An investigation of prospective secondary mathematics teachers conceptions of proof and refutations
by Kathy Jo Riley

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Education
Montana State University

Abstract:
This research study was conducted to investigate prospective secondary mathematics teachers' conceptions of proof and refutations as they were near completion of their preparation program. To research the primary question of the study, the researcher addressed two components of participants' conceptions of proof—1) understanding of the logical underpinnings of proof, and 2) ability to complete mathematical proofs. The researcher developed a questionnaire composed of two parts in order to assess the two components of proof. Both components focused on direct proof indirect proof and refutations. The sample for the study were 23 prospective secondary mathematics teachers that had completed an introduction to proof course, geometry course, and at least two calculus courses.

Results show that only 30% of the prospective teachers correctly answered 9 or more items, of 12 items, for the logical underpinnings of proof. The results show that participants have a weak understanding of the truth of a conditional statement and its related statements (e.g., converse, negation of conditional statement).

Examining prospective teachers' ability to complete mathematical proofs show that only 57% of the participants were able to write a valid direct proof of the Perpendicular Bisector Theorem, a proof common to the high school geometry curriculum. Only 39% of the participants were able to write a valid indirect proof about even integers. Results show that only 39% of the sample recognized and were able to refute a false conjecture about perimeter and area of rectangles.

Results of participants' overall performance on both parts of the questionnaire show that 52% of the sample scored 60% or less on both parts of the questionnaire. The vision of the MAA (1998) and the NCTM (2000) recommendations for teaching reasoning and proof to all students grades K - 12, and in all mathematics content areas, may not be attainable by all of the prospective secondary mathematics teachers in this study. The findings suggest that prospective teachers need more experiences in determining the true values of conjectures and that there is a correlation between an individual's understanding of the logical underpinnings of proof and ability to complete proofs.

AN INVESTIGATION OF PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' CONCEPTIONS OF PROOF AND REFUTATIONS

By

Kathy Jo Riley

A thesis submitted in partial fulfillment
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APPROVAL

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This dissertation has been read by each member of the dissertation committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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Results of participants' overall performance on both parts of the questionnaire show that 52% of the sample scored 60% or less on both parts of the questionnaire. The vision of the MAA (1998) and the NCTM (2000) recommendations for teaching reasoning and proof to *all* students grades K – 12, and in *all* mathematics content areas, may not be attainable by *all* of the prospective secondary mathematics teachers in this study. The findings suggest that prospective teachers need more experiences in determining the true values of conjectures and that there is a correlation between an individual's understanding of the logical underpinnings of proof and ability to complete proofs.

CHAPTER 1

STATEMENT OF THE PROBLEM

Current mathematics education reform efforts call for an increased emphasis in our school curricula on reasoning and proof as a stepping stone toward logical reasoning. The *Principles and Standards for School Mathematics* emphasized the need for opportunities in mathematical reasoning and proof for *all* students grades K – 12 and in *all* mathematics content areas (National Council of Teachers of Mathematics [NCTM], 2000).[1] The document deems these opportunities as essential to understanding mathematics. The NCTM added, "By the end of secondary school, students should be able to understand and produce mathematical proofs—arguments consisting of logically rigorous deductions of conclusions from hypotheses—and should appreciate the value of such arguments" (2000, p. 56).

The NCTM further noted in the Teaching Principle of the *Standards* 2000 that students' understanding of mathematics, their ability to use it to solve problems, and their confidence and disposition toward mathematics are all shaped by the teaching they encounter in school. As a cornerstone to their reform vision, the NCTM emphasized that mathematics teachers must develop and retain the mathematical and pedagogical

1 Herein the *Principles and Standards for School Mathematics* (NCTM, 2000) document is referred to as the *Standards* 2000, whereas the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) are referred to as the *Standards* 1989.

knowledge needed to effectively teach their students. With respect to proof and reasoning, the NCTM emphasized the need for teachers to understand proof and reasoning in order to help students develop critical thinking and reasoning skills.

To establish a backdrop for the study, Chapter One traces the history of proof and reasoning in the mathematics curriculum by examining the recommendations of selected reform efforts. When comparing current reform efforts with those of the past, important trends that emerge are the increased attention given to the preparation of teachers and the goal that proof and reasoning should be taught at all grade levels and in all mathematical content areas. Proof is a very difficult area for high school students—a fact that is noted by the NCTM (2000, p. 56) and supported by many research studies (see Chapter Two). Additionally, research suggests that teachers' content knowledge affects students' performance. The chapter concludes by raising the question of whether prospective secondary teachers are prepared to teach reasoning and proof, as measured by their understanding of these topics, as they embark on their teaching careers.

