y \in C_0 \cap S \), then \( r(R^k(y)) < \infty \) \( \forall k \), and \( \lim_{T \to \infty} \frac{\phi^T(y) - y}{T} \) exists. So to compute \( \rho(y) \), we take a subsequence of return times to \( C_0 \):

\[
v(y) = \left[ \lim_{n \to \infty} \frac{n\sqrt{1 + \alpha^2}}{r(y) + r(R(y)) + r(R^2(y)) + \cdots + r(R^{n-1}(y))} \right] X =: \rho(y) \bar{X},
\]

so by the Ergodic Theorem, since \( R(y) \) is uniquely ergodic,

\[
\frac{1}{\rho(y)} = \frac{1}{\sqrt{1 + \alpha^2}} \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} r(R^k(y)) = \frac{1}{\sqrt{1 + \alpha^2}} \int_0^1 r(y) \, dy
\] (4.15)

everywhere and uniformly on \( S \). By (4.14), we obtain

\[
\frac{1}{\rho(y)} = \frac{1}{\sqrt{1 + \alpha^2}} \int_0^1 \int_0^1 \frac{1}{\Phi} \, dx \, dy = \frac{1}{\sqrt{1 + \alpha^2}} \iint_\Omega \frac{1}{\Phi} \, dx \, dy,
\] (4.16)

which holds even if \( \frac{1}{\Phi} \) is not \( L^1(\Omega) \).

**Theorem 4.6.** Let \( \phi \) be a flow on \( \mathbb{T}^2 \) with one fixed point \( p_0 \).

1. If \( \frac{1}{\Phi} \) is \( L^1(\Omega) \), then \( \phi \) has an ergodic \( \phi \)-invariant measure \( \mu \ll \lambda \), given by (4.7), such that \( \rho(y) = \rho > 0 \) for \( \mu \)-almost all \( y \in \mathbb{T}^2 \). Furthermore,

\[
\rho = \frac{\sqrt{1 + \alpha^2}}{\int_\Omega (1/\Phi) \, dx \, dy} \quad \text{and} \quad \rho(\phi) = [0, \rho] \bar{X}.
\] (4.17)
2. If $\frac{1}{\Phi}$ is not $L^1(\Omega)$, then $\phi$ is uniquely ergodic with the Dirac measure $\delta_{p_0}$, and $\rho(\phi) = \{0\}$.

**Proof.** If $\frac{1}{\Phi}$ is $L^1(\Omega)$, then by (4.16) $\rho(y) = \rho > 0$ on $S$. By Theorem 4.4 there is an ergodic measure $\mu$, given by (4.7). Then, by (4.6),

$$v(p) = \rho(p)1 = \left(\int_{\Omega} \Phi \, d\mu\right) \sqrt{1 + \alpha^2},$$

(4.18)

for $\mu$-almost all $p \in \mathbb{T}^2$. Also, by construction of $\mu$, this $\rho(p) > 0$, so the set of points, for which (4.18) is satisfied, is contained in $S$. But $\rho(p) = \rho$ on $S$, so $\rho$ is realized by the ergodic measure $\mu$. Therefore, it is an extreme point of the rotation set (the other extreme point is 0).

To prove formula (4.17), note that

$$\frac{d\mu}{d\lambda} = \frac{1}{\Phi} \frac{1}{\int_{\Omega} \frac{1}{\Phi}} \, d\lambda,$$

so

$$\rho = \sqrt{1 + \alpha^2} \int_{\Omega} \Phi \, d\mu = \sqrt{1 + \alpha^2} \int_{\Omega} \Phi \frac{1}{\Phi} \frac{1}{\int_{\Omega} \frac{1}{\Phi}} \, d\lambda$$

$$= \sqrt{1 + \alpha^2} \int_{\Omega} \frac{d\lambda}{(1/\Phi)} = \frac{\sqrt{1 + \alpha^2}}{\int_{\Omega} (1/\Phi) \, d\lambda}$$

$$= \frac{\sqrt{1 + \alpha^2}}{\iint_{\Omega} (1/\Phi) \, dx \, dy}.$$

If $\frac{1}{\Phi}$ is not $L^1(\Omega)$, then $\rho(p) = 0$ for all $p \in S$. We will now show that $\rho(p) = 0$ on $\mathbb{T}^2 \setminus S$ as well.
Technically, by construction of $S$, $\forall p \in T^2 \setminus S$ the pointwise rotation set $\rho(\phi, p) = \{0\}$ or $\rho(\phi, p) = A \subset \mathbb{R}_+ \cup \{0\}$. If $A \neq \{0\}$ for some $p$, denote $\rho_0 = \sup\{\rho : \rho \in \rho(\phi, p), p \in T^2 \setminus S\}$. Then it is possible to find a sequence $(p_n, t_n)$, so that

$$\frac{\bar{\phi}^{t_n}(p_n) - p_n}{t_n} \to \rho_0 X, \quad \text{i.e.} \quad \rho(\phi) = [0, \rho_0].$$

But by Theorem 3.15, extreme points of $\rho(\phi)$ are realized by ergodic measures. In particular, $\exists p \in T^2$ with $\rho(\phi, p) = \{p_0\}$. This is a contradiction. Thus $A = \{0\}$ for every $p \in T^2 \setminus S$, and hence on all of $T^2$.

It only remains to be shown that $\phi$ is uniquely ergodic. Suppose that there is another ergodic $\phi$-invariant measure $\mu \neq \delta_{p_0}$, then $\mu$ and $\delta_{p_0}$ are mutually singular. By the Ergodic Theorem, $\lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi(\phi^t(p)) \, dt \equiv \rho(p, \phi)$ exists for $\mu$-almost all $p$, and

$$\int_{T^2} \rho(p) \, d\mu = \int_{T^2} \Phi \, d\mu. \quad (4.19)$$

Since $\Phi = 0$ only at $p_0$ and $\mu \perp \delta_{p_0}$, then $\int_{T^2} \Phi \, d\mu > 0$, and so $\int_{T^2} \rho(p) \, d\mu > 0$, that is, $\rho(p) > 0$ for some $p \in T^2$, which is a contradiction. Hence, $\phi$ is uniquely ergodic. □

**Remark 4.7.** Note that $\frac{1}{\Phi}$ is $L^1(\Omega)$ if and only if $\frac{1}{\Phi}$ is $L^1(T^2)$, and

$$\int\int_{\Omega} (1 / \Phi) \, dx \, dy = \int_{T^2} (1 / \Phi) \, d\lambda,$$

so Theorem 4.6 could be formulated using $\frac{1}{\Phi}$. 

Here we perform the same analysis as in the previous subsection, but for flows on $S_P$. Throughout this subsection, we assume that $\phi$ is a flow on $S_P$ that satisfies (4.1).

**Proposition 4.8.** There exists at most one ergodic $\phi$-invariant measure that has nonzero rotation number.

**Proof.** We follow the idea of Franks and Misiurewicz [29], who proved the same result (see Proposition 4.3) for toral flows.

Suppose that such measure $\mu$ exists. We need to show that it is unique. As in Theorem 4.4, denote $S = \{z \in S_P : \rho(z, \phi) > 0\}$. Let $\tilde{S} = h^{-1}(S)$, and $C_S = \{x \in C : (0, x) \in \tilde{S}\}$. The return map $T$ to a cross-section is the adding machine, with the property that $T^k(k, x) = (0, A^kx)$. Every $(\tau, x) \in \tilde{S}$ can be uniquely represented as $\tilde{\Psi}^t(y)$, where $y \in C_S$, $A^{-k}y = x$ for some $k$ such that $k \leq \tau < k + 1$, and $\tilde{\Psi}^t(y) = \tilde{\phi}^t(T^{-k}(0, y))$, for some $0 \leq t < r(y)$. Here, as before, $r(y)$ is the return time of $y$ to $C_S$. Since $y \in C_S$, then $r(y) < \infty$, and by continuity of the flow, $r(y) \neq 0$. (Passing to $S_P$, we would get the following: if $z = h(\tau, x) \in S$, then $z = \phi^t(z_0)$, for some $z_0 \in C_z \cap S$, $0 \leq t < r(z_0)$.)

Now construct a map $g : \tilde{S} \to Y = \{(y, t) : y \in C_S, 0 \leq t < r(y)\}$, which is a special representation of flow $\phi$. In other words, $\phi$ is metrically isomorphic to the special flow $\phi_Y$ on $Y$ (see Figure 12), so every $\phi$-invariant probability measure $\mu$ on $S$...
corresponds to a $\phi_Y$-invariant measure $\mu_Y$ on $Y$, which is defined by equalities (4.8). But $\phi_Y$ is a direct product of dynamical systems on the base space $C_S$ and on the fibers (see [18]), so $\mu_Y$ is a product of a measure $\nu$ on $C_S$, and the Lebesgue measure on the fibers. This gives us a finite measure $\nu$ on $C_S$, invariant with respect to the adding machine. Also, $\mu$ is determined by $\nu$ and $r(y)$, through equalities (4.8). So if $\mu_1 \neq \mu_2$, then $\nu_1 \neq \nu_2$. If $\mu_1, \mu_2$ are ergodic, then $\int_{C_S} r(y) \, d\nu_i = \mu_i(S) = 1, \; i = 1, 2$.

Then $\frac{\nu_i}{\nu_i(S)}$, $i = 1, 2$ are distinct.

However, if $\nu$ is a normalized invariant measure on $C_S$, then it can be extended to an $A$-invariant measure $\hat{\nu}$ on $C$ by setting $\hat{\nu}(C \setminus C_S) = 0$. If $\nu_1 \neq \nu_2$, then $\hat{\nu}_1 \neq \hat{\nu}_2$. But an invariant probability measure on $C$ is unique. Thus $\nu$ is unique, and so $\mu$ is unique.

