Rotation of flows on generalized solenoids
by Yurii Borisovich Shvetsov

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences
Montana State University

Abstract:
Rotation sets have been defined and studied for maps and flows on various spaces, including the circle, an annulus, and a torus. They have proven useful in the analysis of the dynamics of such maps and flows.

In this dissertation, we analyze ergodicity and rotation properties of flows on one dimensional generalized solenoids, which include true solenoids and substitution tiling spaces. First, covering projections and lifts are constructed for homeomorphisms and flows on solenoids. We argue that only lifts homotopic to the identity should be considered, and show that a lift homotopic to the identity is unique.

The concept of a rotation set is then defined for flows on solenoids. It is shown that a fixed-point free flow is uniquely ergodic, and that its rotation set contains exactly one point. It is also proved that a flow with fixed points may or may not be uniquely ergodic, and as a consequence, the rotation set of such a flow is either a point or an interval. We give a criterion for distinguishing between these two cases. We also construct an example of two fixed-point free flows that have the same rotation set but are not topologically conjugate.

Finally, all the above results are restated for flows on one-dimensional substitution tiling spaces.

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STATEMENT OF PERMISSION TO USE

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ABSTRACT

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In this dissertation, we analyze ergodicity and rotation properties of flows on one-dimensional generalized solenoids, which include true solenoids and substitution tiling spaces. First, covering projections and lifts are constructed for homeomorphisms and flows on solenoids. We argue that only lifts homotopic to the identity should be considered, and show that a lift homotopic to the identity is unique.

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CHAPTER 1

INTRODUCTION

Historical Overview

Rotation properties of both discrete and continuous dynamical systems have attracted a great deal of interest throughout last century as well as nowadays. In many cases, studying such properties allows one to make important conclusions about typical orbits of a dynamical system, and about long-term behavior of such orbits.

The idea of a *rotation number* was first introduced by Poincaré in Chapter 15 of the third of his memoirs [58]. The concept was inspired by his study of toral flows whose return maps are circle homeomorphisms. Rotation number is a measure of the asymptotic average rotation rate of a point under the homeomorphism. Poincaré proved that the rotation number of an orientation preserving circle homeomorphism exists and is independent of the point on the circle. He also gave a complete description of the possible behavior of orbits of circle homeomorphisms: if a homeomorphism has a rational rotation number, then it has a periodic orbit, and if a homeomorphism has an irrational rotation number, then the ω-limit set of every point on the circle is either the entire circle or a Cantor subset of the circle. Poincaré also posed a question of what conditions a given homeomorphism or diffeomorphism should satisfy in order to be equivalent to a rotation. This question led him to the result commonly known as Poincaré Classification (see, for instance, [38]), in which the concept of rotation number plays a central role.

Since this concept has proven so useful, many generalizations of the rotation number have been made. In 1979, Newhouse, Palis, and Takens [56] introduced the *rotation interval* for circle maps of degree 1, and Ito [37] proved that the rotation interval of a degree-one circle map is closed. [3] contains a comprehensive treatment of rotation intervals and a detailed historical overview on the subject.

The concept of a *rotation vector* is a generalization of the rotation number for homeomorphisms (or flows) on higher-dimensional tori. Given a homeomorphism $f : \mathbb{T}^m \to \mathbb{T}^m$ and a lift $F : \mathbb{R}^m \to \mathbb{R}^m$ of f, the *rotation set* of $x \in \mathbb{R}^m$ is

$$\rho(x, F) = \left\{ \text{limit points of } \left(\frac{F^n(x) - x}{n} \right)_{n=1}^{\infty} \right\},$$

and the *pointwise rotation set* of F is

$$\rho_p(F) = \bigcup_{x \in \mathbb{R}^m} \rho(x, F).$$

An element $v \in \rho_p(F)$ is called a *rotation vector.*

The pointwise rotation set describes the average rotation speed and direction of individual orbits of F, but it lacks some important properties. It need not be connected or convex. Also, it is not known if $\rho_p(F)$ is closed for every homeomorphism of $\mathbb{T}^2$. Swanson and Walker [67] have constructed an example of an analytic diffeomorphism on $\mathbb{T}^3$ whose pointwise rotation set is not closed. Barge and Walker [9] provided examples of C^∞ diffeomorphisms on $\mathbb{T}^m$ $(m \geq 3)$ that have rotation sets with nonempty interior. Further, they showed that a rotation set with interior does not guarantee the existence of periodic orbits. Dumonceaux [22] obtained results pertaining to rotation sets on $\mathbb{T}^m$, where $m \geq 3$. A good survey of the main results concerning rotation vectors for toral maps and flows is provided in [70].

Rather than defining a rotation set as the union of rotation sets of individual orbits, Misiurewicz and Ziemian [52] proposed a more general definition. If $f : \mathbb{T}^m \to \mathbb{T}^m$ is a continuous map of the m-torus, homotopic to the identity, and a lift $F : \mathbb{R}^m \to \mathbb{R}^m$ of f, then the *rotation set* $\rho(F)$ is the set of all limits of convergent sequences

$$\left(\frac{F^{n_i}(x_i) - x_i}{n_i}\right)_{i=1}^{\infty},$$

where $x_i \in \mathbb{R}^m$ and $n_i \to \infty$. This generalized definition (sometimes referred to as the MZ rotation set) has been adopted in subsequent works. Misiurewicz and Ziemian [52] proved that the rotation set is compact and connected. Handel [35] showed that a periodic-point free homeomorphism of $\mathbb{T}^2$, homotopic to the identity, cannot have a rotation set with interior. Llibre and MacKay [45] analyzed the dynamics of homeomorphisms of the torus that have rotation sets with nonempty two-dimensional interior. Their results were subsequently extended in [53]. Franks [28] has shown that every vector with rational coordinates, that lies in the interior of the rotation set, is realized by some periodic orbit. Kwapisz [41] demonstrated that every convex polygon with rational vertices is realized as a rotation set of some homeomorphism of $\mathbb{T}^2$.