Proof and Reasoning in the Mathematics Curriculum

Prior Reform Efforts

In the 1930's, Fawcett addressed the need for reform within the mathematics curriculum (1938). Much like the mathematical reforms that would follow, Fawcett was concerned about the development of critical and reflective thought. According to Fawcett, the area of the curriculum most appropriate for addressing these concerns was geometry, since it was accepted as the primary course for acquainting students with the nature of deductive thought. Fawcett described geometric proof as a means for cultivating critical and reflective thought—a convention that would become standard in the high school mathematics curriculum.

Fawcett's report indicated that the transfer of logical reasoning to situations outside of geometry was dependent on efficient instruction. Fawcett added, however, that most geometry teachers were not teaching in a manner that facilitated this transfer. The current reform efforts of the NCTM seem to reflect Fawcett's concern about the development of critical thinking skills and the need for improvement in teacher preparation at the collegiate level. One of the most significant recommendations of the *Standards* 2000 was that proof and reasoning should be taught in all areas of school mathematics and not restricted to the geometry course.

In the late 1950's and early 1960's, efforts to reform school mathematics were referred to as the New Math movement. The New Math movement included the School Mathematics Study Group (SMSG), which was founded in 1958 as a national organization devoted to the improvement of mathematics programs. The SMSG's objectives included the development of an improved mathematics curriculum that would emphasize basic skills and promote a deeper understanding of basic concepts and the axiomatic structure of mathematics, logic, and methods of deductive proof. (Begle, 1968; Hanna, 1991). The New Math movement strove to align mathematics instruction in the public schools with the standards of practice by mathematicians. To establish their visions of an improved curriculum, the SMSG wrote and published textbooks that they deemed to be pedagogically and mathematically sound. These texts were to serve as models for what should be used in teaching school mathematics (Begle, 1968).

Unfortunately, the New Math movement and its emphasis on proof is widely considered a failure. Hanna and Jahnke (1993) attributed the demise of the New Math to exaggerated formalism, unsuccessful teaching, and a critical public, all of which eventually led to a critical reassessment of mathematics education. Likewise, Hirschi (1977) attributed the failure to too much emphasis on curriculum and too little on teacher preparation. Instead of considering the training of teachers, Hirshi maintains the New Math movement tried to establish reform through the introduction of textbooks:

> If we had moved in a more cautious manner by a massive retraining of mathematics teachers at the inservice and college level, I believe we could have warded off most of our present difficulties (1977, p. 244).

More recent reform efforts are often traced to the publication of the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). The *Standards* 1989 set forth goals as a guideline for the learning and teaching of school mathematics. The document emphasized a dynamic form of literacy instead of the rigor of deductive proof (NCTM, 1989). "Making conjectures, gathering evidence, and building an argument to support such notions are fundamental to doing mathematics" (NCTM, 1989, p. 7). Tall (1992) noted that the *Standards* 1989 recommended increased attention should be given to deductive arguments expressed orally and in paragraph form and decreased attention to Euclidean geometry as a complete axiomatic system and two-column proofs. Many in the mathematics education and mathematics community, however, view the *Standards* 1989 as having a relaxed emphasis on deductive proof (Chazan, 1989).

Current Reform Efforts

A significant change from the *Standards* 1989 to the *Standards* 2000 document was the latter document's increased emphasis on reasoning and proof. In 1996, the NCTM's Commission on the Future of the *Standards* asked a group within the Mathematical Association of America (MAA) to respond to the role of proof and reasoning in school mathematics (Ross, 1998).[2] The MAA Task Force was asked what mathematical reasoning skills should be emphasized, how the *Standards* 2000 should address proof, and how to address topics within mathematical structure (Ross, 1998).

The Task Force emphasized that the foundation of mathematics is reasoning. They stated that the goals of school mathematics should include opportunities to learn logical reasoning, develop valid arguments or proofs, and criticize the arguments of others (Ross, 1998). According to the Task Force, students should understand that science verifies through observation and mathematics verifies through logical reasoning. The distinction among illustrations, conjectures, and proofs should be emphasized and students should understand that mathematical results are valid only after they have been carefully proved.

The MAA Task Force outlined seven key points of reasoning and proof that students should learn beginning in their eighth-grade mathematics course:

1. distinguish between inductive and deductive reasoning and explain when each is appropriate;
2. understand the meaning of logical implication, in particular, be able to identify the hypothesis and conclusion in a deduction;

[2] The group formed was referred to as the President's Task Force on the NCTM *Standards* herein referred to as the Task Force or MAA Task Force.

3. test an assertion with examples;
4. realize that one counterexample is enough to show that an assertion is false;
5. clearly recognize that the truth of an assertion in a few cases does not allow one to conclude that it is true in all cases;
6. recognize whether something is being proved or is merely being given a plausibility argument;
7. identify logical errors in chains of reasoning involving more than one step (Ross, 1998, p. 254).