\textbf{Theorem 4.9.} If an ergodic $\phi$-invariant measure $\mu$ with nonzero rotation number exists, then it is absolutely continuous with respect to the Haar measure $\lambda$. Moreover,
it is given by

\[ \mu(B) = \frac{\int_B (1/\bar{\Phi}) \, d\lambda}{\int_{\Omega} (1/\bar{\Phi}) \, d\lambda}, \]  

(4.20)

where \( \Omega \) is the fundamental domain in \( \mathbb{R} \times C \).

**Proof.** The proof is identical to that of Theorem 4.4, except that the base space of the special flow \( C_0 \) is the Cantor set \( C_\varepsilon \), the invariant measure in it is the Bernoulli measure, and the map in the base space in the adding machine, which is uniquely ergodic by Theorem 3.18. The measure \( \lambda \) in the covering space \( \mathbb{R} \times C \) is the product of the Bernoulli measure \( \beta \) in \( C \) and the Lebesgue measure on the arc components \( \mathbb{R} \times \{x\}, \ x \in C \).

By the same argument as in Theorem 4.6, we obtain the following two theorems, where by \( L^1(\Omega) \) we mean \( L^1(\Omega) \) with respect to \( \lambda \).

**Theorem 4.10.** Let \( \phi \) be a flow with one fixed point \( p_0 \), \( \bar{\Phi} \) a lift of \( \Phi \), then:

1. If \( \frac{1}{\bar{\Phi}} \) is \( L^1(\Omega) \), then \( \phi \) is not uniquely ergodic. It has a \( \phi \)-invariant ergodic measure \( \mu \), given by (4.20), that has nonzero rotation number. Such a measure is unique.

2. If \( \frac{1}{\bar{\Phi}} \) is not \( L^1(\Omega) \), then \( \phi \) is uniquely ergodic, with the Dirac measure \( \delta_{p_0} \) being the only \( \phi \)-invariant ergodic probability measure on \( S_\mathcal{F} \).

**Theorem 4.11.** Let \( \phi \) be a flow with fixed points, \( \bar{\Phi} \) a lift of \( \Phi \), then the following is satisfied:
1. If $\frac{1}{F}$ is $L^1(\Omega)$, then

$$\rho(\phi) = [0, \rho], \text{ where } \rho = \frac{1}{\int_\Omega (1/\Phi) \, d\lambda}. \quad (4.21)$$

2. If $\frac{1}{F}$ is not $L^1(\Omega)$, then

$$\rho(\phi) = \{0\}. \quad (4.22)$$

**Remark 4.12.** With trivial adjustments, the same results can be obtained for flows with more than one fixed point.

**Measure-Theoretic Realization of Points in the Rotation Set**

The following proposition shows that if the rotation set of flow $\phi$ is nontrivial, that is, $\rho(\phi) = [0, \rho]$, then every point in the rotation set is realized with some $\phi$-invariant measure.

**Proposition 4.13.** Let $\phi$ be a flow on $\mathbb{T}^2$ whose rotation set is nontrivial, i.e. $\rho(\phi) = [0, \rho]$ for some $\rho > 0$. Then for every $r \in \rho(\phi)$ there exists a $\phi$-invariant measure $\nu$, such that

$$r = \int_{s_p} d_{arc}(\phi^1(z), Z) \, d\nu. \quad (4.23)$$

**Proof.** Let $\mu$ be the ergodic measure with $\rho = \int_{s_p} d_{arc}(\phi^1(z), Z) \, d\mu$, $r = \alpha \rho$, $\alpha \in (0, 1)$. Take

$$\nu = \alpha \mu + (1 - \alpha) \delta_{z_0},$$
then
\[ r = \int \alpha \left( \phi^1(z), z \right) d\nu. \]

Pointwise Rotation Sets and Katok's Example

What can be said about the pointwise rotation set \( \rho_p(\phi) \) if the rotation set \( \rho(\phi) = [0, \rho] \)? In other words, which points in the rotation set are realized as rotation averages of some flow lines? We give a partial answer to this question by adapting the construction of Katok's Example for the flows on solenoids.

Katok's Example, briefly sketched in [35] and [42], is an example of an irrational toral flow that has flow lines with rotation number equal to any given number \( r \) inside the rotation set. In other words, if \( \phi \) is a flow on \( \mathbb{T}^2 \) that satisfies (4.4), \( \Phi(p_0) = 0 \) for a unique point \( p_0 \in \mathbb{T}^2 \), and \( \frac{1}{\Phi} \) is \( L^1(\mathbb{T}^2) \) (thus \( \rho(\phi) = [0, \rho] \)), then Katok's Example describes the process of adjusting return times to the cross-section through the fixed point, \( C_{p_0} \), so that the resulting flow has flow lines with \( \rho(p) = a \) for any given \( 0 \leq a \leq \rho \).

For completeness, we give a detailed description of Katok's Example for toral flows (since we could not locate such a detailed treatment of this example in literature). In our exposition, we follow the outline of the construction given in [71].

Let \( \phi \) be a flow on \( \mathbb{T}^2 \) that satisfies (4.4), \( \Phi(p_0) = 0 \) for a unique point \( p_0 \in \mathbb{T}^2 \), and \( \frac{1}{\Phi} \) is \( L^1(\mathbb{T}^2) \). Then by Theorem 4.6, \( \rho(\phi) = [0, \rho]X \). Furthermore, by Theorem
4.4, \( \phi \) has an ergodic measure \( \mu \) such that \( \rho(p) = \rho \mu \)-almost everywhere on \( \mathbb{T}^2 \), where

\[
\rho = \frac{\sqrt{1 + \alpha^2}}{\int_{\mathbb{T}^2} (1 / \Phi) \, d\lambda},
\]

and \( \mu << \lambda \).

Let \( C_{p_0} \) denote the cross-section of the torus through the point \( p_0 \), and let \( r(x) \) be the return time of \( x \in C_{p_0} \) to \( C_{p_0} \). The Poincaré map of \( \phi \) in \( C_{p_0} \) is a rotation by \( \alpha \in \mathbb{R} \setminus \mathbb{Q} : \, x_k = x + k\alpha \). By formulas (4.15), (4.16),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} r(s + k\alpha) = \frac{1}{\sqrt{1 + \alpha^2}} \int_{\mathbb{T}^2} \frac{1}{\Phi} \, d\lambda =: A,
\]

for \( \mu \)-almost all flow lines, and hence for \( \lambda \)-almost all \( s \in C_{p_0} \), since \( \mu << \lambda \). Clearly, by (4.24),

\[
\rho = \frac{1}{A}.
\]

We would like to adjust \( r(x) \), \( x \in C_{p_0} \), so that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} r(x_0 + k\alpha) = B > A,
\]

for some \( B \in \mathbb{R}_+ \), \( x_0 \in C_{p_0} \). Then \( \rho(x_0, \phi) = \frac{1}{B} < \rho \). (As above, \( \rho(x, \phi) \) is such that \( v(x, \phi) = \rho(x, \phi) \bar{X} \).) This is done by adding a sequence of tent functions to \( r \), which 'slow down' the flow on the flow line of \( x_0 \).

Choose \( x_0 \in C_{p_0} \), and denote \( x_n = x_0 + n\alpha \), the \( n \)-th return of \( x_0 \) to \( C_{p_0} \). Since \( \alpha \) is irrational, the orbit \( \{x_n\}_{n=0}^{\infty} \) is dense in \( C_{p_0} \), so there is a subsequence \( x_{n_i} \to p_0 \), \( \frac{n_{i+1}}{n_i} \to 1 \) (see Figure 13). Now we construct a sequence

\[
a_n = \begin{cases} 
0 & : n \notin \{n_i\}_{i=1}^{\infty} \\
y_i & : n = n_i \text{ for some } i
\end{cases}
\]
Figure 13. Adjusting return times by tent functions in Katok’s Example.

such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k = c, \tag{4.25}
\]

for a given constant \( c \). For each \( i \), define \( \phi_i \) to be the tent function with height \( y_i \) at \( x_{ni} \) and with integral less than \( \frac{1}{M^i} \), where \( M \gg 1 \). That is,

\[
\text{supp } \phi_i = (x_{ni} - \delta_i, x_{ni} + \delta_i), \quad \text{where } \delta_i \leq \frac{1}{2 \cdot M^i y_i}. \tag{4.26}
\]

For every \( i \), find \( L_i \) such that

\[
\forall l \geq L_i \quad \frac{1}{l} \sum_{k=0}^{l-1} \phi_i(t + k\alpha) \leq \frac{2}{M^i} \quad \forall t.
\]

Note that \( L_i \) does not depend on \( t \), since the sequence in the above formula converges uniformly in \( t \) by the Ergodic Theorem.
Replace $\phi_i$ by $\psi_i$, defined by narrowing the support of $\phi_i$, so that

$$
\psi_i(x_{n_i}) = \phi_i(x_{n_i}), \quad \psi_i(x_n) = 0 \quad \forall n \leq L_i, \, n \neq n_i.
$$

Thus the effect of adding $\psi_i$ to $r$ is completely localized to $x_{n_i}$. Let

$$
\hat{r} = r + \sum_{i=1}^{\infty} \psi_i \tag{4.27}
$$

(see Figure 13). Then $\hat{r}(x) \to \infty$ as $x \to p_0$, and it has no other discontinuities. Also,

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat{r}(x_k) = \lim_{n \to \infty} \left( \frac{1}{n} \sum_{k=0}^{n-1} r(x_k) + \frac{1}{n} \sum_{k=0}^{n-1} \sum_{i} \psi_i(x_k) \right)
$$

$$
= A + \left( c + \sum_{i=1}^{\infty} \int_{C_{p_0}} \psi_i(x) \, dx \right) \leq A + c + \sum_{i=1}^{\infty} \frac{1}{M^i} \tag{4.28}
$$

$$
= A + c + \frac{1}{M - 1} =: B.
$$

Let $\psi$ be the flow on $T^2$ generated by return times $\hat{r}$, then by (4.28), $\rho(x_0, \psi) = \frac{1}{B} = a < \rho$ on the flow line of $x_0$.