Franks and Misiurewicz [29] gave a complete characterization of rotation sets for the flows on $\mathbb{T}^2$. They proved that a rotation set of a toral flow is either a single point, a segment of a line through 0 with rational slope, or a line segment with irrational slope and one end point equal to 0.

In 1957, Schwartzman [63] introduced the concept of *asymptotic cycles*, which is a generalization of the rotation set for C^1 flows on compact differentiable manifolds. The rotation vector for a fixed-point free flow on $\mathbb{T}^2$ is a coordinate representation of the asymptotic cycle with respect to the standard basis in the first cohomology group [38].

Because of the way it is defined, the concept of rotation set is closely related to ergodic properties of a map or a flow. We extensively use such interdependence in this dissertation.

At the same time, significant progress has been made in the study of hyperbolic attractors. While solenoids (also called the Vietoris - van Dantzig solenoids) have been known to topologists for quite some time, they first appeared in the dynamical systems setting in 1967, in Smale's paper [64]. Smale considered two types of diffeomorphisms of the torus: the DE (Derived from Expanding) diffeomorphisms, which have solenoids as attractors, and the DA (Derived from Anosov) diffeomorphisms, whose attractors were later shown to be homeomorphic to substitution tiling spaces. Williams in a series of articles [72, 73, 74] introduced and developed the idea of representing such attractors as inverse limits. He defined *generalized solenoids* to be inverse limits of branched one-dimensional manifolds, with bonding maps that are generalizations of expanding circle maps. This inverse-limit viewpoint has been dominant in subsequent studies. Plykin [57] reformulated Smale's and Williams' results using slightly different initial assumptions.

Following Williams' work, there has been steady interest in solenoids during the last three decades. McCord [50] proved that solenoids are homogeneous. Aarts and Fokkink [1] obtained necessary and sufficient conditions for two solenoids to be homeomorphic. Hagopian [34] showed that a continuum is a solenoid if and only if it is homogeneous and all of its proper subcontinua are arcs. De Man [46] proved that any two composants of any two solenoids are homeomorphic. Various algebraic properties of solenoids were studied by Yi in a series of papers [75, 76, 77].

Multiple results have been obtained regarding maps and flows on solenoids. Some properties of continuous maps on solenoids were discovered by Ustinov [68]. Keesling [39] determined the precise topological structure of the group of homeomorphisms of a solenoid. Kwapisz [43] obtained an explicit decomposition of orientation preserving homeomorphisms on solenoids. Clark [17] investigated solenoidal homeomorphisms homotopic to the identity, and showed that a certain class of solenoids admits no expansive homeomorphisms.

As far as flows on solenoids are concerned, Aarts and Martens [2] obtained a relevant result by showing that a one-dimensional, separable and metrizable space is an orientable matchbox manifold if and only if it is the phase space of some fixed-point free flow. Since a solenoid is a matchbox manifold, fixed-point free flows on solenoids coincide with the suspensions of homeomorphisms of the underlying zero-dimensional space C.

Clark investigated flows on solenoids of any finite dimension. In [14] he introduced *linear* flows on solenoids and addressed the question of when two linear flows are equivalent. He then extended his research, giving in [16] a method of constructing flows on k-dimensional solenoids from a given flow on the torus $\mathbb{T}^k$, and showing in [15] that flows on solenoids are generically not almost periodic.

Many of the aforementioned contributions deal with generalized solenoids in Williams' setting. *Substitution tiling spaces*, which in fact form a subclass of Williams' generalized solenoids, have also been studied separately, since they exhibit many peculiar phenomena that are absent in true solenoids. The study of substitution tiling spaces has grown out of certain problems of symbolic dynamics, namely, the analysis of substitutions on finite alphabets. One of the well-known contributions in this area is a paper by Coven and Keane [19], who analyzed topological and measure-theoretic properties of substitutions of constant length on two symbols. Dekking [21] gave a complete spectral classification for substitutions of constant length. Gottschalk [30], Martin [49], and others considered the so called substitution minimal flows, which in essence are actions on a subshift of finitely many symbols by the infinite cyclic transformation group generated by the left shift homeomorphism. It was a natural step to replace this transformation group by the reals, which led to the introduction of substitution tiling spaces.

Substitution tilings of Euclidean spaces, obtained through the process of inflation and substitution of *prototiles*, have been studied by many researchers, including Kenyon [40], Mozes [54], and others. Radin and Sadun [60] developed an algebraic invariant, related to a subgroup of rotations, that helps determine when two substitution tiling systems are dynamically equivalent. Another noteworthy result is that of Solomyak [65], who looked into the spectral properties of tiling systems arising from self-affine tilings of $\mathbb{R}^d$, and proved that such systems are uniquely ergodic.

One of the first to look at substitution tiling spaces from the inverse limit viewpoint, was the paper by Anderson and Putnam [4]. They showed that substitution tiling spaces are a special case of expanding attractors, thereby providing a relation between substitution tiling theory and hyperbolic attractors. Using the inverse limit approach, they also computed cohomology for tiling spaces in one- and two-dimensional case. Sadun and Williams [62] later showed that a tiling space forms a bundle over a manifold whose fiber is a Cantor set. They also demonstrated that tiling spaces on $\mathbb{R}^d$ are suspensions of $\mathbb{Z}^d$ subshifts. A related result of local nature was obtained by Gutek [32]. This in turn establishes a connection between the topology of substitution tiling spaces and homeomorphisms of Cantor sets, which are studied in [26] and [23]. Such homeomorphisms break into two classes: the adding machines, whose suspensions are true solenoids, and subshifts generated by primitive substitutions, whose suspensions are substitution tiling spaces [7].

Among recent contributions, the work of Barge and Diamond [7] is of particular interest to us. In this article, they investigate the topological structure of one-dimensional substitution tiling spaces. First, they prove that all one-dimensional substitution tiling spaces have a finite and nonempty collection of 'asymptotic composants', thus providing a geometric insight into the structure of tiling spaces. The authors then proceed to establish a condition for two tiling spaces to be homeomorphic, using the concept of *weak equivalence* (see also [8]). The consequence of their results and [72, 73] is the following classification.

Every orientable hyperbolic one-dimensional attractor is either homogeneous, and hence a true solenoid, or nonhomogeneous, and hence a one-dimensional substitution tiling space.

