



Semiclassical effects on black holes
by Brett Eric Taylor

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics
Montana State University
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Abstract:

In an attempt to understand the effects quantum mechanical processes will have in general relativity without knowing the theory of quantum gravity, many relativists have turned to semiclassical gravity. The term semiclassical arises from the fact that the Einstein equations from classical general relativity have been converted so that they are partly quantum mechanical. In particular, any matter fields present in the system are quantized. Semiclassical gravity has provided a number of important results, not the least of which is the discovery that black holes are thermodynamic objects. Semiclassical gravity will be used in this thesis to investigate the effects that quantum mechanical processes will have on black holes.

In the first section zero temperature black hole solutions will attempt to be found using semiclassical gravity. These zero temperature solutions would be remnants, black holes at the endpoint of their evaporation. In particular the solutions will be charged black holes for which the classical solution has a temperature very near zero, or equivalently one whose charge is nearly equal to its mass. A black hole which has a charge very nearly equal to its mass will then be placed in the presence of either a massless or a massive quantized scalar field. The presence of this field can perturb the temperature of the black hole from nearly zero to precisely zero for particular values of the coupling constant.

The second section will concern the evolution of an evaporating black hole which is initially rotating. The conventional viewpoint of black hole evolution is that a black hole which is initially highly rotating will lose most of its angular momentum by the time it has lost approximately half of its mass. This result was determined by calculating the scattering amplitude of spin 1/2, spin 1 and spin 2 fields off of the black hole. A hole which is radiating solely into a massless scalar field will not approach a state where $J/M^2 = 0$. The addition of a scalar field to the nonzero spin fields is found to have a small effect on the evolution.

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CONVENTIONS

The sign conventions in this thesis are those of Misner, Thorne and Wheeler[1]. We use natural units, such that $G = c = \hbar = k_B = 1$, except where otherwise specified. In Chapter 3 we will follow the notation conventions of Page [2] and Teukolsky and Press [3].

ABSTRACT

In an attempt to understand the effects quantum mechanical processes will have in general relativity without knowing the theory of quantum gravity, many relativists have turned to semiclassical gravity. The term semiclassical arises from the fact that the Einstein equations from classical general relativity have been converted so that they are partly quantum mechanical. In particular, any matter fields present in the system are quantized. Semiclassical gravity has provided a number of important results, not the least of which is the discovery that black holes are thermodynamic objects. Semiclassical gravity will be used in this thesis to investigate the effects that quantum mechanical processes will have on black holes.

In the first section zero temperature black hole solutions will attempt to be found using semiclassical gravity. These zero temperature solutions would be remnants, black holes at the endpoint of their evaporation. In particular the solutions will be charged black holes for which the classical solution has a temperature very near zero, or equivalently one whose charge is nearly equal to its mass. A black hole which has a charge very nearly equal to its mass will then be placed in the presence of either a massless or a massive quantized scalar field. The presence of this field can perturb the temperature of the black hole from nearly zero to precisely zero for particular values of the coupling constant.

The second section will concern the evolution of an evaporating black hole which is initially rotating. The conventional viewpoint of black hole evolution is that a black hole which is initially highly rotating will lose most of its angular momentum by the time it has lost approximately half of its mass. This result was determined by calculating the scattering amplitude of spin 1/2, spin 1 and spin 2 fields off of the black hole. A hole which is radiating solely into a massless scalar field will not approach a state where $J/M^2 = 0$. The addition of a scalar field to the nonzero spin fields is found to have a small effect on the evolution.

CHAPTER 1

INTRODUCTION

1.1 Semiclassical methods

The work presented in this thesis is concerned with how quantum mechanical effects will influence the equilibrium states and non-equilibrium evolution of black holes. Since there is no working theory of quantum gravity another approach must be utilized. The community is divided between those who are trying to take the full leap to quantum gravity by studying string theory, and now the more general approach of M-theory, and those who are taking a more conservative approach. The conservative approach is to utilize the semiclassical theory of gravity.

Semiclassical gravity is constructed by starting with the purely classical Einstein field equations

$$G^\mu{}_\nu = 8\pi T^\mu{}_\nu . \tag{1.1}$$

The left-hand side of Eq. (1.1) contains terms that describe the geometry of spacetime while the right-hand side contains the information describing the stress-energy of the

matter fields present in the spacetime. There is no known way to quantize geometry so in semiclassical gravity the left-hand side of the Einstein field equations will be kept classical. Methods of quantizing matter fields, however, are known so the right-hand side of the field equations will be replaced by the expectation value of the stress-energy operator for the quantized matter fields present. The right-hand side may also contain classical background stress-energy terms due to the nature of the geometry. To represent this we will write the semiclassical field equations as

$$G^\mu{}_\nu = 8\pi [\langle T^\mu{}_\nu \rangle + T^\mu{}_\nu] , \quad (1.2)$$

where the classical pieces are represented without bra-kets. Semiclassical methods have been used as a successful tool in understanding the evolution of evaporating black holes. Of particular importance is the discovery that black holes follow the laws of thermodynamics. This result will be discussed in more detail in Section 1.3.