**Remark 4.14.** 1. By taking $x_0 = p_0$, we can make $\rho(p, \psi) = \{0, a\}$ on the $X$-flow line of $p_0$. 2. By taking a sequence $\{y_i\}$ so that $\frac{1}{n} \sum_{k=0}^{n-1} a_k = \infty$, we get $\rho(x_0, \psi) = \{0\}$ on the flow line of $x_0$. 3. By varying $c$ in 4.25 and $M$ in 4.26, we can obtain any specified $B > A$, and hence any $0 \leq a \leq \rho$.

We now describe the construction of Katok's Example on a solenoid. First, consider the dyadic solenoid $S_2$.

Let $\phi$ is a flow on $S_2$ that satisfies (4.1), $\Phi(z_0) = 0$ for a unique point $z_0 \in S_2$, and $\frac{1}{\Phi}$ is $L^1(S_2)$. For simplicity, suppose that $z_0 \in C_\mu$. By Theorem 4.11, $\rho(\phi) = [0, \rho]$. 
Furthermore, by Theorem 4.10, $\phi$ has an ergodic measure $\mu$ such that $\rho(p) = \rho$

$\mu$-almost everywhere on $\mathbb{T}^2$, where

$$\rho = \frac{1}{\int_{S_2} (1/\Phi) \, d\lambda}, \quad (4.29)$$

and $\mu \ll \lambda$.

As before, let $r(x)$ be the return time of $x \in C_\varepsilon$ to $C_\varepsilon$. The Poincaré map of $\phi$ in

$C_\varepsilon$ is the adding machine $A$. By an argument similar to (4.15), (4.16),

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} r(A^k x) = \int_{S_2} \frac{1}{\Phi} \, d\lambda =: A,$$

for $\mu$-almost all flow lines, and hence for $\lambda$-almost all $x \in C_\varepsilon$, since $\mu \ll \lambda$. Clearly, by (4.29), $\rho = \frac{1}{A}$.

As on the torus, we would like to adjust $r(x)$, $x \in C_\varepsilon$, so that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} r(A^k x_0) = B > A,$$

for some $B \in \mathbb{R}_+$, $x_0 \in C_\varepsilon$. Then $\rho(x_0, \phi) = \frac{1}{B} < \rho$.

Choose $x_0 \in C_\varepsilon$, and denote $x_n = A^n x_0$, the $n$-th return of $x_0$ to $C_\varepsilon$. The orbit of $x_0$ under $A$ is dense in $C_\varepsilon$, so we can find a subsequence $x_{n_i} \to x_0$. By the properties of the solenoid, $\frac{n_{i+1}}{n_i} = 2 \forall i$. Now we construct a sequence

$$a_n = \begin{cases} 0 & : n \notin \{n_i\}_{i=1}^{\infty} \\ y_i & : n = n_i \text{ for some } i \end{cases},$$

such that

$$\lim_{i \to \infty} Y_{n_i} = c, \quad \text{where} \quad Y_n = \frac{1}{n} \sum_{k=0}^{n-1} a_k. \quad (4.30)$$
Since $\frac{n_{i+1}}{n_i}$ does not converge to 1, the sequence $Y_n$ does not have a limit, but rather a set of limit points. So by construction of $y_i$,

$$\limsup_{n \to \infty} Y_n = \lim_{i \to \infty} Y_{n_i} = c, \quad \liminf_{n \to \infty} Y_n = \lim_{i \to \infty} Y_{n_i+1-1} = \lim_{i \to \infty} \frac{1}{2n_i - 1} \sum_{k=1}^{i} y_k = \frac{c}{2},$$

so the set of limit points of $Y_n$ is $[c/2, c]$.

For each $i$, define $\theta_i$ to be the step function with height $y_i$ at $x_{n_i}$ and with integral less than $\frac{1}{M^i}$, where $M >> 1$. That is,

$$\theta_i(x) = \begin{cases} y_i & : x \in U_i(x_{n_i}) \\ 0 & : \text{otherwise} \end{cases}, \quad (4.31)$$

where $U_i(x_{n_i})$ is a clopen set containing $x_{n_i}$ with $\beta(U_i(x_{n_i})) \leq \frac{1}{M^i}$. Then for every $i$,

$$\int_{C_x} \theta_i \, d\beta \leq \frac{1}{M^i},$$

so

$$\exists L_i \quad \forall l \geq L_i \quad \frac{1}{l} \sum_{k=0}^{l-1} \theta_i(A^k x) \leq \frac{2}{M^i}$$

for all $x \in C_x$. Replace $\theta_i$ by $\xi_i$, defined by narrowing the support of $\theta_i$ to a smaller clopen set, so that

$$\xi_i(x_{n_i}) = \theta_i, \quad \xi_i(x_n) = 0 \quad \forall n \leq L_i, \ n \neq n_i.$$

Thus the effect of adding $\xi_i$ to $r$ is completely localized to $x_{n_i}$. Define

$$\tilde{r} = r + \sum_{i=1}^{\infty} \xi_i. \quad (4.32)$$

Then $\tilde{r}(x) \to \infty$ as $x \to x_0$, and it has no other discontinuities. Also,

$$\frac{1}{n} \sum_{k=0}^{n-1} \tilde{r}(x_k) = \frac{1}{n} \sum_{k} r(x_k) + \frac{1}{n} \sum_{k} \sum_{i} \xi_i(x_k)$$

$$\to A + \left( \lim Y_n + \sum_{i=1}^{\infty} \int_{C_x} \xi_i(x) \, d\beta \right) \leq A + \lim Y_n + \frac{1}{M-1}. \quad (4.33)$$
So \( \frac{1}{n} \sum_{k=0}^{n-1} \hat{f}(x_k) \) has the set of limit points

\[
\left[ A + \frac{c}{2} + \frac{1}{M-1}, A + c + \frac{1}{M-1} \right] =: [b_1, b_2].
\] *(4.34)*

Now let \( \psi \) be a flow on \( S^2 \) generated by \( \hat{f} \). Then on the arc component of \( x_0 \),

\[
\rho(x_0, \psi) = \left\{ \text{limit points of } 1 \left/ \left( \frac{1}{n} \sum_{k=0}^{n-1} \hat{f}(x_k) \right) \right. \right\} = \left[ \frac{1}{b_2}, \frac{1}{b_1} \right],
\] *(4.35)*

where \( 0 \leq \frac{1}{b_2} \leq \frac{1}{b_1} \leq \rho \).

**Remark 4.15.** Instead of \( S^2 \), let us consider a flow on \( S_P \), where \( P = (p_1, p_2, \ldots) \) is a bounded sequence, not necessarily periodic, i.e. \( p_i \leq p \ \forall i \). Let \( p = \sup_i p_i \).

Then in the construction above, we have \( \frac{n_{i+1}}{n_i} = p_i \). As above, by construction of \( y_i \),

\[
Y_{n_i} = \frac{1}{n_i} \sum_{k=1}^{i} y_k \to c. \quad \text{Also, } \frac{Y_{n_i}}{Y_{n_{i+1}-1}} = \frac{m_i n_i - 1}{n_i} \to m, \quad \text{so } Y_{n_{i+1}-1} \to \frac{c}{m}.
\]

Thus we obtain a flow line with rotation set \( \left[ \frac{1}{b_2}, \frac{1}{b_1} \right] \), where

\[
b_1 = A + \frac{c}{m} + \frac{1}{M-1}, \quad b_2 = A + c + \frac{1}{M-1}.
\]

**Remark 4.16.** If \( P = (p_1, p_2, \ldots) \) is aperiodic and unbounded, that is, \( \sup_i p_i = \infty \), then there is a subsequence \( Y_{n_k} \to 0 \), and we obtain a flow line with rotation set \( \left[ \frac{1}{b_2}, \frac{1}{b_1} \right] \), where \( b_2 = A + c + \frac{1}{M-1} \) and \( \frac{1}{b_1} \) is as close to \( \rho \) as we like.

**Remark 4.17.** As on the torus, by varying \( c \) in (4.30) and \( M \) in (4.31), we can obtain flow lines with various intervals in \([0, \rho]\) as pointwise rotation sets.
CHAPTER 5

ROTATION OF FLOWS ON SUBSTITUTION TILING SPACES

Definitions and Preliminaries

As we have mentioned in Chapter 1, every orientable hyperbolic one-dimensional attractor is either homogeneous, and hence a true solenoid; or nonhomogeneous, and then a one-dimensional substitution tiling space ([7]). Due to this fact, and because substitution tiling spaces have attracted a lot of attention recently, we show in this chapter how to extend the results of Chapters 3 and 4 to substitution tiling spaces, without doing too much extra work. Then, the results of this thesis would apply to all orientable one-dimensional hyperbolic attractors.

First, a number of necessary definitions will be stated, following Barge and Diamond [7].