In the latter case, the inhomogeneity exhibits itself in the asymptotic composants.

Among the results of geometric nature, we also mention Canterini and Siegel [13]. For Pisot substitution systems, they give an explicit continuous semi-conjugacy between the shift on the system and a translation on the torus. Queffélec [59], along with a comprehensive review of the theory, offers various new results and poses some interesting questions, including the one partially answered in [13]. Another, more recent compendium on the subject has been written by Arnoux *et al* [5].

Results

The goal of this thesis is to analyze rotation properties of flows on one-dimensional generalized solenoids, including true solenoids (which we refer to simply as solenoids), and substitution tiling spaces. Such analysis is carried out by means of investigating ergodicity and invariant measures of the flows. Fixed-point free flows and flows with fixed points are treated separately, because they exhibit substantially different ergodic properties and rotation behavior.

As we show in Chapter 3, a fixed-point free flow on a solenoid is uniquely ergodic, and its rotation set consists of just one point (which we can call the *rotation number*). We also give an explicit formula for computation of the rotation number, using return times of the flow to some cross-section of the solenoid. This formula proves handy in constructing an example of a flow that is not topologically conjugate to a linear flow.

The flows with fixed points, as we discover in Chapter 4, can manifest two types of behavior. Some of them are uniquely ergodic, and consequently have a trivial rotation set, while others admit more than one ergodic measure, and their rotation sets are intervals. We develop criteria for distinguishing between these two types of behavior, using the 'speed' of the flow.

We also address the problem of describing pointwise rotation sets for flows on solenoids. After a detailed discussion of Katok's example in its original setting (on the torus), we adapt it for the flows on solenoids. Katok's example provides an algorithm for the construction of flows whose pointwise rotation set contains points other than the extremes of the MZ rotation set. The question remains whether it is possible to construct a flow with a unique interior point of $\rho(\phi)$ being contained in $\rho_p(\phi)$, that is, with $\rho_p(\phi) = \{0, r, \rho\}$, where $\rho(\phi) = [0, \rho]$, $r \in (0, \rho)$.

Finally, we discuss the question of whether two fixed-point free flows having the same rotation number are actually topologically conjugate. Our findings show that this is not always the case. We construct an explicit example of a flow that is not topologically conjugate to a linear flow with the same rotation number.

The following theorems summarize the main results of this thesis concerning rotation and ergodicity of flows on solenoids.

Let ϕ denote a continuous flow on the solenoid $S_{\mathcal{P}}$, $\mathcal{P} = (p_1, p_2, \dots)$, and let $\tilde{\phi}$ be the lift of ϕ to $\mathbb{R} \times C$, so that $h \circ \tilde{\phi} = \phi \circ h$. Denote by $r(x)$ the return time of the flow to the cross-section of the identity $C_{\underline{e}}$, that is, $r(x)$ is such that $\phi^{r(x)}(h(0, x)) \in C_{\underline{e}}$. Let also β denote the Bernoulli measure on the Cantor set C. Ω denotes the fundamental domain in $\mathbb{R} \times C$, and $\mathcal{B}(S_{\mathcal{P}})$ the σ-algebra of Borel sets on $S_{\mathcal{P}}$.

THEOREM A. *Let ϕ be a continuous fixed-point free flow on $S_{\mathcal{P}}$, then it is uniquely ergodic, with the invariant measure given by*

$$\mu(B) = \frac{1}{\bar{r}_C} \int_C \tau(x) \, \mathrm{d}\beta, \quad \forall B \in \mathcal{B}(S_{\mathcal{P}})$$

where $\bar{r}_C = \int_C r(x) \, \mathrm{d}\beta$, $\tau(x)$ is the 'time-length' of the arc component of B with the Cantor-set coordinate equal to x.

THEOREM B. *If ϕ be a continuous fixed-point free flow on $S_{\mathcal{P}}$, then the rotation set of ϕ is $\rho(\phi) = \left\{ \frac{1}{\bar{r}_C} \right\}$.*

Now let ϕ be a differentiable flow on $S_{\mathcal{P}}$ with fixed points, such that $\frac{\mathrm{d}}{\mathrm{d}t}\,\phi^t(\dot{\underline{z}}) = \Phi(\underline{z})$, and let $\tilde{\Phi}$ be the lift of Φ to the covering space $\mathbb{R}\times C$. Ω denotes the fundamental domain in $\mathbb{R}\times C$. By $L^1(\Omega)$ we mean $L^1(\Omega)$ with respect to the Lebesgue measure λ.

THEOREM C. *Let ϕ be a flow on $S_{\mathcal{P}}$ with one fixed point p_0, $\tilde{\Phi}$ a lift of Φ, then:*

1. *If $\frac{1}{\tilde{\Phi}}$ is $L^1(\Omega)$, then ϕ is not uniquely ergodic. It has a ϕ-invariant ergodic measure μ, given by $\mu(B) = \frac{\int_B \left(1/\tilde{\Phi}\right)\mathrm{d}\lambda}{\int_\Omega \left(1/\tilde{\Phi}\right)\mathrm{d}\lambda}$, such that $\rho(\underline{z}, \phi) = \rho > 0$ for μ-almost all $\underline{z} \in S_{\mathcal{P}}$. Furthermore, such a measure is unique.*

2. *If $\frac{1}{\tilde{\Phi}}$ is not $L^1(\Omega)$, then ϕ is uniquely ergodic, with the Dirac measure δ_{p_0} being the only ϕ-invariant probability measure on $S_{\mathcal{P}}$.*

THEOREM D. *Let ϕ be a flow on $S_{\mathcal{P}}$ with fixed points, $\tilde{\Phi}$ a lift of Φ, then the following is satisfied:*

1. *If $\frac{1}{\tilde{\Phi}}$ is $L^1(\Omega)$, then $\rho(\phi) = [0, \rho]$,* where $\rho = \frac{1}{\int_\Omega \left(1/\tilde{\Phi}\right)\mathrm{d}\lambda}$.
2. *If $\frac{1}{\tilde{\Phi}}$ is not $L^1(\Omega)$, then $\rho(\phi) = \{0\}$.*

Essentially the same results for flows on substitution tiling spaces are stated in Chapter 5.