The Task Force also recommended that the *Standards* 2000 document include material on logic and mathematical language appropriate for the different grade levels. This material includes mathematical sentences with basic connectives in logic, quantifiers such as "all", and true and false statements, and strategies for justifying or refuting statements.[3]

In 2000, the NCTM released the *Principles and Standards for School Mathematics* after considering recommendations from the MAA. This document included a new Reasoning and Proof Standard, which provided a guideline for the education of students in pre-kindergarten through grade 12. The Reasoning and Proof Standard recommended that students in *all* grade levels should be able to establish the validity of a true conjecture and find a counterexample that demonstrates a conjecture is false (NCTM, 2000).

The explicit goals of NCTM 2000 Reasoning and Proof Standard for grades 9 – 12 recommended that students should be able to:

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs; and
- select and use various types of reasoning and methods of proof (2000, p. 342).

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[3] The five basic connectives in logic are: 1) not, 2) and, 3) or, 4) "if – then" statements, and 5) "if and only if" statements (Esty, 2000, p. 150).

In addition to these goals, the *Standards* 2000 recommended that proof should be a focus of *all content areas* of grades 9 – 12, and that students should understand and be able to use proof techniques. These techniques include direct proof, indirect proof, and also the understanding that a counterexample refutes a false conjecture.

The *Standards* 2000 state that opportunities to explore mathematical ideas and ponder questions such as "Why does that work?" help students develop conjectures. A natural way for students to justify their conjectures can be through mathematical reasoning and proof. According to Sowder and Harel (1998), if we are to promote mathematics as reasoning, students must be continuously exposed to the value of arguments that focus on the *why* of mathematics results and not just on the results themselves.

In summary, current reform efforts in mathematics education strongly support reasoning and proof for *all* students and in *all* mathematics courses. These recommendations appear to be "lofty goals", given results of prior research studies that have shown students' abilities to understand proof is inadequate. For example, Senk's (1985) study found that only 30% of a sample of 1,520 high school students in a full-year geometry class that emphasized traditional two-column proofs, achieved a 75% mastery level of proof. The new visions of mathematics classrooms bear little resemblance to the classrooms in which prospective teachers themselves were students. Therefore, prospective secondary teachers may need a more thorough, and possibly different, preparation if the NCTM vision is to be realized. The catalyst for building stronger understandings of reasoning and proof within a mathematics classroom is the teacher.

Teacher Preparation

The *Standards* 2000 proposed students will better understand mathematics if they develop their mathematical reasoning and proof skills. The NCTM further note that teachers themselves need to understand proof and reasoning in order to help students develop critical thinking and reasoning skills. The NCTM's *Professional Standards for Teaching Mathematics* (1991) emphasized that mathematics teachers' experiences while learning mathematics have a powerful impact on the education they provide their students. The need for teachers to promote the development of their students' understanding of proof suggests that teachers themselves must have a robust understanding of proof. This robust understanding develops through opportunities to explore, conjecture, develop mathematical arguments, validate possible solutions, and recognize connections among mathematical ideas (NCTM, 1991).

The National Commission on Teaching and America's Future [Glenn Commission] published a report in which they reiterate the NCTM's concerns on the quality of mathematics and science teaching in the nation (2000).[4] The report identified possible solutions, suggesting that students must improve their performance in science and mathematics and that the most direct route to improving mathematics achievement for all students is better teaching. The report stated, "...better teaching is the lever for change" (Glenn Commission, 2000, p. 14). It stressed that *better teaching* implied the

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4 The report was entitled *"Before It's Too Late: A Report to the Nation from the National Commission on Mathematics and Science Teaching for the 21st Century"* (2000).

need for professional teacher development programs to *better prepare* mathematics teachers to meet the needs and demands in school mathematics. "A focused professional development experience led by qualified teachers, mentors, and colleagues is the indispensable foundation for competence and high-quality teaching" (Glenn Commission, 2000, p. 14).

In 2001 the Conference Board of the Mathematical Sciences (CBMS) published a report entitled *The Mathematical Education of Teachers*.[5] The CBMS acknowledged that teachers must develop mathematical knowledge for teaching because this knowledge allows teachers to evaluate their students' work and recognize the sources of student understanding and errors. In the area of proof, the CBMS emphasized that prospective teachers at all grade levels need experience justifying conjectures and added, "Future high school teachers must develop a sound understanding of what it means to write a formal proof" (CBMS, 2001, p. 14).

Mathematics Teachers' Knowledge

The emphasis on teacher preparation raises the question of what knowledge teachers need to teach mathematics. "Teachers need to know and use 'mathematics for teaching' that combines mathematical knowledge and pedagogical knowledge" (NCTM, 2000, p. 370). Preparing teachers to enact recommendations set forth by the NCTM has been a focus of reform efforts in teacher preparation programs. In *Knowing and Learning Mathematics for Teaching*, the Mathematical Sciences Education Board

[5] The CBMS is affiliated with the Mathematical Association of America.