**Definition 5.1.** Let \( A = \{1, 2, \ldots, d\} \) be a finite alphabet. \( A^* \) will denote the collection of finite nonempty words with letters in \( A \). A *substitution* is a map \( \sigma : A \to A^* \), it has an associated transition matrix \( A_\sigma = (a_{ij})_{i,j \in A} \), in which \( a_{ij} \) is the number of occurrences of \( i \) in the word \( \sigma(j) \).

**Definition 5.2.** A substitution \( \sigma \) is called *primitive* if there is a natural number \( k \) such that \( A_\sigma^k \) has all positive entries, that is, \( \forall i, j \in A \) \( i \) is contained in \( \sigma^k(j) \).
Let $X_\sigma$ denote the set of allowable bi-infinite words for $\sigma$. That is, $w \in X_\sigma$ if and only if for each finite subword $w'$ of $w$, there are $i \in A$ and $n \in \mathbb{N}$ such that $w'$ is a subword of $\sigma^n(i)$. For the topology on $X_\sigma$, we take the subspace topology inherited from the product topology on $\prod_{i=1}^{\infty} \mathbb{Z}$, with the discrete topology on every factor in the product. If $\sigma$ is primitive and nonperiodic, then $X_\sigma$ has a Cantor set structure. We identify the $0^{th}$ coordinate in a bi-infinite word $w$ by either an indexing, as in $w = \ldots w_{-1} w_{0} w_{1} \ldots$, or by use of a decimal point. The substitution $\sigma : A \rightarrow A^*$ extends to $\sigma : X_\sigma \rightarrow X_\sigma$, where
\[
\sigma(\ldots w_{-1} w_{0} w_{1} \ldots) = \ldots \sigma(w_{-1}) \cdot \sigma(w_{0}) \sigma(w_{1}) \ldots
\]
The word $w$ is $\sigma$-periodic if for some $m \in \mathbb{N}$, $\sigma^m(w) = w$. A primitive substitution is nonperiodic if each $\sigma$-periodic bi-infinite word is not periodic under the natural shift map.

**Definition 5.3.** A substitution $\sigma$ is called proper ([23]), if there exists an integer $p > 0$ and two letters $r, l \in A$ such that:

(i) for every $a \in A$, $r$ is the last letter of $\sigma^p(a)$;

(ii) for every $a \in A$, $l$ is the first letter of $\sigma^p(a)$.

A proper substitution has exactly one periodic, hence fixed, bi-infinite word. Durand, Host, and Skau [23] show that every substitution dynamical system is isomorphic to the system associated with some proper substitution.
Example 5.1. Let \( A = \{1, 2\} \), and define \( \sigma(1) = 112, \ \sigma(2) = 12 \). Then \( A_\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \), and the fixed word (generated by 2.1) is

\[
...112112112112112112... \]

The following well-known proposition will be useful to us in the subsequent exposition.

Proposition 5.4. A fixed word \( u \) of a primitive substitution is uniformly recurrent, i.e. every letter \( a \in A \) recurs with bounded interval, and the bound does not depend on \( a \). More precisely, \( \exists K \ \forall a \in A \ \ a \) occurs in \( u_{n+1} \ldots u_{n+K} \).

Proof. Let \( m \) be such that \( \sigma^m \) has positive transition matrix, that is, \( \forall i, j \in A \ \ \sigma^m(i) \) contains \( j \). Let \( u_n = a \), then \( a \in \sigma^m(u_0) \) for some \( k \). Denote \( [u_0 \ldots u_l] = \sigma^{m(k_1)}(u_0) \), then

\[
a = u_n \in \sigma^m(\sigma^{m(k-1)}(u_0)) = \sigma^m(u_0 \ldots u_l) = [\sigma^m(u_0) \ldots \sigma^m(u_l)].
\]

1) If \( u_n = a \in \sigma^m(u_j), \ j < l \), then \( j + 1 \leq l \) and \( \sigma^m(u_{j+1}) \) contains \( a \). Hence, it recurs after at most \( |\sigma^m(u_j)| + |\sigma^m(u_{j+1})| \leq 2L \), where \( L = \max_{j \in A} |\sigma^m(j)| \).

2) If \( u_n = a \in \sigma^m(u_i) \), then let \( [u_0 \ldots u_s] = \sigma^m(u_0) \) and consider

\[
\sigma^{m(k+1)}(u_0) = \sigma^{mk}(\sigma^m(u_0)) = \sigma^{mk}(u_0)\sigma^{mk}(u_1)\ldots\sigma^{mk}(u_s)
\]

\[
= \sigma^m(u_0)\ldots\sigma^m(u_i)\sigma^m(j_1)\ldots\sigma^m(j_{s_1}),
\]

where \( [j_1 \ldots j_{s_1}] = \sigma^{m(k-1)}(u_1) \).
Now $a \in \sigma^m(u_i)$ by assumption, and $a \in \sigma^m(j_1)$ by positivity of $A_{\sigma^m}$. So $a$ recurs after at most $|\sigma^m(u_i)| + |\sigma^m(j_1)| \leq 2L$ letters. Thus $2L$ is the uniform bound on the recurrence of any letter in $A$. □

By Perron-Frobenius Theorem, if $\sigma$ is primitive, its transition matrix $A_\sigma$ has a simple, real, positive eigenvalue $\lambda_\sigma$, called the Perron-Frobenius eigenvalue, that is larger in modulus than its remaining eigenvalues. For $\lambda_\sigma$, consider the associated positive left eigenvector $\bar{v}_\sigma$ with entries $(l_1, \ldots, l_d)$. The intervals $P_i = [0, l_i]$, $i = 1, \ldots, d$, are called prototiles.

A tiling $T$ of $\mathbb{R}$ by the prototiles is a collection $T = \{T_i\}_{i=-\infty}^{\infty}$ of tiles $T_i$ for which $\bigcup_{i=-\infty}^{\infty} T_i = \mathbb{R}$, each $T_i$ is a translate of some $P_j$, and $T_i \cap T_j$ is a singleton for each $i \neq j$.

The map $F_\sigma$, called ‘inflation and substitution’, takes a tiling $T = \{T_i\}_{i=-\infty}^{\infty}$ of $\mathbb{R}$ by prototiles to a new tiling, $F_\sigma(T)$, of $\mathbb{R}$ by prototiles defined by inflating, substituting, and suitably translating each $T_i$. A translation of a tiling is defined as

$$\{T_i\}_{i=-\infty}^{\infty} + t = \{T_i + t\}_{i=-\infty}^{\infty}, \quad (5.1)$$

which amounts to translating all tiles by $t$.

There is a natural topology on the collection of tilings of $\mathbb{R}$: two tilings are ‘close’ if one of them and a small translate of the other are identical in a large neighborhood of zero (see [4]).
Definition 5.5. The tiling space $\mathcal{T}_\sigma$, associated with $\sigma$, is the closure of the orbit (under translation) of the tiling that corresponds to the fixed word of $\sigma$.

There is a natural flow on $\varphi^t : \mathcal{T}_\sigma \rightarrow \mathcal{T}_\sigma$, defined by

$$\varphi^t(T) = T - t,$$ \hspace{1cm} (5.2)

which amounts to shifting the origin to the right, or translating all tiles of $T$ to the left by $t$. Each $T \in \mathcal{T}_\sigma$ is uniformly recurrent under $\varphi$ and satisfies $\text{cl}(\text{orb}_\varphi(T)) = \mathcal{T}_\sigma$ (i.e. $\varphi$ is minimal on $\mathcal{T}_\sigma$). It follows that $\mathcal{T}_\sigma$ is a continuum. Furthermore, $F_\sigma : \mathcal{T}_\sigma \rightarrow \mathcal{T}_\sigma$ is a homeomorphism.

A composant of a point $x$ in a topological space $X$ is the union of the proper compact connected subsets of $X$ containing $x$. In tiling spaces, composants and arc components are identical. Given a substitution $\sigma$ which is primitive and nonperiodic, the composants of $\mathcal{T}_\sigma$ coincide with the orbits of the natural flow. For $\sigma$ proper, $\mathcal{T}_\sigma$ has a nonzero number of (and finitely many) asymptotic composants, hence $\mathcal{T}_\sigma$ is not homogeneous.

Define a map $\mathcal{I} : X_\sigma \rightarrow C_0$ by setting $\mathcal{I}(\ldots i_{-1} . i_0i_1\ldots) = T$, where $T = \ldots T_{i_{-1}} . T_{i_0}T_{i_1}\ldots \in C_0$, where each $T_{i_k}$ is a translate of the prototile $P_{i_k} = [0, l_{i_k}]$. We shall say that a tiling $T \in C_0$ corresponds to an allowable word $w \in X_\sigma$ if $\mathcal{I}(w) = T$.

Since every substitution tiling space is homeomorphic to a tiling space associated with a proper substitution, we shall assume that the substitution $\sigma$ under consideration is proper. Thus $X_\sigma$ has exactly one fixed word $u$. 
Denote \( T(0) \) the tiling that corresponds to the fixed word \( u \) of \( \sigma \). Also define the cross-section \( C_0 \) of \( \mathcal{T}_\sigma \) by setting \( T \in C_0 \) if and only if the origin is aligned with the edge of a tile in \( T \). Then there is a natural correspondence between tilings in \( C_0 \) and allowable bi-infinite words in \( X_\sigma \). Thus we shall identify \( C_0 \) with \( X_\sigma \).