Structure of this Dissertation

This dissertation is organized into five chapters. In Chapter 1, we give an account of the history of development of the concepts of rotation sets, solenoids, and substitution tilling spaces, which we bring together in the remainder of the thesis. Here we also state the main results of the thesis.

Chapter 2 opens with basic definitions concerning solenoids, and a summary of the most important properties of solenoids. We also introduce the universal cover that can be used for both maps and flows on solenoids, and discuss typical covering projections associated with this universal cover. Then the focus of the discussion turns to homeomorphisms of solenoids, as well as their lifts. We devote a separate section to homeomorphisms homotopic to the identity. Such homeomorphisms include time-one maps of flows, therefore, their properties are extensively used in subsequent chapters. In the last section of the chapter, a known result ([43]) is stated about the decomposition of a homeomorphism of a solenoid into a composition of three maps: a homeomorphism homotopic to the identity, a group multiplication by a fixed element of the solenoid, and a shift on the inverse limit space.

In the subsequent chapters, we adopt the following method of presenting the results. Some proofs are given for the case of a flow on the dyadic solenoid, S_2, in order to maximize clarity of presentation. The majority of such proofs remain essentially the same, with obvious replacements in notation, for general solenoids $S_{\mathcal{P}}$. This being the case, the corresponding results for $S_{\mathcal{P}}$ are stated without proof, or with comments that indicate the changes that need to be made in the proofs for S_2. Whenever a proof in the general case differs substantially from the dyadic case, it is given in its most general form.

In Chapter 3, we analyze fixed-point free flows on solenoids. After a brief discussion of basic characteristics of flows and their lifts, the definitions of rotation sets are given, and a few propositions are proved, establishing some facts about rotation sets of solenoidal flows. These facts, although redundant for fixed-point free flows, prove useful for flows with fixed points, discussed in Chapter 4. The next two sections of Chapter 3 are intended for the proof of the main results of the chapter, Theorems A and B. With the unique ergodicity and triviality of the rotation set thus established, we discuss some desirable geometric properties of fixed-point free flows, and show that they are not always present. This discussion leads us to an example that illustrates a breakdown in the link between the equality of rotation sets and topological conjugacy of the corresponding flows.

Chapter 4 deals with solenoidal flows with fixed points, and some related material. The results of this chapter are developed for the case of a flow with one fixed point. Here, we show that such flows may be either uniquely or non-uniquely ergodic, obtain criteria for that to happen, and, applying these facts to rotation sets, prove the other two main results of the thesis, Theorem C and Theorem D. Due to a certain similarity between solenoidal flows and flows on the torus $\mathbb{T}^2$ that follow a foliation of the torus by vectors of irrational slope, we treat corresponding results on the torus first, translating them later to the solenoid. Also in Chapter 4, we state a theorem on the measure-theoretic realization of points in the rotation set, and discuss possibilities for the pointwise rotation set, illustrating the discussion with a famous Katok's example, adapted to the solenoid.

Finally, in Chapter 5 we restate the results of the previous two chapters for flows on substitution tiling spaces, indicating the changes that need to be made in the proofs, and giving new proofs whenever necessary. The chapter opens with definitions and background information on substitution tiling spaces, followed by the discussion of the universal cover and lifts. Next, we outline two different approaches to defining a rotation set for flows on substitution tiling spaces, and show their equivalence for primitive substitutions. Then, unique ergodicity of the underlying subshift is proved (see [59]), and Theorems A, B, C, and D are reformulated for substitution tiling spaces.

CHAPTER 2

HOMEOMORPHISMS OF SOLENOIDS AND THEIR LIFTS

Solenoids and Their Properties

Definitions and Notation

We shall now define a solenoid and state the most important properties of solenoids. To begin with, we give an intuitive explanation of these spaces.

A solenoid can be visualized in the following fashion. Let us imagine that we start with a rubber doughnut, then stretch it, making it thinner, and fold it, so it wraps around its original shape twice. The resulting double doughnut is also stretched and folded in the same way. This process is continued infinitely many times. In the limit, we obtain a solenoid.

The process described above lies at the core of one of the first definitions of a solenoid, proposed by van Dantzig [20].

Definition 2.1 (van Dantzig). Let $\mathbf{T}^2 = \mathbb{D}^2 \times S^1$ be a solid torus, and let $\mathcal{P} = (p_1, p_2, \ldots)$ be a sequence of natural numbers, such that $p_i \geq 2\ \forall i$. Suppose that $\forall k \geq 2 \quad F_k : \mathbf{T}^2 \to F_k\left(\mathbf{T}^2\right)$ is a homeomorphism given by

$$F_k\left(u, z\right) = \left(f_k\left(u, z\right), z^k\right), \quad u \in \mathbb{D}^2,\ z \in S^1,$$

where $f_k(u, z)$ is a contraction in the first argument. Define a solenoid as

$$S_{\mathcal{P}} = \bigcap_{k \geq 1} F_{p_k} \circ F_{p_{k-1}} \circ \cdots \circ F_{p_1} \left(\mathbf{T}^2\right). \tag{2.1}$$

Even though this definition is geometric and intuitive, it is not very practical. Another, more recent and convenient approach is to define solenoids through inverse limits.

DEFINITION 2.2 (WILLIAMS). Let S^1 be a unit circle, $\mathcal{P} = (p_1, p_2, \ldots)$ a sequence of natural numbers, such that $p_i \geq 2 \, \forall i$, and let $f_j : S^1 \to S^1$ be given by $f_j(z) = z^{p_j}$. Define the *$\mathcal{P}$-adic solenoid* as the following inverse limit:

$$\Sigma_{\mathcal{P}} = \varprojlim \left\{S^1, f_j\right\} = \left\{ (z_1, z_2, \ldots) \in \prod_{j=1}^{\infty} S^1 \ : \ z_j = f_j(z_{j+1}) \right\}. \tag{2.2}$$

These two definitions are in fact equivalent, as the following proposition shows.