(MSEB) state, "Many experts now agree that reforming teacher preparation in post-secondary institutions is central to sustaining and deepening efforts to provide quality mathematics education for all students" (National Research Council, 2001, p. 1). In steps toward reform of teacher preparation programs, the MSEB posed two questions: 1) "What is the mathematical knowledge teachers need to know to teach well?"; and 2) "How can teachers develop the mathematical knowledge they need to teach well?" (2001, p.2).

The concern of what knowledge is necessary to facilitate recommended changes within school mathematics has been the focus of many in mathematics education (Ball, 1989; Cooney, 1999; Fischbein, 1990; Galbraith, 1982; Shulman 1986). Ball (1989) stated that pre-service teachers are not likely to know and understand mathematics in ways that they will need in order to teach. She added that prospective teachers' schooling shapes their understanding of mathematics. "As this is the mathematics they will *teach*, what they have learned about the subject matter in elementary and high school turns out to be a significant component of their preparation for teaching" (Ball, 1989, p. 3).

Galbraith (1982) voiced similar concerns about the conceptions of secondary mathematics teachers and what these conceptions imply about their ability to teach school mathematics. He stated, "Concern has been expressed for the *recycling effect* induced when students lacking in some essential mathematical background, return to the education system as teachers" (1982, p. 91, emphasis added).

Concern about teachers' knowledge was also raised by Cooney (1999), who noted that previous conceptions of teachers' knowledge were based primarily on their knowledge of mathematics, and that reforming mathematics teacher education basically meant requiring teachers to take more mathematics courses (1999). Mathematical knowledge is needed for teachers to be able to teach students. In addition, teachers must have the ability to transfer their knowledge to a form that students are able to understand and learn. Cooney (1999) and Shulman (1986) raised concerns that even though the pre-service teachers have a strong mathematical background, the translation of that mathematical knowledge into meaningful tasks for students is often suspect. Both Galbraith (1982) and Cooney (1999) emphasized the need for teacher preparation programs to help teachers make connections with the mathematics they eventually will be teaching.

Evidence supports the idea that a teacher's *sound* conceptual understanding of mathematics can influence students' learning in a positive way (Fennema & Franke, 1992). Researchers often characterize conceptual understanding in terms of two constructs: *concept definition* and *concept image* (Tall and Vinner, 1981). Chapter Two of this research study addresses these constructs in detail and their relationship to an individual's conceptual understanding of proof. Examining the skills that prospective teachers need leads to the question of what conceptions they possess as they complete their preparation program. In other words, are prospective teachers' conceptions of proof consistent with what they will need to teach reasoning and proof? This research study examines prospective secondary teachers' conceptions of proof and refutations.

Purpose of the Study

Many in the mathematics education community have emphasized the importance of teaching students logical reasoning and formal proofs, and providing opportunities to examine conjectures in all content areas within school mathematics (Hanna, 1997; MAA, 1998; NCTM, 2000; Schoenfeld, 1994; Sowder & Harel, 1998). The goals that the current reform effort sets forth for reasoning and proof are similar to goals of prior reforms, such as those from the New Math era. A key factor in the failure of the New Math movement was the lack of preparation of teachers (Hanna & Jahnke, 1993; Hirshi, 1977), a lesson not lost on subsequent reform efforts. The importance of preparing teachers to teach proof and reasoning has been widely recognized (Glenn Commission, 2000; MAA, 1998; MSEB, 2001; NCTM, 2000). One of the significant areas in the preparation of teachers is the teachers' mathematical content knowledge (Ball, 1989; Cooney, 1999; Galbraith, 1982; MSEB, 2001; NCTM, 2000; Shulman, 1986).

Studies have shown that students' inadequacies and misconceptions in the area of proof and reasoning are widespread at the secondary level (Balacheff, 1987; Bell, 1976; Burke, 1984; Chazan, 1989, 1993; Fischbein & Kedem, 1982; Galindo, 1997; Galindo, et. al., 1998; Healy & Hoyles, 1998; Schoenfeld, 1989; Senk, 1985, 1989; Williams, 1979). If professional development programs are to help prospective teachers change this cycle of inadequate understanding of proof, then educators of prospective teachers need to continually intervene in prospective teachers' experiences at the university level (Galbraith, 1982; Ball, 1989). Prospective teachers must understand and be able to teach proof and reasoning in areas recommended by the NCTM (2000) *Standards*. Thus, the current reform recommendations in proof and reasoning, and the increased awareness of the preparation of teachers, establish a need for research to describe prospective secondary teachers' conceptions of proof as they are completing their teacher preparation program.

Many of the research studies in the area of proof and reasoning have focused on high school students or college students that are not majoring in education. Research that has focused primarily on elementary or secondary teachers' conceptions of proof is sparse and inconclusive. There has been little research in the area of prospective secondary teachers' conceptions of proof as they are completing their preparation programs, especially since the release of the *Standards* 2000. Also, very few studies exist that research prospective secondary teachers' cognitive understanding of proof by examining their understanding of the logical underpinnings of proof and their ability to complete proof.