Suppose that \( T = \ldots T_{i-1} \cdot T_{i_0} \cdot T_{i_1} \ldots \in C_0 \), where each \( T_{i_k} \) is a translate of the prototile \( P_{i_k} = \left[ 0, l_{i_k} \right] \), corresponds to \( \ldots w_{-1} \cdot w_0 \cdot w_1 \ldots \in X_\sigma \). Then \( \varphi^{l_0}(T) \in C_0 \), and it corresponds to \( \ldots w_0 \cdot w_1 \cdot w_2 \ldots \in X_\sigma \). So the natural flow on \( \mathcal{T}_\sigma \) induces a homeomorphism on \( X_\sigma \), which is the left shift:

\[
S(\ldots w_{-1} \cdot w_0 \cdot w_1 \ldots) = (\ldots w_0 \cdot w_1 \cdot w_2 \ldots). \tag{5.3}
\]

\( X_\sigma \) is actually the closure of the orbit of the fixed word \( u \) under the left shift, and \( \mathcal{T}_\sigma \) is a suspension of \( S : X_\sigma \to X_\sigma \) (see [2]).

If \( \sigma \) is proper, then, adopting Williams' approach ([72], [73]), \( \mathcal{T}_\sigma \) can be viewed as \( \mathcal{T}_\sigma = \lim f \), where \( f \) is a map, associated with \( \sigma \), on a wedge of circles of lengths \( l_1, \ldots, l_d \).

### Universal Cover and Lifts

As in the case of solenoids, we take \( \mathbb{R} \times C \) as the covering space for \( \mathcal{T}_\sigma \). Here we set \( C = X_\sigma \). Recall that there is a natural flow on \( \mathbb{R} \times C \), given by

\[
\varphi^t(\tau, x) = (\tau + t, x).
\]

We define a map \( h : \mathbb{R} \times C \to \mathcal{T}_\sigma \) so that:
Figure 14. Covering projection for the tiling space.

(a) $h$ maps $\{0\} \times C$ one-to-one onto $C_0$ by $h(0, w) = (w)$ (hence $h(0, u) = T_0$.)

(b) $h(\varphi^t(\tau, x)) = \varphi^t(h(\tau, x)) \ \forall (\tau, x) \in \mathbb{R} \times C$, that is, the diagram in Figure 14 commutes.

The map $h$ that satisfies (a) – (c) is continuous and unique. Similarly to the covering projection for solenoids, one can show that the map $h$ is a covering map. The following property of $h$, which follows from its definition, is an equivalent of Proposition 2.16 for solenoids.

Let $l(x)$ be the smallest positive number such that $T - h(l(x), x) \in C_0$. This means that the length of the tile $T_0$ of $T$ is equal to $l(x)$, and $T$ corresponds to $sx \in X_\sigma$. Thus we obtain

$$h(l(x), x) = h(0, Sx).$$

So a fundamental domain in $\mathbb{R} \times C$ is $\Omega = \{(\tau, x): x \in X_\sigma, 0 \leq \tau < l(x)\}$.

If $T = h(0, x)$, denote $l_1(x), l_2(x), \ldots$ to be the lengths of tiles $T_0, T_1, \ldots$ in $T$, and $l_{-1}(x), l_{-2}(x), \ldots$ to be the lengths of tiles $T_{-1}, T_{-2}, \ldots$. In the notation of (5.4),
\( l(x) = l_1(x) \). Also, set \( l_0(x) = 0 \). Then for any \( x \in X_\sigma, n \in \mathbb{Z}, \)

\[
h(l_n(x), x) = h(l_{n-1}(x), Sx),
\]

(5.5)

or, more generally,

\[
h(l_n(x), x) = h(0, S^nx).
\]

(5.6)

This also gives us a tiling of the arc component \( \mathbb{R} \times \{x\} \) of \( \mathbb{R} \times C \) by tiles of lengths \( l_1, l_2 - l_1, l_3 - l_2, \ldots \), which corresponds to the allowable word \( x \in X_\sigma \). Thus the tiling space structure of \( T_\sigma \) is 'pulled back' to the covering space \( \mathbb{R} \times C \).

**Proposition 5.6.** \( l(x) \) is continuous on \( X_\sigma \).

**Proof.** Since range \( l(x) = \{l_1, \ldots, l_d\} \) is discrete and finite, we need to show that \( l(x_1) = l(x_2) \) if \( x_1, x_2 \) are close enough. Any \( x \in X_\sigma \) can be written as \( x = \ldots x_{-1} \cdot x_0x_1 \ldots \). By definition of the metric on \( X_\sigma \), \( x_1, x_2 \) are close if and only if \( x_i^{(1)} = x_i^{(2)}, \ |i| \leq N \) for some \( N \). Choose \( x_1, x_2 \) so that \( x_0^{(1)} = x_0^{(2)} \), then the corresponding tilings \( T_{x_1}, T_{x_2} \) both contain \([0, l_0^{(1)}]\) as the first tile to the right of 0, so \( l(x_1) = l(x_2) = l_{x_0^{(1)}} \).

**Remark 5.7.** By choosing \( x_1, x_2 \) so that \( x_i^{(1)} = x_i^{(2)}, \ i = 0, 1, \ldots, n \), we can show that \( l_n(x) \) is continuous for any \( n \). Similarly, \( l_{-n} \) is continuous for any \( n \).

**Proposition 5.8.** \( l_n(Sx) = l_{n+1}(x) - l_1(x) \).
PROOF. Since $S$ is a left shift, then $l_1(Sx) = l_2(x) - l_1(x)$ (the length of the second tile to the right of 0 in $h(0, x)$), so

$$l_2(Sx) = (l_2(x) - l_2(x)) + l_1(Sx) = l_3(x) - l_1(x)$$

... 

$$l_n(Sx) = (l_{n+1}(x) - l_n(x)) + l_{n-1}(Sx) = l_{n+1}(x) - l_1(x).$$

\[ \square \]

COROLLARY 5.9. $l_n(S^k x) = l_{n+k}(x) - l_k(x)$.

PROPOSITION 5.10. $h(t + l_1(x), x) = h(t, Sx)$ $\forall t \in \mathbb{R}, x \in X_\sigma$.

PROOF. $h(t + l_1(x), x) = h(\phi^t(l_1(x), x)) = \phi^t \circ h(l_1(x), x) = \phi^t \circ h(0, Sx) = h \circ \phi^t(0, Sx) = h(t, Sx)$. \[ \square \]

COROLLARY 5.11. $h(t + l_n(x), x) = h(t, S^n x)$ $\forall t \in \mathbb{R}, x \in X_\sigma, n \in \mathbb{Z}$. Conversely, if $h(t, x) = h(s, y)$, then there is a $k = k(x)$ such that $(t + l_k(x), x) = (s, S_k y)$.

PROPOSITION 5.12. If $f : T_\sigma \to T_\sigma$ is a homeomorphism, then it has a lift \( \tilde{f} : \mathbb{R} \times C \to \mathbb{R} \times C \), such that $f \circ h = h \circ \tilde{f}$.

PROOF. The proof is similar to that of Theorem 2.29, with the following adjustment. Define the open sets $U_0, U_1$ such that for every $x \in X_\sigma$,

$$\left[0, \frac{l_1(x)}{2}\right] \times \{x\} \subset \tilde{U}_0, \quad \left[\frac{l_1(x)}{2}, l_1(x)\right] \times \{x\} \subset \tilde{U}_1.$$

Then proceed as in the proof of Theorem 2.29. \[ \square \]
For the same reason as on solenoids, since rotation sets computed with different lifts differ by any arbitrary constant, we restrict attention to the lifts homotopic to the identity. The following can be proved similar to the corresponding results on solenoids.

**Proposition 5.13.** A homeomorphism of $\mathcal{T}_\sigma$ homotopic to the identity has a lift to $\mathbb{R} \times C$ homotopic to the identity.

**Proposition 5.14.** A lift homotopic to the identity is unique.

Let $\tilde{f}$ be a lift of $f$. Denote $(s, y) = \tilde{f}(t, x)$. By Corollary 5.11, $f \circ h(t + l_n(x)) = f \circ h(t, S^n x)$. Since $f \circ h = h \circ \tilde{f}$, then $h \circ \tilde{f}(t + l_n(x)) = h \circ \tilde{f}(t, S^n x)$. Denote $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$, then again by Corollary 5.11, there is a $k_n(x)$ such that

$$\tilde{f}_1(t + l_n(x), x) = \tilde{f}_1(t, S^n x) + k_n(x),$$

$$\tilde{f}_2(t + l_n(x), x) = S^{-k_n} \tilde{f}_2(t, S^n x).$$

Technically, $k_n$ depends not on $x$, but on $\tilde{f}_2(t, S^n x)$.

We now turn to the flows on substitution tiling spaces. As on solenoids, a flow on $\mathcal{T}_\sigma$ is said to be positively oriented if its orientation coincides with the orientation of the natural flow. Otherwise, it is called negatively oriented.

A flow $\phi$ on $\mathbb{R} \times C$ is called a lift of $\phi$, provided

$$\phi^t(h(\tau, x)) = h(\phi^t(\tau, x)) \quad \forall (\tau, x) \in \mathbb{R} \times C, \ t \in \mathbb{R}. \quad (5.7)$$
The lift \( \hat{\phi} \) of \( \phi \) is unique, since for every \( t \in \mathbb{R} \), \( \hat{\phi}^t \) is a lift of the homeomorphism \( \phi^t \), and \( \hat{\phi}^t \) is homotopic to the identity. Hence, it is unique.

**Two Approaches to Defining Rotation Sets**

There are two different approaches to defining rotation sets for flows on substitution tiling spaces. The first approach is the same as the one we adopted for flows on solenoids, defining the rotation set through the lift of the flow to the covering space, and measuring the average displacement of a point under the flow.