PROPOSITION 2.3. *$S_{\mathcal{P}}$ and $\Sigma_{\mathcal{P}}$ are homeomorphic.*

PROOF. Consider the diagram in Figure 1. Here, $p\,(u, z) = z$ is a projection map, it induces the map $\hat{p} : \varprojlim \left\{\mathbf{T}^2, F_{p_j}\right\} \to \Sigma_{\mathcal{P}}$, given by

$$\hat{p}\left((u_1, z_1), (u_2, z_2), \ldots\right) = (z_1, z_2, \ldots).$$

By a straightforward argument, one shows that $\pi_1 : \varprojlim \left\{\mathbf{T}^2, F_{p_j}\right\} \to S_{\mathcal{P}}$, given by $\pi_1\left((u_1, z_1), (u_2, z_2), \ldots\right) = (u_1, z_1)$ (projection onto the first coordinate), is a homeomorphism. It can also be shown that $\hat{p}$ is a bijection. Clearly, $\hat{p}$ is continuous,

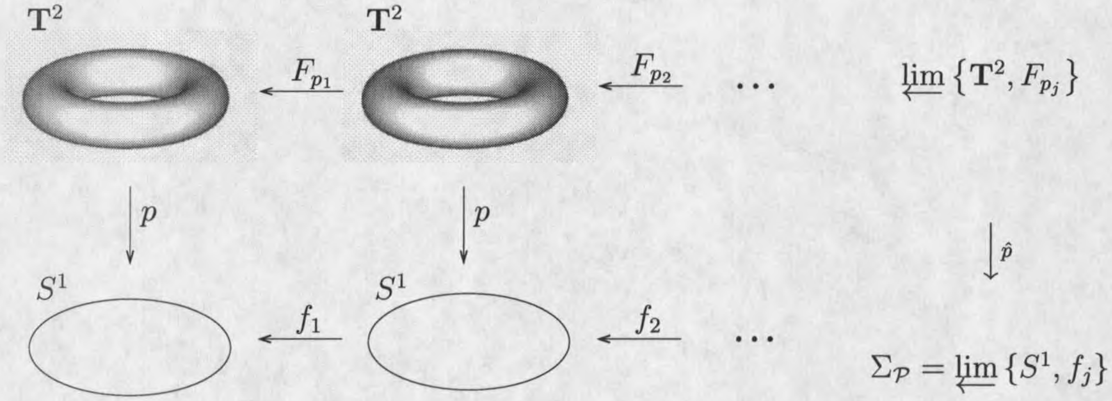

Figure 1. Homeomorphism between $S_{\mathcal{P}}$ and $\Sigma_{\mathcal{P}}$.

since the projections on every coordinate are continuous. Then $\hat{p}^{-1}$ is also continuous, and hence $\hat{p}$ is a homeomorphism. It follows that $\hat{p} \circ \pi_1^{-1} : \ S_{\mathcal{P}} \to \Sigma_{\mathcal{P}}$ is a homeomorphism. □

The second definition is easily generalized to include inverse limits of bouquéts of circles [72, 73], which we treat in Chapter 5, and inverse limits of tori that yield solenoids of higher dimensions. Clark [14] defines solenoids as inverse limits of maps of $\mathbb{T}^n$:

$$\Sigma_{\bar{M}} = \varprojlim\{\mathbb{T}^n, f_j\}, \tag{2.3}$$

where $\bar{M} = \{M_1, M_2, \dots\}$ is a sequence of $n \times n$ matrices with integer entries and nonzero determinants, $f_j : \ \mathbb{T}^n \to \mathbb{T}^n$ are automorphisms represented by the matrices M_j. In this setting, $\Sigma_{\bar{M}} \subset \prod_{j=1}^{\infty} \mathbb{T}^n$.

Such solenoids occur as attractors of hyperbolic dynamical systems. Examples of such dynamical systems are given in [24], [38], and [61].

The fact that solenoids occur as attractors justifies the importance of studying them, since properties of an attractor can give an insight into the behavior of the dynamical system. In this dissertation, we concentrate on one-dimensional solenoids.

NOTATION. Throughout the rest of this dissertation, we denote a $\mathcal{P}$-adic solenoid by $S_{\mathcal{P}}$, and use Definition 2.2. Furthermore, by $C_{\underline{w}}$ we denote the cross-section of $S_{\mathcal{P}}$ through the element $\underline{w} = (w_1, w_2, \ldots) \in S_{\mathcal{P}}$:

$$C_{\underline{w}} \equiv C_{w_1} = \{(z_1, z_2, \ldots) \in S_{\mathcal{P}} \ : \ z_1 = w_1\},$$

and by $S_{\underline{w}}$ we denote the arc component that contains $\underline{w} \in S_{\mathcal{P}}$. We equip $S_{\mathcal{P}}$ with the subspace topology from the product topology on $\prod_{i=1}^{\infty} S^1$.

PROPOSITION 2.4. *$S_{\mathcal{P}}$ is a compact abelian topological group, with the group operation (denoted '$*$') given by factor-wise multiplication, and the identity element $\underline{e} = (1, 1, \ldots)$.*

PROOF. Compactness follows from $S_{\mathcal{P}}$ being a continuum. The fact that $S_{\mathcal{P}}$ is a topological group follows because f_j is a topological group homomorphism for all j, and the inverse limit of topological group homomorphisms is a topological group. $\square$

In the next subsection and throughout this thesis, we use the concept of an *adding machine*, which we now introduce. Roughly, the adding machine is 'addition with carry': add 1 to the first coordinate and carry to the right. Our more precise definition follows that of [10].