The need for research examining prospective teachers' conceptions as they are completing their teaching training programs, is brought about by the recent reform efforts and the lofty goals of these efforts. This study intends to focus on prospective secondary teachers' conceptual knowledge about the nature of proof. To gain better insight into an individual's conceptions of proof, this study will investigate not only the ability to complete proofs but also an individual's understanding of the logical underpinnings of proof. An examination of prospective teachers' conceptions of proof helps the mathematics education community evaluate the preparation of prospective teachers in this important area of mathematics education.

Statement of the Problem

This study investigates prospective secondary mathematics teachers' conceptions of proof and refutations. In keeping with the traditions of mathematics education research, conceptions are defined operationally as an individual's knowledge about the subject matter (Fennema & Franke, 1992; Knuth, 1999; Thompson, 1992).[6]

The fundamental purpose of proof, according to Layman (2002), is to show that the premises lead, by way of valid rules of inference, to the conclusion. To investigate the primary question of this study, the researcher addressed two criteria of conceptions of proof—1) understanding of the logical underpinnings of proof, and 2) ability to complete mathematical proofs. The logical underpinnings of proof are defined as the laws of logic that underlie proof. The research study first examined an individual's *understanding* of the logical underpinnings of proof of several common proof schemes emphasized by the NCTM *Standards* 2000 document—direct proof, indirect proof (proof by contradiction), and refutation of a false conjecture. Secondly, the study examined prospective secondary mathematics teachers' *abilities to complete* mathematical proofs that include methods that are emphasized by the *Standards* 2000—direct proof, indirect proof, and refutation of a false conjecture.

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[6] An overview of conceptions and logical underpinnings of proof is discussed in Chapter Two.

Research Questions

Primary Research Question

What are prospective secondary mathematics teachers' conceptions of proof and refutations?

Secondary Research Questions

1. What are prospective secondary mathematics teachers' understandings of the logical underpinnings of proof and refutations?
   a. What are their understandings of the logical underpinnings of direct proof?
   b. What are their understandings of the logical underpinnings of indirect proof?
   c. What are their understandings of the logical underpinnings of refutations?
   d. Can they demonstrate an understanding of the laws of logic that underlie proof?
2. What are prospective secondary mathematics teachers' abilities to complete mathematical proofs?
   a. What are their abilities to complete direct proof?
   b. What are their abilities to complete indirect proof?
   c. What are their abilities to refute a false conjecture?

Definition of Terms

For the purpose of this research, the following definitions were used in the study (Burke, 1984; Esty, 2000; Hershkowitz, 1990; Karush, 1962; Layman, 2002; Mish & Morse, 2000; Moore, 1990; Serra, 1993; Tall & Vinner, 1981; Williams, 1979).

Ability	Competence in doing; skill; natural aptitude or acquired proficiency.
Argument	A set of statements, one of which called the conclusion, is affirmed on the basis of other statements which are called premises.
Axiom (Postulate)	A mathematical proposition assumed to be true.
Complex Statement	A statement that is a simple (prime) statement (statement without connectives) or a statement containing one or more connectives; also called compound statement.
Concept Definition	A concept's formal mathematical definition with critical attributes and non-critical attributes; a criterion for classifying the structural characteristics of an object.
Concept Image	The total cognitive structure that an individual associates with the concept, which includes all the mental pictures, associated properties, and processes.
Concept Usage	The methods an individual operates with in generating or using examples or in completing a proof.
Conditional Statement	A statement written in symbolic form $P \rightarrow Q$ which means if you are given premise or statement P, then you can conclude Q; written as *If P, then Q*.
Conjecture	A mathematical statement which has neither been proved nor refuted by a counterexample.
Connectives	The five connectives in logic are: 1) negation (not); 2) conjunction (and); 3) disjunction (or); 4) conditional (if *H*, then *C*); and 5) biconditional (if and only if).

Converse	The converse of the statement *P implies Q* where *P* and *Q* are statements, is the statement *Q implies P*; written as *If Q, then P*.
Conversion	Reasoning from the converse is called conversion. This is an invalid pattern of reasoning whereby one concludes *P* when given *P implies Q* and *Q*.
Contradiction	A compound statement form which is false for all truth values of its components.
Contrapositive	The contrapositive of a statement *P implies Q* where *P* and *Q* are statements, is the statement *not-Q implies not-P*; written as *If not-Q , then not-P*. A conditional statement and its contrapositive are logically equivalent.
Counterexample	An instance in mathematics that falsifies an assertion. A counterexample is used in proving the negation of a statement by finding at least one example in which the generalization is false. A counterexample demonstrates an argument form is invalid by showing the form can lead from true premises to a false conclusion.
Deductive Proof	A chain of reasoning based upon accepted assumptions called axioms (postulates), definitions, and/or previously proven propositions, which, provided the accepted rules of logic are followed, demonstrates that a conclusion is necessarily true if the postulates on which the argument is based are accepted as true.
Definition	A description of a new term by accepted undefined terms and previously defined terms.
Empirical Evidence	Evidence that is based on observation and experiments; the basis of inductive reasoning.
Generalization	A statement that asserts that something is always the case.
Indirect Proof	A proof that employs the Proof by Contradiction or the Proof by Contrapositive method to prove *If P, then Q*. The steps of an Indirect Proof use the negation of *C* (or the negation of a component of *C*) as a hypothesis.