**Definition 5.15.** Let \( \phi \) be a flow on \( \mathcal{T}_\sigma \), and \( \hat{\phi} \) its lift to \( \mathbb{R} \times C \). The rotation set of \( \phi \) at the point \( T \in \mathcal{T}_\sigma \) is

\[
\rho(T, \phi) = \left\{ \text{limit points of } \frac{\pi_1 \hat{\phi}^t (\tau, x) - \tau}{t_i}, \ \forall t_i \in \mathbb{R}, \ t_i \to \infty \right\}, \tag{5.8}
\]

where \( h(\tau, x) = T \).

**Definition 5.16.** The pointwise rotation set of flow \( \phi \) is

\[
\rho_p(\phi) = \bigcup_{T \in \mathcal{T}_\sigma} \rho(T, \phi). \tag{5.9}
\]

**Definition 5.17.** Let \( \phi \) be a flow on \( \mathcal{T}_\sigma \), and \( \hat{\phi} \) its lift to \( \mathbb{R} \times C \). The MZ rotation set (or simply the rotation set) of \( \phi \) is

\[
\rho(\phi) = \left\{ \text{limit points of } \frac{\pi_1 \hat{\phi}^t (\tau_i, x_i) - \tau}{t_i}, \ \forall t_i \in \mathbb{R}, \ t_i \to \infty, \ (\tau_i, x_i) \in \mathbb{R} \times C \right\}.
\]

Similarly we define rotation sets for homeomorphisms:
DEFINITION 5.18. Let $f$ be a homeomorphism of $\mathcal{T}_\sigma$, and $F$ its lift to $\mathbb{R} \times C$. The MZ rotation set of $F$ is

$$\rho(F) = \left\{ \text{limit points of } \frac{\pi_1 F^{k_i}(\tau_i, x_i) - \tau_i}{k_i}, \forall k_i \in \mathbb{Z}, k_i \to \infty, (\tau_i, x_i) \in \mathbb{R} \times C \right\}.$$ 

The rotation set defined in such a way possesses all the properties discussed in Chapter 3.

The second approach to defining a rotation set is to involve the tiling space structure of $\mathcal{T}_\sigma$. Suppose that $\sigma$ is a primitive substitution. As above, denote $\lambda_\sigma$ its Perron – Frobenius eigenvalue, and $\vec{l}_\lambda = (l_1, \ldots, l_d)$ the positive left eigenvector associated with $\lambda_\sigma$. $P_i = [0, l_i], i = 1, \ldots, d$ are prototiles.

DEFINITION 5.19. For every $T \in \mathcal{T}_\sigma$, define $p_i(T, \phi, t), i = 1, 2, \ldots, d$ to be the number of copies of $P_i$ passed in moving from $T$ to $\phi^t(T)$. The rotation vector of $\phi$ at the point $T$ is

$$\bar{\rho}(T, \phi) = \lim_{t \to \infty} \frac{1}{t} \left( p_i(T, \phi, t) \right)_{i=1}^d. \quad (5.10)$$

Clearly, $\bar{\rho}(T, \phi) \in \mathbb{R}^d$. If $\sigma$ is primitive, the geometric representation of a flow on $\mathcal{T}_\sigma$ is a flow on $\mathbb{R}^d / \mathbb{Z}^d$, in the direction of the right eigenvector $\vec{r}_\lambda$, associated with $\lambda_\sigma$. We can write $\vec{r}_\lambda = (\alpha_1 l_1, \alpha_2 l_2, \ldots, \alpha_d l_d)'$.

Now $\forall T \in \mathcal{T}_\sigma$, $\pi_1 \phi^t(\tau, x) - \tau = d_{\text{arc}}(T, \phi^t(T)) = \gamma(t) ||\vec{r}_\lambda||$, for some function $\gamma(t)$, depending on $T$. Then

$$\rho(T, \phi) = \lim_{t \to \infty} \frac{\gamma(t)}{t} ||\vec{r}_\lambda|| =: \rho ||\vec{r}_\lambda||,$$
and so
\[ \hat{\rho}(T, \phi) = \lim_{t \to \infty} \frac{1}{t} (\gamma(t) \alpha_t)_{t=1}^d = (\rho \alpha_1, \ldots, \rho \alpha_d)'. \]

Thus, knowing \( \rho(T, \phi) \), we can find \( \hat{\rho}(T, \phi) \), and vice versa. So it is sufficient to analyze one of them. For consistency reasons, we choose the first approach, i.e. Definitions 5.15 – 5.17. In the following two sections, we reformulate the results for solenoidal flows, obtained in Chapters 3, 4, for flows on substitution tiling spaces.

**Unique Ergodicity of the Subshift \( X_\sigma \)**

As we have previously mentioned, a flow on a solenoid is metrically isomorphic to a special flow, for which the base space is the Cantor set, with the adding machine as the underlying homeomorphism. The same is true for flows on substitution tiling spaces. Here, the underlying homeomorphism of the base space is the left shift \( S \).

For irrational toral flows, it is irrational rotation on \( S^1 \), which is the base space in this case. In all these cases, the results we can obtain regarding ergodicity and rotation sets of flows, grow out of the fact that the underlying homeomorphism in the base space of the special flow is uniquely ergodic. In other words, unique ergodicity of the left shift \( S \) on \( X_\sigma \) lies in the core of the theorems we state in the next section.

Even though it has been known for some time (see [59]), because of its importance we present a complete proof of the unique ergodicity of the subshift \( X_\sigma \), following the ideas of [59], but with more details and different notation. Throughout this section, we assume that \( \sigma \) is a primitive substitution.
DEFINITION 5.20. Let $w$ be an allowable bi-infinite word. The frequency of symbol $i$ in $w$ is defined as follows:

$$\text{freq}(i, w) = \lim_{n \to \infty} \frac{1}{2n} \sum_{j=-n}^{n-1} \chi_i(S^j(w)),$$

(5.11)

where $\chi_i(w) = \begin{cases} 1 & : w_0 = i \\ 0 & : \text{otherwise} \end{cases}$

Clearly,

$$\text{freq}(i, w) = \frac{1}{2} \left( \text{freq}^+(i, w) + \text{freq}^-(i, w) \right),$$

(5.12)

where

$$\text{freq}^+(i, w) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_i(S^j(w)),$$

$$\text{freq}^-(i, w) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=-n}^{-1} \chi_i(S^j(w)),$$

provided all limits exist.

Denote $L_i(w)$ the number of occurrences of $i$ in $w$. If $w = u$ is a fixed word of $\sigma$, then

$$\text{freq}^+(i, u) = \lim_{n \to \infty} \frac{1}{|\sigma^n(u)|} (L_i(\sigma^n(u))),$$

$$\text{freq}^-(i, u) = \lim_{n \to \infty} \frac{1}{|\sigma^n(u)|} (L_i(\sigma^n(u))),$$

PROPOSITION 5.21. For every $i \in \mathcal{A}$, $\text{freq}(i, j)$ exists and is the same $\forall j \in \mathcal{A}$.

PROOF. If $A_\sigma$ is the transition matrix for $\sigma$, then $A^n_\sigma$ is the transition matrix for $\sigma^n$. Denote its elements by $a^{(n)}_{ij}$. $a'$ will denote the transpose of a vector $a$. 

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Let \( i, j \in \mathcal{A} \). Denote \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0)' \) – the \( i \)-th standard basis vector, \( i = 1, \ldots, d \). Then \( e_i' A^n = (a_{i_1}^{(n)}, a_{i_2}^{(n)}, \ldots, a_{i_d}^{(n)}) \) – the number of occurrences of \( i \) in \( \sigma^n(1), \sigma^n(2), \ldots, \sigma^n(d) \). So the number of occurrences of \( i \) in \( \sigma^n(j) \) is \( e_i' A^n e_j \).

The length of the word \( \sigma^n(j) \), that is, the number of symbols in it, is \( |\sigma^n(j)| = \|A^n e_j\|_1 = \sum_{k=1}^d a_{kj}^{(n)} \). Now

\[
\frac{1}{|\sigma^n(j)|} (L_i(\sigma^n(j))) = \frac{e_i' A^n e_j}{\|A^n e_j\|_1} = e_i' \left( \frac{A^n e_j}{\|A^n e_j\|_1} \right).
\]

Now we take a limit as \( n \to \infty \). By the Perron – Frobenius Theorem,

\[
\frac{A^n e_j}{\|A^n e_j\|_1} \to r_\lambda,
\]

where \( r_\lambda \) is the normalized right eigenvector of the Perron – Frobenius eigenvalue \( \lambda_\sigma \).

Therefore,

\[
\text{freq}(i, j) = e_i' r_\lambda = (r_\lambda)_i =: r_i,
\]

and it does not depend on \( j \).

**Corollary 5.22.** \( \text{freq}(i, u) = r_i \quad \forall \ i \in \mathcal{A} \).

Next we would like to show that the frequency of any finite subword of \( u \) exists and is well-defined.

**Definition 5.23.** Let \( x = x_1 x_2 \ldots x_m \) be a subword of an allowable bi-infinite word \( w \). The frequency of occurrence of \( x \) in \( w \) is

\[
\text{freq}(x, w) = \lim_{n \to \infty} \frac{1}{2n} \sum_{j=-n}^{n-1} \chi_x(S^j(w)),
\]

(5.14)
where

\[
\chi_x(w) = \begin{cases} 
1 & : w_1 = x_1, \ldots, w_m = x_m \\
0 & : \text{otherwise}
\end{cases}
\]

Clearly, as for the frequency of a symbol in (5.12), the frequency of a word is a sum of the forward and backward frequency.