DEFINITION 2.5. Consider the Cantor set $C_{\mathcal{P}} = \prod_{i=1}^{\infty} (\mathbb{Z} \bmod p_i)$, where $\mathcal{P} = (p_1, p_2, \dots)$. The adding machine operator $A: \ C_{\mathcal{P}} \to C_{\mathcal{P}}$ is defined by

$$A: \ (\dots, \alpha_k, \dots) \mapsto (\dots, \gamma_k, \dots),$$
$$\text{so that } \gamma_k = \begin{cases} 0 & : \ 1 \le k < r \\ \alpha_r + 1 & : \ k = r \\ \alpha_k & : \ k > r \end{cases}, \text{ where } \alpha_k = \begin{cases} p_k - 1 & : \ 1 \le k < r \\ \alpha_r & : \ k = r \\ \alpha_k & : \ k > r \end{cases},$$
$$\alpha_r < p_r - 1, \quad 1 \le r \le \infty.$$

Let $\underline{z} = (z_1, z_2, \dots, z_j, \dots) \in S_{\mathcal{P}}$, where $\mathcal{P} = (p_1, p_2, \dots)$, then

$$z_1 = z_2^{p_1}, \ z_2 = z_3^{p_2}, \ \dots, \ z_j = z_{j+1}^{p_j}, \ \dots$$

For every value of $z_j \in S^1$ there are p_j points z_{j+1} in S^1 with $z_{j+1}^{p_j} = z_j$. In other words, if the value of z_j is known, we can order the p_j-roots of it and use the number of the root to describe the value of z_{j+1}.

Define a function $\chi : C_{\underline{e}} \to C_{\mathcal{P}}$ by

$$\chi(\underline{z}) = (s_1, s_2, \dots, s_j, \dots) \quad \text{if and only if} \quad \underline{z} = (1, z_2^{(s_1)}, \dots, z_{j+1}^{(s_j)}, \dots) \tag{2.4}$$

where $z_{j+1}^{(s_j)}$ is the (s_j+1)-st of the p_j-th roots of z_j, counting counterclockwise, starting with the root that has the smallest argument. More precisely,

$$s_j = \frac{(\arg z_{j+1})p_j - \arg z_j}{2\pi}.$$

We can call χ the *$\mathcal{P}$-adic representation* of $C_{\underline{e}}$.

Then the adding machine $\hat{A}$ on $C_{\underline{e}}$, induced by A, is given by $\hat{A} = \chi^{-1} A \chi$. Abusing notation, we shall denote this adding machine also by A.

In the following discussion, we shall also use the concept of a natural flow on a solenoid, as defined in [16].

DEFINITION 2.6. The *natural flow* on a solenoid $S_{\mathcal{P}}$ is defined as follows:

$$\varphi^t(\underline{z}) = \underline{z} * \pi_{\mathcal{P}}(t), \tag{2.5}$$

where $\pi_{\mathcal{P}}: \mathbb{R} \to S_{\underline{e}}$ is the continuous isomorphism onto the arc component of the identity $S_{\underline{e}}$, given by

$$\pi_{\mathcal{P}}(t) = \left(\exp(2\pi i t), \exp\left(\frac{2\pi i t}{p_1}\right), \exp\left(\frac{2\pi i t}{p_1 p_2}\right), \ldots, \exp\left(\frac{2\pi i t}{p_1 p_2 \ldots p_j}\right), \ldots\right). \tag{2.6}$$

REMARK 2.7. In the case of the dyadic solenoid S_2, we have

$$\pi_2(t) = \left(\exp(2\pi i t), \exp(\pi i t), \exp\left(\frac{\pi}{2} i t\right), \ldots, \exp\left(\frac{\pi}{2^{j-1}} i t\right), \ldots\right). \tag{2.7}$$

REMARK 2.8. The Poincaré map of the natural flow is conjugate to the adding machine on the cross-section $C_{\underline{e}} = \{\underline{z} \in S_{\mathcal{P}} : z_1 = 1\}$.

REMARK 2.9. The topology on $S_{\mathcal{P}}$ is induced by the metric

$$\underline{d}(\underline{z}, \underline{w}) = \sum_{j=1}^{\infty} \frac{1}{2^j} d(z_j, w_j), \tag{2.8}$$

where $d(z_j, w_j)$ is a metric on S^1.

Properties of Solenoids

We now list some properties of solenoids.

PROPOSITION 2.10. *$S_{\mathcal{P}}$ is a matchbox manifold, i.e. locally a product of a Cantor set with an arc.*

PROOF. Any segment of an arc component can be defined as

$$S = \left\{\varphi^t\left(\underline{z}\right) \ : \ 0 \leq t \leq T, \ \underline{z} \in S_{\mathcal{P}}\right\}.$$

It immediately follows that S is homeomorphic to an arc $[0, T]$. Thus we only need to show that a cross-section $C_{\underline{w}} = \{\underline{z} \in S_{\mathcal{P}} \ : \ z_1 = w_1 =: w\}$ of $S_{\mathcal{P}}$ is a Cantor set.

To that end, consider the spaces

$$X_1 = \{w\}, \quad X_2 = f_1^{-1}(X_1), \quad X_3 = f_2^{-1}(X_2), \quad \ldots$$

Every X_i (Figure 2) is contained in S^1, is finite, and the subspace topology on it is discrete.

Then we represent C_w as an inverse limit, $C_w = \varprojlim\{X_i, g_i\}$, where $g_i = f_i\,|_{X_{i+1}}$ for all i. Note that $g_i(z) = z^{n_i}$ and $\forall\, x_i \in X_i, \ k \geq i$,

$$\begin{aligned} \lim_{k\to\infty} \operatorname{card}\left(g_k^{-1} \circ \cdots \circ g_i^{-1}\left(\{x_i\}\right)\right) &= \lim_{k\to\infty} \prod_{j=i}^{k} p_j \\ &\geq \lim_{k\to\infty} 2^{k-i-1} = \infty \end{aligned}$$

and apply the well-known Theorem 2.11. □

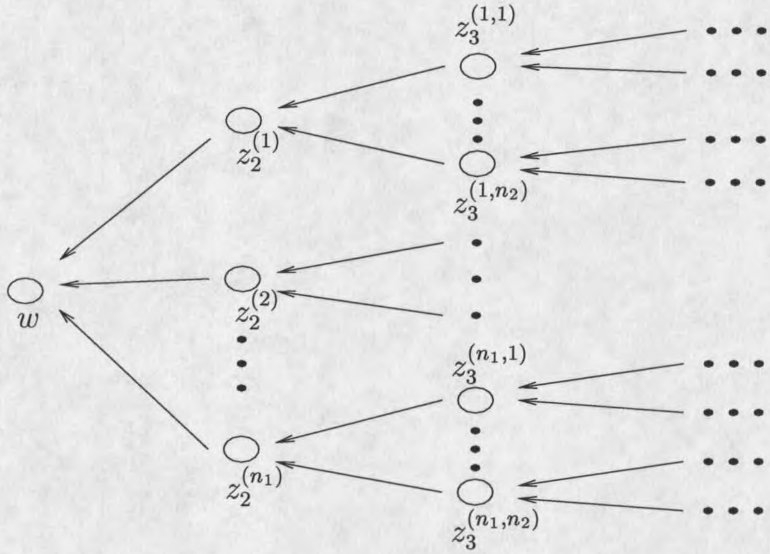

Figure 2. Finite approximations of C_w.