Inductive Reasoning	Inductive reasoning is a process of observing data or empirical evidence, recognizing patterns, and making inferences about those observations.
Inference	In logic, inference is a single-step deduction from premises.
Inverse	The inverse of the statement *P implies Q* where *P* and *Q* are statements, is the statement *not P implies not Q*; written as *If ~P, then ~Q*.
Logic	Logic is the study of methods for evaluating mathematical arguments.
Logically Equivalent	Two statements *P* and *Q* are logically equivalent if *P* and *Q* have the same truth values whenever all prime statements in one have the same values as corresponding prime statements in the other.
Logical Underpinnings of Proof	Laws of logic that underlie proof; the methods that underlie the basis of proof techniques and evaluation of mathematical arguments.
Modus Ponens	Modus Ponens is a valid rule of inference that allows an individual to conclude *Q* when given *P implies Q* and *P*.
Modus Tollens	Modus Tollens is a valid rule of inference that allows an individual to conclude *not P* when given *P implies Q* and *not Q*.
Premise (Hypothesis)	Statements on the basis on which a conclusion is affirmed. In logic, the proposition *P* in a conditional statement *If P, then Q*.
Proof (Valid Argument)	An argument that follows that if the premises are true, then the conclusion is true. The validity of an argument is guaranteed by its form and does not depend on the content of its subject matter.
Proof by Contradiction	To prove *If P, then Q*, the Proof by Contradiction method assumes that *P* is true and assumes the negation of *Q* is true, and then deduces any contradiction.

Proof by Contrapositive	To prove *If P, then Q*, the Proof by Contrapositive method consists of giving a direct proof of the contrapositive of the statement (*If not-Q, then not-P*).
Proof by Counterexample	A mathematical method of proving an assertion is false by the demonstration of the existence of a counterexample.
Proposition	A proposition (or statement) is a sentence that is either true or false.
Refutation	The process of proving a statement is false or wrong by argument or evidence.
Statement	A statement is a sentence that is either true or false; also called a proposition. The truth value of a statement is true if the statement is true and false if the statement is false.
Understanding	Comprehension; a mental grasp; to grasp the meaning of; knowing.

CHAPTER 2

LITERATURE REVIEW

Introduction

The focus of this study was to determine prospective secondary mathematics teachers' conceptions of proof and refutations. This chapter provides a synthesis of the literature on proof and refutations that are related to this study. The themes for this chapter are: the underpinnings of proof; the nature and role of proof; concept understanding; research on conceptions of proof; and theoretical framework of proof schemes. A summary of the results and methodological principles of the research studies conclude this chapter.

Underpinnings of Proof

According to Barnier and Feldman (1990) "A basic knowledge of logic is indispensable for analyzing and constructing proofs" (p. 1). Logic is the study of methods for evaluating mathematical arguments where arguments are defined as a set of statements, one of which, called the conclusion, is affirmed on the basis of the others, which are called the premises or hypotheses (Layman, 2002). The logical underpinnings of proof include propositional logic, which is the study of certain kinds of statements. Propositional logic involves forming complex statements from simple statements and then determining the truth value of the complex statements from the truth values of the simple statements. According to Fendel and Resek (1990), the two primary ideas of formal mathematical logic are implication and negation. Implication is the causal relationship in a conditional statement between its hypothesis and conclusion, whereas negation provides a link between the two types of quantifiers (universal and existential), and is an essential element in understanding the connection between a conditional statement and a counterexample.

Logical underpinnings form the basis of proof techniques for the construction and analysis of proofs (Barnier & Feldman, 1990; Epp, in press). The basic idea of proof (valid argument) is to infer or derive the conclusion from the hypothesis through valid rules of inference (Nolt, Rohatyn, & Varzi, 1998). Therefore, a valid argument is one in which the truth of the conclusion is absolutely guaranteed, given the truth of the hypothesis. Rules of inference, also referred to as methods of proof, are used to construct proofs. Table 1 displays three common methods of proof of the conditional statement, *If P, then Q* (Fletcher & Patty, 1996, p. 27).