**PROPOSITION 5.24.** \(\text{freq}(x, y)\) exists and is well-defined.

**PROOF.** Following [59], we shall prove the statement for forward frequencies (a proof for backward frequencies is similar.) To that end, we consider one-sided infinite allowable words.

Let \(x = x_1 \ldots x_m\), then \(|x| = m\). If \(m = 1\), the statement is proved in Proposition 5.21. So let \(m \geq 2\). Denote \(A_m\) the set of all words in \(y\) of length \(m\). We shall construct a new substitution \(\sigma\) on the alphabet \(A_m\).

Define \(\sigma\) as follows. If \(w \in A_m\) and if

\[
\sigma(w) = \sigma(w_1 \ldots w_m) = y_1 \ldots y_{|\sigma(w_1)|} y_{|\sigma(w_1)|+1} \ldots \ldots y_{|\sigma(w_m)|},
\]

then set

\[
\sigma(w) = (y_1 \ldots y_m)(y_2 \ldots y_{m+1}) \ldots (y_{|\sigma(w_1)|} \ldots y_{|\sigma(w_1)|+m-1}). \tag{5.15}
\]

For \(\sigma\) defined by (5.15), we have \(|\sigma(w)| = |\sigma(w_1)|\), and we extend \(\sigma\) to \(A_m^*\) in the usual way.

**LEMMA 5.25.** \(\sigma\) has a fixed point \(\hat{u} = (u_1 \ldots u_m)(u_2 \ldots u_{m+1}) \ldots\), where \(u = u_1u_2 \ldots\) is a (one-sided) fixed word of \(\sigma\).
PROOF. Denote $\omega = u_1 \ldots u_m$. Since $\sigma(u) = u$, then $\sigma(\omega) = u_1 \ldots u_{|\sigma(\omega)|}$, and so

$$\hat{\sigma}(\omega) = (u_1 \ldots u_m)(u_2 \ldots u_{m+1}) \ldots (u_{|\sigma(u_1)|} \ldots u_{|\sigma(u_1)|+m-1}) = \omega \ldots .$$

Similarly,

$$\hat{\sigma}^n(\omega) = (u_1 \ldots u_m)(u_2 \ldots u_{m+1}) \ldots (u_{|\sigma^n(u_1)|} \ldots u_{|\sigma^n(u_1)|+m-1}) = \hat{\sigma}^{n-1}(\omega) \ldots .$$

So for every $n$, $\hat{\sigma}^n(\omega)$ starts with $\hat{\sigma}^{n-1}(\omega)$, which we denote by $\hat{u}_n$. Taking $\lim_{n \to \infty}$, we get $\hat{\sigma}^n(\omega) = \hat{\sigma}(\hat{u}_n) = \hat{u}_n \ldots $. But $\hat{\sigma}(\hat{u}) \to \hat{u}$, so $\hat{\sigma}(\hat{u}) = \hat{u}$.

**Lemma 5.26.** $\hat{\sigma}$ is primitive if $\sigma$ is primitive.

**Proof.** Clearly, we have the following:

(i) $\lim_{n \to \infty} |\hat{\sigma}^n(w)| = \infty \quad \forall w \in A_m$, since $|\hat{\sigma}^n(w)| \geq |\sigma^n(w_0)| \to \infty$.

(ii) $\exists \alpha \in A_m$ such that $\hat{\sigma}(\alpha)$ starts with $\alpha$, which follows from the proof of Lemma 5.25.

We need to show that $\exists k \forall v, w \in A_m \quad w \subset \hat{\sigma}^k(v)$. Let $v, w \in A_m$ be two 'symbols' of the alphabet $A_m$. By construction of $A_m$, $v, w \subset u$, so $\exists \alpha \in A$ such that $w \subset \sigma^p(\alpha)$ for some $p \in \mathbb{N}$. The substitution $\sigma$ is primitive, so $\exists n_0 \forall n \geq n_0 \quad \sigma^n(v_1)$ contains $\alpha$, where $v = v_1 \ldots v_m$. Thus $w \subset \sigma^n(v_1)$.

By construction of $\hat{\sigma}$, $\hat{\sigma}^k(v)$ contains all words of length $m$ in $\sigma^k(v_1)$, including $w$, if $k \geq n + p$. Note that $n \geq n_0$, $n_0$ does not depend on $v, w$. However, $p$ depends on $w$. Take $P = \max_{w \in A_m} p, \quad k \geq n + P$, then $w \subset \hat{\sigma}^k(v)$, and $k$ does not depend on $v, w$. □
Now back to the proof of Proposition 5.24, we apply Proposition 5.21 to get
\[
\text{freq}(x, \hat{u}) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_x(S^j(\hat{u})) = r_x \quad \forall x \in \mathcal{A}_m,
\]
where \( \hat{S} \) is the left shift for substitution \( \hat{\sigma} \). On the other hand, \( \hat{u} \) consists of overlapping words of \( u \), so
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_x(S^j(\hat{u})) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_x(S^j(u)) = \text{freq}^+(x, u)
\]
as a word. Hence \( \text{freq}^+(x, u) = r_x \), i.e. the frequency exists and is well-defined. \( \square \)

**Remark 5.27.** 1. See [59] for a proof using a different (but equivalent) definition of frequency.

2. The fixed word \( u \) in the proof of Proposition 5.24 could be replaced by any allowable word. Lemma 5.25 would not hold in that case, but Lemma 5.26 and Proposition 5.24 would still hold.

Thus, we have the following result.

**Proposition 5.28.** For any finite word \( x \in \mathcal{A}^* \) and any allowable bi-infinite word \( w \), \( \text{freq}(x, w) \) exists and does not depend on \( w \).

In other words, for any \( x \in \mathcal{A}^* \), the sequence of functions
\[
f_{n,x}(w) = \frac{1}{n} \sum_{j=0}^{n-1} \chi_x(S^j(w)) \to f_x(w) \quad \forall w \in X_\sigma,
\]
for some function \( f_x \). For unique ergodicity, we need to verify that the convergence is uniform.
PROPOSITION 5.29. For any word \( x = x_1 \ldots x_m \), the time averages of \( \chi_x \) converge uniformly on the orbit of the fixed word \( u \), that is,
\[
\frac{1}{n} \sum_{j=0}^{n-1} \chi_x (S^j(S^k(u))) \to r_x \equiv f_x(u) \tag{5.17}
\]
uniformly in \( k \in \mathbb{Z} \).

PROOF. Again, we follow [59], adding more detail and using our notation. The statement in (5.17) is equivalent to the uniform convergence of
\[
\frac{1}{N+1} \sum_{j=0}^{N} \chi_x (S^j(S^k(u))) = \frac{L_x(u_k \ldots u_{k+N})}{N+1}.
\]
Proposition 5.24 shows that this quantity converges to \( r_x \) for any \( k \) pointwise. We shall place a uniform lower bound on the rate of convergence.

We try to compare \( u_k \ldots u_{k+N} \) to a word \( \sigma^n(\omega) \) for some \( \omega \in A^* \) and \( n \geq 1 \). By properties of the fixed word \( u \), for a given \( k \in \mathbb{Z} \) and large enough \( N > 0 \), there exist \( j, l > 0 \) such that
\[
u_k \ldots u_{k+N} = b_0 \sigma^n(u_j \ldots u_{j+l}) b_1, \tag{5.18}
\]
where \( b_0, b_1 \) are words in \( u \) such that
\[
|b_i| = \|b_i\|_1 \leq \sup_{\alpha} |\sigma^n(\alpha)| =: \bar{t}_n, \quad i = 0, 1.
\]
Thus we obtain
\[
N + 1 = |b_0| + |b_1| + \sum_{i=j}^{j+l} |\sigma^n(u_i)|. \tag{5.19}
\]
This implies that

$$L_x(u_k \ldots u_{k+N}) - (N + 1)r_x = L_x(u_k \ldots u_{k+N}) - r_x(|b_0| + |b_1|) - r_x \sum_{i=j}^{j+l} |\sigma^n(u_i)|.$$  

By Proposition 5.24, $\frac{L_x(\sigma^n(u_i))}{|\sigma^n(u_i)|} \rightarrow \text{freq}(x, u) = r_x$, that is,

$$\forall \varepsilon > 0 \exists \tilde{N}_{i} \forall n > \tilde{N}_{i} |L_x(\sigma^n(u_i)) - r_x| |\sigma^n(u_i)| \leq \varepsilon |\sigma^n(u_i)|. \quad (5.20)$$

From this we get

$$\left| L_x(u_k \ldots u_{k+N}) - r_x \sum_{i=j}^{j+l} |\sigma^n(u_i)| \right|$$

$$\leq |L_x(u_k \ldots u_{k+N}) - \sum_{i=j}^{j+l} L_x(\sigma^n(u_i))| + \left| \sum_{i=j}^{j+l} L_x(\sigma^n(u_i)) - r_x \sum_{i=j}^{j+l} |\sigma^n(u_i)| \right|$$

$$\leq |L_x(u_k \ldots u_{k+N}) - \sum_{i=j}^{j+l} L_x(\sigma^n(u_i))| + \varepsilon \sum_{i=j}^{j+l} |\sigma^n(u_i)|$$

$$\leq |L_x(u_k \ldots u_{k+N}) - \sum_{i=j}^{j+l} L_x(\sigma^n(u_i))| + \varepsilon (N + 1), \quad (5.21)$$

since $\sigma^n(u_j \ldots u_{j+l}) \subset u_k \ldots u_{k+N}$. On the other hand,

$$L_x(u_k \ldots u_{k+N}) = L_x(b_0 \sigma^n(u_j \ldots u_{j+l}) b_1)$$

$$\leq L_x(b_0) + L_x(b_1) + \sum_{i=j}^{j+l} L_x(\sigma^n(u_i)) + \sum_{i=j-1}^{j+l} L_x(\sigma^n(u_iu_{i+1})), \quad (5.22)$$

where the last term accounts for the possibility of $x$ overlapping two consecutive words. (Note that this is sufficient, we do not need to account for overlapping three consecutive words, by (5.25) below.) The indexing takes into account that $\sigma^n(u_{j-1})$ intersects $b_0$, and $\sigma^n(u_{j+l+1})$ intersects $b_1$.