THEOREM 2.11. *Let* $\{X_i, f_i\}$ *be an inverse sequence, where every* X_i *is a finite space with the discrete topology, every* f_i *is onto, and for each* i, *if* $x_i \in X_i$, *then*

$$\lim_{k\to\infty} \text{card}\left(f_n^{-1} \circ \cdots \circ f_i^{-1}\left(\{x_i\}\right)\right) = \infty$$

(i.e. the number of preimages of any x_i *tends to* ∞*). Then* $X = \varprojlim\{X_i, f_i\}$ *is a Cantor set.*

A discussion of further properties of solenoids that are consequences of Proposition 2.10 can be found in [25].

PROPOSITION 2.12. $S_{\mathcal{P}}$ *is a homogeneous continuum, i.e. for any two points in* $S_{\mathcal{P}}$, *there is a homeomorphism of* $S_{\mathcal{P}}$ *that swaps these two points.*

PROOF. This statement follows from Proposition 2.4 and the fact that all topological groups are homogeneous. □

PROPOSITION 2.13. *Every nondegenerate proper subcontinuum of* $S_{\mathcal{P}}$ *is an arc.*

PROOF. The proof is easy and well-known. See [55]. □

We note that the last two propositions actually characterize solenoids. Hagopian [34] proved the following theorem:

THEOREM 2.14 (HAGOPIAN). *M is a homogeneous continuum and every proper subcontinuum of M is an arc if and only if M is a solenoid.*

Mardesic and Segal [48] showed that a solenoid is a circle-like continuum. Moreover, every homogeneous circle-like continuum that contains an arc is a solenoid [11].

NOTATION. By $S(\underline{z})$ we denote the arc component of $S_{\mathcal{P}}$ containing $\underline{z}$. Set

$$S^{+}(\underline{z}_0) = \{\, \varphi^t(\underline{z}_0) \quad : \quad t \geq 0 \,\},$$

$$S^{-}(\underline{z}_0) = \{\, \varphi^t(\underline{z}_0) \quad : \quad t \leq 0 \,\}.$$

$d_{\text{arc}}(\underline{x}, \underline{y})$ denotes the signed distance between $\underline{x}$ and $\underline{y}$ along the arc component of $S_{\mathcal{P}}$:

$$d_{\text{arc}}(\underline{x}, \underline{y}) = l \quad \text{if and only if} \quad \underline{y} = \varphi^l(\underline{x}). \tag{2.9}$$

Universal Cover and Covering Projections

As we mentioned in the previous section, a solenoid $S_{\mathcal{P}}$ is locally a product of an arc with a Cantor set. So the natural choice for a covering space would be the product of the real line and a Cantor set, $\mathbb{R} \times C$, where $C = C_{\underline{e}}$. Let $h: \mathbb{R} \times C \to S_{\mathcal{P}}$ denote the desired covering projection. We would like h to satisfy the following properties:

1. h maps $[0,1) \times C$ one-to-one onto $S_{\mathcal{P}}$.
2. h is a covering map (in particular, it is continuous and a local homeomorphism.)
3. h is orientation preserving.
4. The lift of the natural flow φ by h is a linear flow on $\mathbb{R} \times C$, that is, a flow with a constant speed across all arc components of $\mathbb{R} \times C$.

REMARK 2.15. A *cylinder set* of C is a set of the form:

$$U_{q_1 q_2 \ldots q_k} = \chi^{-1}\left\{x \in \prod_i (\mathbb{Z} \bmod p_i) \quad : \quad x_1 = q_1, x_2 = q_2, \ldots, x_k = q_k \text{ fixed}\right\} \quad (2.10)$$

We now turn to the construction of the covering projection. Define h as follows:

$$h(t, x) = \varphi^t(x). \quad (2.11)$$

By definition, h maps $[0,1)\times C$ (or, for that matter, any set $[t, t+1)\times C, t \in \mathbb{R}$) one-to-one onto $S_{\mathcal{P}}$. Also by definition of h, the lift of φ^t to $\mathbb{R}\times C$ is $\tilde{\varphi}^t : (\tau, x) \mapsto (\tau + t, x)$. Since $\tilde{\varphi}|_{\mathbb{R}\times\{x\}}$ and $h|_{\{0\}\times C}$ are continuous, and since $\mathbb{R}\times C$ has the product topology, we know h is continuous. Also, if $\mid t_2 - t_1 \mid < 1$, then $h \mid_{[t_1,t_2]\times C}$ is one-to-one onto its image, $(h \mid_{[t_1,t_2]\times C})^{-1}$ is continuous, since φ^{-1}, $(h \mid_{\{0\}\times C})^{-1}$ are continuous. Thus h is a local homeomorphism. The following propositions will assist us in showing that h is a covering map.

The map h defined above possesses another intrinsic property, which we now describe. For simplicity, we shall denote a point $(0, x) \in \{0\} \times C$ by x. Now by Remark 2.8,

$$h(1, x) = \varphi^1 h(0, x) = h(0, Ax). \tag{2.12}$$

This implies the following two propositions.