Table 1. Methods of Proof

Direct Proof	**Contrapositive Proof**	**Contradiction Proof**
Assume *P*. ⋮ (logical sequence of steps) ⋮ Conclude *Q*.	Assume *not Q*. ⋮ (logical sequence of steps) ⋮ Conclude *not P*.	Assume *P* and *not Q*. ⋮ (logical sequence of steps) ⋮ Conclude *R and not R*.

According to Barnier and Feldman (1990) *semantics* is a set of rules used to determine the truth or falsity of statements. An easy way to summarize semantic rules is by using a *truth table*. Truth tables help distinguish if two statements are logically equivalent—have the same truth values whenever the main columns of their standard truth tables are identical. Table 2 represents the truth table for conditional statements (Fletcher & Patty, 1996, p. 4). Each row of the table represents a possible combination of the truth values of the simple statements and conditional statement.

Table 2. Truth Table for a Conditional Statement

Hypothesis **H**	Conclusion **C**	Conditional Statement **H** → **C**
True	True	True
True	False	False
False	True	True
False	False	True

Barnier and Feldman (1990) suggest that to understand a proof one must know: 1) the goal of the proof; 2) the hypotheses; 3) the necessary facts and definitions of the content area; and 4) previously proved facts or laws of logic to be used in the proof. According to Epp (in press), goals for a first course in reasoning and proof for university students should include helping students appreciate the role of definitions in mathematical proof and reasoning and also develop an ability to evaluate the truth or falsity of mathematical statements. She further suggests that an introductory unit on the principles of logical reasoning provides a supportive framework in which students can draw from while learning various aspects of proof and disproof. Epp (in press) states that for future teachers to effectively guide the development of students' reasoning abilities, as is expected by the NCTM 2000 *Standards*, they need to have an understanding of what constitutes a valid argument and what it means for statements of various forms to be true or false (2002, p. 8).

The Nature and Role of Proof

The conceptual framework for research in proof is addressed by examining the *nature* of proof and the *role* of proof (Borko, Peressini, & Romagnano, 2000). The *nature* of proof refers to an individual's explanations, justifications, and elaborations in order to make a conjecture more convincing (Lakatos, 1976; Borko, et.al 2000). The primary *role* of proof in mathematics is to validate propositions, but an additional role is to explain the meaning and the mathematical basis of the theorem being proved (Hanna, 1998).

The Nature of Proof

Fawcett (1938) stated that a pupil understands the nature of deductive proof when he or she understands:

1. The place and significance of undefined concepts in proving any conclusion.
2. The necessity for clearly defined terms and their effect on the conclusion.
3. The necessity for assumptions or unproved propositions.
4. That no demonstration proves anything that is not implied by the assumptions (p.10).

In addition to understanding these four ideas, it is assumed the student will also understand that the conclusions that are established can have universal validity only if the definitions and assumptions that imply these conclusions have universal validity.

Lakatos (1976) examined the nature of proof in *Proof and Refutations*, through a portrayal of a classroom debate among fictional students about the truth and meaning of a geometry theorem. Students presented arguments about what a proof of a theorem means in mathematics. Through reading the debate, one gains Lakatos's insight that mathematics develops by a process of "conscious guessing" about relationships.[7] According to Lakatos, proof follows a "zig-zag" path that is prodded by counterexamples. Once counterexamples are discovered, proof moves from

> ... the naive conjecture to the premises and then turns back again to delete the naive conjecture and replace it by the theorem. Naive conjecture and counterexamples do not appear in the fully fledged deductive structure: the zig-zag of discovery cannot be discerned in the end-product (Lakatos, 1976, p. 42).

An important part of Lakatos's book was the classroom discussion, in which students made conjectures and participated in a dialogue about the truth or falsity of the conjecture. Lakatos emphasized that students take a risk when they present their arguments and allow them to be evaluated by their peers.

Bell (1976) viewed the meaning of proof as having three senses: (a) *verification* or *justification*, which is concerned with the truth of a proposition; (b) *illumination*, which conveys insight into why a proposition is true; and (c) *systematisation*, the organization of results into a deductive system of axioms, major concepts and theorems.

Balacheff (1987) classified proofs as *pragmatic proofs* and *intellectual proofs*. Pragmatic proofs are naive empirical arguments or arguments based on crucial

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7 Lakatos referred to conjectures as "conscious guesses" (1976).

experiments. Intellectual proofs assert the truth of a mathematical statement based on a process of reasoning that is usually in the making of a general argument.

To understand student' conceptions of the nature of mathematical proof, Harel and Sowder (1998) distinguished proof schemes. According to Harel and Sowder (1998) "A person's proof scheme consists of what constitutes ascertaining and persuading for that person" (p. 244). Harel and Sowder (1998) distinguished between three proof categories or as they referred to as proof schemes—*external conviction proof scheme*, *empirical proof scheme*, and *analytical proof scheme*. They asserted that a person's evidence and justification of an observation is based on "...logical and deductive arguments, empirical evidence, intuitions, personal beliefs, an authority, ..., social conventions, or any other knowledge the person considers relevant to the truth of the observation" (p. 243).