Now $x$ may intersect the words $\sigma^n(u_i)$ and $\sigma^n(u_{i+1})$ at most $|x| = m$ times, so

$$\sum_{i=j}^{j+l} L_x(\sigma^n(u_iu_{i+1})) \leq m(l + 1). \quad (5.23)$$
Since $|b_i| \leq \bar{t}_n$ and by (5.19), (5.21), (5.22), it follows that

\[
|L_x(u_k \ldots u_{k+N}) - \tau_x(N+1)| \\
\leq L_x(u_k \ldots u_{k+N}) - \tau_x \left( \sum_{i=j}^{j+1} |\sigma^n(u_i)| + \tau_x(|b_0| + |b_1|) \right) \\
\leq L_x(b_0) + L_x(b_1) + \sum_{i=j-1}^{j+1} L_x(\sigma^n(u_iu_{i+1})) + \varepsilon(N+1) + 2\tau_x\bar{t}_n \\
\leq (2\tau_x + 2)\bar{t}_n + m(l+1) + \varepsilon(N+1).
\]

(5.24)

Since $|\sigma^n(\alpha)|$ is increasing and tends to $\infty$ as $n \to \infty$ for any $\alpha \in \mathcal{A}$, we can find

\[
\hat{N} \geq \max_{j \leq i \leq j+1} \hat{N}_i
\]
such that

\[
\forall n > \hat{N} \quad |x| = m \leq \varepsilon \inf_{\alpha} |\sigma^n(\alpha)| =: \varepsilon \bar{t}_n.
\]

(5.25)

Also, take $N >> n$ so that (5.18) is satisfied. Then

\[
m(i+1) \leq \varepsilon \bar{t}_n (l+1) \leq \varepsilon (N+1).
\]

From (5.24) we obtain

\[
\left| \frac{L_x(u_k \ldots u_{k+N})}{N+1} - \tau_x \right| \leq \frac{(2\tau_x + 2)\bar{t}_n}{N+1} + 2\varepsilon \leq \frac{4\bar{t}_n}{N+1} + 2\varepsilon,
\]

which does not depend on $k$. Thus the convergence is uniform on the orbit of $u$ under the shift.

PROPOSITION 5.30. The convergence in (5.16) is uniform on $X_{\sigma}$.

PROOF. Let $w \in X_{\sigma}$. We need to show that

\[
f_{N,\sigma}(w) = \frac{1}{N} \sum_{j=0}^{N-1} \chi_x\left(S^j(w)\right) \to f_x(w)
\]
uniformly in $w$. The orbit of $w$ is dense in $X_\sigma$, so for any $w$ and $N$ there is a $k \in \mathbb{Z}$ such that $w_0 \ldots w_N = u_k \ldots u_{k+N}$. So

$$\frac{L_x(w_0 \ldots w_N)}{N + 1} = \frac{L_x(u_k \ldots u_{k+N})}{N + 1},$$

therefore,

$$\left| \frac{L_x(w_0 \ldots w_N)}{N + 1} - r_x \right| \leq \frac{4\bar{r}_n}{N + 1} + \epsilon$$

for a given $\epsilon$ and appropriately chosen $n, N$ (see (5.26)). This estimate does not depend on $k$, therefore, it does not depend on $N$ and $w \in X_\sigma$. So the convergence is uniform on $X_\sigma$. \Box

Finally, from Propositions 5.28, 5.29, and 5.30 we obtain the main result of the section.

**Theorem 5.31.** The left shift homeomorphism $S$ on the substitutive system $X_\sigma$ is uniquely ergodic.

**Ergodicity and Rotation of Flows on Tiling Spaces**

**Definition 5.32.** The return time of $x \in X_\sigma$ to $X_\sigma$ under the flow $\phi$ is

$$r(x) = \min \{ \tau > 0 : \phi^\tau(0, x) = (l(x), x), \text{ so that } h(l(x), x) = h(0, y), y \in X_\sigma \}.$$

**Definition 5.33.** The average return time to $X_\sigma$ under $\phi$ is

$$\bar{r}_{X_\sigma} = \int_{X_\sigma} r(x) \, d\beta,$$

where $\beta$ is the unique $S$-invariant ergodic measure on $X_\sigma$. 

\begin{equation}
(5.27)
\end{equation}
We now reformulate the main results of Chapters 3, 4 for flows on substitution tiling spaces.

**THEOREM 5.34.** A fixed-point free flow \( \phi \) on \( T_\sigma \) is uniquely ergodic, with the unique ergodic measure given by

\[
\mu(B) = \frac{1}{\bar{r}_{X_\sigma}} \int_{X_\sigma} \tau_B(x) \, d\beta, \quad \forall \, B = h(\bar{B}), \, \bar{B} \in B(\Omega),
\]

where \( \bar{r}_{X_\sigma} \) is defined by (5.27), \( \tau_B(x) = \lambda\{t \in [0, r(x)) : (t, x) \in \bar{B}\} \) is the ‘time-length’ of the fiber with Cantor-set coordinate \( x \), \( \beta \) is the \( S \)-invariant ergodic measure on \( X_\sigma \).

**PROOF.** We proceed in the same fashion as in Chapter 3. First, similarly to the proof of Theorem 3.22, we show that every \( \phi \)-invariant measure \( \mu \) on \( T_\sigma \) is given by

\[
\mu(B) = \int_{X_\sigma} \tau_B(x) \, d\bar{\beta},
\]

\( \bar{\beta} \) some measure on \( X_\sigma \). and that \( \bar{\beta} \) is invariant with respect to the left shift \( S \) on \( X_\sigma \).

Then by Theorem 5.31, since such a measure on \( X_\sigma \) is unique, it follows that \( \bar{\beta} = \beta \), where \( \beta \) is the unique \( S \)-invariant ergodic measure on \( X_\sigma \). We conclude that \( \mu \) is unique and is given by 5.28.

A consequence of this result is the following theorem, which is proved in the same way as Theorem 3.24.
Theorem 5.35. If $\phi$ is a fixed-point free flow on $\mathcal{T}_\sigma$, then its rotation set is

$$\rho(\phi) = \{\rho\}, \quad \text{where} \quad \rho = \frac{1}{r_{X_\sigma}} = \frac{1}{\int_{X_\sigma} r(x) \, d\beta}.$$ (5.29)

We now turn to the flows on $\mathcal{T}_\sigma$ with fixed points. As in Chapter 4, flow $\phi$ is assumed to satisfy an initial-value problem

$$\frac{d}{dt} \phi^t(T) = \Phi(\phi^t(T))$$ (5.30)
$$\phi^0(T) = T$$ (5.31)

where $\Phi : \mathcal{T}_\sigma \to \mathbb{R}_+ \cup \{0\}$ is a function that has a lift $\bar{\Phi}$. Furthermore, $\lambda$ denotes the measure on $\mathbb{R} \times C$, which is the product of the unique $S$-invariant measure $\beta$ on $C$ and the Lebesgue measure on the arc components $\mathbb{R} \times \{x\}$, $x \in C$.

Proposition 5.36. There exists at most one ergodic $\phi$-invariant measure that has nonzero rotation number.

Theorem 5.37. If an ergodic $\phi$-invariant measure $\mu$ with nonzero rotation number exists, then it is absolutely continuous with respect to the measure $\lambda$. Moreover, it is given by

$$\mu(B) = \frac{\int_B (1/\bar{\Phi}) \, d\lambda}{\int_\Omega (1/\bar{\Phi}) \, d\lambda},$$ (5.32)

where $\Omega$ is the fundamental domain in $\mathbb{R} \times C$.

In the following two theorems, again as in Chapter 4, by $L^1(\Omega)$ we mean $L^1(\Omega)$ with respect to $\lambda$. 
**Theorem 5.38.** Let $\phi$ be a flow with one fixed point $p_0$, $\tilde{\Phi}$ a lift of $\Phi$, then:

1. If $\frac{1}{\tilde{\Phi}}$ is $L^1(\Omega)$, then $\phi$ is not uniquely ergodic. It has a $\phi$-invariant ergodic measure $\mu$, given by (5.32), that has nonzero rotation number. Such a measure is unique.

2. If $\frac{1}{\tilde{\Phi}}$ is not $L^1(\Omega)$, then $\phi$ is uniquely ergodic, with the Dirac measure $\delta_{p_0}$ being the only $\phi$-invariant ergodic probability measure on $\mathcal{T}_\sigma$.

**Theorem 5.39.** Let $\phi$ be a flow with fixed points, $\tilde{\Phi}$ a lift of $\Phi$, then each of the following is satisfied:

1. If $\frac{1}{\tilde{\Phi}}$ is $L^1(\Omega)$, then

   $$\rho(\phi) = [0, \rho], \quad \text{where} \quad \rho = \frac{1}{\int_{\Omega} \frac{1}{\tilde{\Phi}}}.$$

   (5.33)

2. If $\frac{1}{\tilde{\Phi}}$ is not $L^1(\Omega)$, then

   $$\rho(\phi) = \{0\}.$$

   (5.34)
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