PROPOSITION 2.16. *If h is the map defined by (2.11), then*

$$h(t + 1, x) = h(t, Ax) \quad \forall (t, x) \in \mathbb{R} \times C. \tag{2.13}$$

PROPOSITION 2.17. *If h is the map defined by (2.11), then*

$$h(t + n, x) = h(t, A^n x) \quad \forall (t, x) \in \mathbb{R} \times C, \forall n \in \mathbb{Z}. \tag{2.14}$$

Conversely, if $(t_1, x_1), (t_2, x_2) \in h^{-1}\{\underline{z}\}$, then

$$(t_2, x_2) = (t_1 - n, A^n x_1), \quad \textit{for some } n \in \mathbb{Z}. \tag{2.15}$$

For convenience, we define a map

$$T(t, x) = (t + 1, A^{-1}x). \tag{2.16}$$

If $\underline{p} \in S_{\mathcal{P}}$, take an open neighborhood U of $\underline{p}$, of diameter less than $\frac{1}{2}$. Let $\tilde{U} = h^{-1}|_{[0,1)\times C}(U) \subset [0,1) \times C$. Then the collection $\left\{T^n(\tilde{U})\right\}_{n=-\infty}^{\infty}$ evenly covers U. Each $T^n(\tilde{U}) \subset [t, t+1) \times C$ for some $t \in \mathbb{R}$. Also, $h|_{[t,t+1)\times C}$: $[t, t+1) \times C \to S_{\mathcal{P}}$ is a homeomorphism, so h maps every $T^n(\tilde{U})$ homeomorphically onto U. It follows that h is a covering map.

Note also that $h^{-1}(C_{\underline{e}}) = \mathbb{Z} \times C$.

Lifts of Solenoidal Maps

DEFINITION 2.18. A homeomorphism $f: S_{\mathcal{P}} \to S_{\mathcal{P}}$ is called orientation preserving if whenever $\underline{y} = \underline{x} * \pi_{\mathcal{P}}(t)$, $t > 0$, we have $f(\underline{y}) = f(\underline{x}) * \pi_{\mathcal{P}}(\tau)$, $\tau > 0$. It is called orientation reversing, if $t > 0$, $\tau < 0$, respectively.

In other words, $\forall t > 0 \quad \forall \underline{x} \in S_{\mathcal{P}} \quad f(\varphi^t(\underline{x})) = \varphi^\tau(f(\underline{x}))$, $\tau > 0$.

DEFINITION 2.19. Let $f: S_{\mathcal{P}} \to S_{\mathcal{P}}$ be an orientation preserving homeomorphism on $S_{\mathcal{P}}$. A lift of f is a map $F: \mathbb{R} \times C \to \mathbb{R} \times C$, such that F is continuous and

$$h \circ F = f \circ h, \tag{2.17}$$

where $h: \mathbb{R} \times C \to S_{\mathcal{P}}$ is the covering projection.

Note that in general, a lift of f is not unique, however, as we show in the next section, a lift of a homeomorphism homotopic to the identity, that is also homotopic to the identity, is in fact unique. We also note that a lift of a homeomorphism is not necessarily a homeomorphism, as the following example demonstrates.

EXAMPLE 2.1. Consider the identity homeomorphism $f(\underline{z}) = \underline{z}$ of S_2. Decompose C into a union of two clopen sets, C_0, C_1, that permute under the adding machine. Define a lift F, so that

$$[0,1) \times C_0 \overset{F}{\mapsto} [0,1) \times C_0, \quad [0,1) \times C_1 \overset{F}{\mapsto} [1,2) \times C_0.$$

Then F is a lift of f, but it is not a homeomorphism.

PROPOSITION 2.20. *Let* $F: \mathbb{R} \times C \to \mathbb{R} \times C$ *be a lift of* $f: S_{\mathcal{P}} \to S_{\mathcal{P}}$, *then*

$$F\left(t+n, A^{-n}x\right) = \left(s + k_n(x), A^{-k_n(x)}y\right), \tag{2.18}$$

where $(s,y) = F(t,x)$, *and* $k_n(x)$ *is an integer that depends on* $x \in C$, $n \in \mathbb{Z}$. *In particular,*

$$F\left(t+1, A^{-1}x\right) = \left(s + k(x), A^{-k(x)}y\right). \tag{2.19}$$

PROOF. By properties of the covering projection h, we have $h(t+n, A^{-n}x) = h(t,x)$. So $f \circ h(t+n, A^{-n}x) = f \circ h(t,x)$. By Equation (2.17),

$$h \circ F(t+n, A^{-n}x) = f \circ h(t+n, A^{-n}x) = f \circ h(t,x) = h \circ F(t,x).$$

Now again by properties of h, $\exists k = k_n(x)$, such that

$$F\left(t+n, A^{-n}x\right) = \left(s + k_n(x), A^{-k_n(x)}y\right).$$

$\square$

Remark 2.21. Using the function T, defined in (2.16), Equation (2.19) can be rewritten as

$$F \circ T(t,x) = T^{k(x)} \circ F(t,x). \tag{2.20}$$

Proposition 2.22. *A composition of lifts is a lift, i.e. if F is a lift of f, and G is a lift of g, then $G \circ F$ is a lift of $g \circ f$.*

Proof. $h \circ G \circ F = g \circ h \circ F = g \circ f \circ h.$ □

For our purposes, lifts of homeomorphisms are more important than lifts of any continuous maps, therefore, we establish several properties of such lifts. In the following lemmas, it is assumed that $f: S_{\mathcal{P}} \to S_{\mathcal{P}}$ is an orientation preserving homeomorphism, and F a lift of f.

Lemma 2.23. *F maps arc components to arc components, i.e. if $t_1, t_2 \in \mathbb{R}, x \in C$, then $F(t_1,x) = (s_1,y), \quad F(t_2,x) = (s_2,y)$ for some $y \in C$.*

Proof. The set $[t_1,t_2] \times \{x\}$ is arc connected, so its image under a continuous map F must also be arc connected. Thus $F(t_1,x)$ and $F(t_2,x)$ belong to the same arc component. □

Lemma 2.24. *F is 'strictly' orientation preserving on the arc components $\mathbb{R} \times \{x\}, x \in C$. That is, if $(s_1,y) = F(t_1,x)$, $(s_2,y) = F(t_2,x)$, and $t_1 < t_2$, then $s_1 < s_2$.*