The Role of Proof

In mathematics, one of the main functions of proof is to validate propositions (Alibert & Thomas, 1991; Balacheff, 1987; Bell, 1976; de Villiers, 1999; Fawcett, 1938; Hanna, 1998; Harel & Sowder, 1998; Schoenfeld, 1994). Schoenfeld (1994) stated that one of the most wonderful features of proof is that it yields certainty. “When you have a proof of something you know it *has to be* true, and why" (Schoenfeld, 1994, p. 74).

Mathematical proof also plays a role in convincing and explaining mathematical ideas (Hanna, 1998; Hersh, 1993; Schoenfeld, 1994). "The best proof, even in the eyes of practicing mathematicians, is one that not only establishes the truth of a theorem but also helps understand it" (Hanna, 1998, p. 9). Proof provides insight into *why* the theorem is true.

Many researchers noted the importance of considering proof as both a cognitive (verification, convincing, or explanation) and a social process (Alibert & Thomas, 1991; Bell, 1976; de Villiers, 1999; Hanna, 1991, 1998; Harel &Sowder, 1998; Lampert, 1990; Simon & Blume, 1996; Vernaud, 1990; Yackel et. al, 2000). The role of proof as a social activity is when individuals determine the truth or falsity of mathematical statements to each other, promoting an understanding of *why* the statement is true or false. The manner that Alibert and Thomas (1991) described proof was as "...a means of convincing oneself whilst trying to convince others" (p. 215).

There are implications that an individual's concept image is influenced by the classroom social interactions as in Lakatos (1976) in which he created an imaginary classroom and narrated the social interactions amongst students as they tackle their concept image of proof. According to Lampert (1990), for students to expose their exploratory thinking to others with the hope that by engaging in an exchange of ideas in the classroom discussion, takes courage and modesty but students may end up with better ideas in end.

Bell (1976) also acknowledged the importance of a student's own activity of arriving at their conviction, of making a verification, and in communicating their convictions of results to others. In addition, Bell suggested the development of the proof process grows out of an internal testing of a mathematical idea that is either accepted or rejected, which in turn accompanies the development of a generalization. According to

Bell (1976), gradually the proof process becomes more externalized. This internal testing that Bell suggested is coined by others as a person's *concept image*.

Concept Understanding

Concept Image and Concept Definition

According to Piaget (1967) the last stage of cognitive development—Formal Operations—involves the transition from concrete to formal thinking, or what Piaget referred to as "hypothetico-deductive" thinking (p. 62). "Formal thought is 'hypothetico-deductive' in the sense that it permits one to draw conclusions from pure hypotheses and not merely from actual observations" (Piaget, 1967, p. 63). Piaget (1967) added that these conclusions have a validity independent of their factual truth which explains why formal thinking represents so much difficulty and so much more mental work than concrete thinking. In the formal stage of cognitive development, an individual constructs new operations and operations of propositional logic (Piaget, 1972).

While Piaget's cognitive stages of development were based on the maturation of the student, Pierre van Hiele and Dieke van Hiele-Geldof based their five levels of geometric thought on the experiences of the student (van Hiele, 1986). According to the van Hiele's (1986), the fourth level, Formal Logic, includes understanding the notions of mathematical definitions, axioms, and theorems and the ability to write formal proofs. The fifth and highest level of geometric reasoning is the Nature of Logical Laws, which is highly abstract. In this level, students can recognize the relationships between different axiomatic systems (e.g., Euclidean and non-Euclidean geometry).

To understand the cognitive aspects of proof in mathematics, we look outside of psychology. Fischbein (1990) suggested a particular system of concepts for use in mathematics and mathematics education was needed in addition to the general ones inspired by psychology. Fischbein stated "Even Piagetian stage theory and the respective findings concerning mathematical concepts (number, space, chance, function, etc.) cannot be translated directly in curriculum terms" (1990, p. 9). Two of the concepts that Fischbein referred to are: *concept image* and *concept definition.* According to Tall (1992), "Mathematical proof as a human activity requires not only an understanding of the concept definitions and the logical processes, but also insight into how and why it works" (p. 506). To understand the cognitive aspects of an individual's conceptual understanding of a mathematical idea, this research study examines the individual's *concept image* and the *concept definition.*

According to Tall and Vinner (1981), when comparing mathematics to other fields, mathematics is usually thought of as a subject of great precision where concepts can be defined accurately so as to provide a solid foundation of mathematical theory. Tall and Vinner suggested that the psychological reality of this idea rests instead on the fact that many concepts which are used in mathematics are not formally defined at all, but instead are learned by experience and usage in appropriate contexts. To understand how these processes work, Tall and Vinner formulated a distinction between mathematical concepts as they are formally defined and the cognitive processes by which they are learned (1981). Tall and Vinner stated,