1.2 Black holes and naked singularities

Before discussing the thermodynamics of black holes, a description of black holes and a related set of objects, naked singularities, is required. A black hole is a region of spacetime which is not connected to future null infinity. The size of a black hole is typically described by the radius of the event horizon. The event horizon is the boundary of the past of future null infinity for asymptotically flat spacetimes. In this work all the spacetimes to be considered will be asymptotically flat. The location of

the event horizon can be found in static spacetimes by finding where the time-time component of the metric, g_{tt} , is zero. In a general black hole spacetime one must use global techniques to find the event horizon.

The simplest black hole solution is the Schwarzschild black hole. Physically this vacuum solution describes a black hole that is static and spherically symmetric. The metric which describes this spacetime is

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (1.3)$$

where M is the mass of the black hole, r is a coordinate which determines the area of a 2-sphere and $d\Omega^2$ is the metric on the 2-sphere. The event horizon of a Schwarzschild black hole is located at

$$r = 2M. \quad (1.4)$$

A second type of black hole can be created by taking a Schwarzschild black hole and adding an electric charge to it. This type of black hole is described by the Reissner-Nordström metric:

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \quad (1.5)$$

Here $|Q|$ describes the electric charge of the black hole and as in the Schwarzschild case M is the mass of the black hole. The Reissner-Nordström geometry has two horizons, an outer event horizon, r_+ , and an inner horizon, r_- . Their locations are

given by

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}. \quad (1.6)$$

Realistic black holes which form due to stellar collapse are expected to be rotating [4]. The Kerr metric describes this type of rotating black hole. Its global structure is different than that of the Schwarzschild black hole as it is stationary rather than static and is not spherically symmetric. A static spacetime is one in which all metric components are independent of t and has the characteristic that the geometry is unchanged under time reversal, $t \rightarrow -t$. A stationary spacetime is one which changes under time reversal. The Kerr metric, in Boyer-Lindquist coordinates, is given by

$$ds^2 = -\frac{\Delta}{\Sigma} [dt - a \sin^2 \theta d\phi]^2 + \frac{\sin^2 \theta}{\Sigma} [(r^2 + a^2)d\phi - a dt]^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2, \quad (1.7)$$

where $\Delta = r^2 - 2Mr + a^2$, $\Sigma = r^2 + a^2 \cos^2 \theta$, a is the specific angular momentum of the black hole given by $a = J/M$ where J is the angular momentum of the black hole, and θ and ϕ are the polar and azimuthal angles respectively.

In addition a Kerr black hole has a region called the ergosphere. The outer boundary of this region is located at

$$r_{\text{ergo}} = M + \sqrt{M^2 - a^2 \cos^2 \theta}, \quad (1.8)$$

Inside this region an observer cannot remain at constant values of ϕ due to frame dragging by the black hole. In 1969, Penrose showed that it was possible to extract

energy from the ergosphere [5]. Imagine throwing a small piece of matter from infinity into the black hole. Once the ingoing matter crosses into the ergosphere allow the mass to break into two pieces, one of which escapes back out to infinity and the other which crosses the event horizon. The change in mass of the black hole is exactly equal to the energy as measured at infinity for the piece of matter which crossed the event horizon. The energy is just

$$E = -p_\mu \xi_{(t)}^\mu, \quad (1.9)$$

where p is the 4-momentum of the matter and $\xi_{(t)}$ is the timelike Killing vector, $\partial/\partial t$. Outside of the ergosphere the Killing vector is timelike as is the 4-momentum so the energy is necessarily positive. However inside the ergosphere the Killing vector is spacelike. For certain choices of timelike momentum vectors the energy of the piece absorbed by the black hole is now negative, reducing the mass of the black hole. By conservation of energy, the energy of the piece that escapes to infinity must then be greater than the total mass thrown in towards the hole. In a similar manner it is also possible to remove energy by scattering a wave off of a Kerr black hole. An ingoing wave could be scattered into an outgoing state with more energy than it entered with, removing energy from the black hole. This process is known as superradiant scattering. The Kerr metric also has an inner and outer horizon. Their locations are given by

$$r_\pm = M \pm \sqrt{M^2 - a^2}. \quad (1.10)$$

Another class of objects which will be encountered are naked singularities. Naked

singularities are objects which have a curvature singularity which is not hidden behind an event horizon. Nature appears to conspire to prevent us from seeing any naked singularities by always forcing an event horizon to clothe all but initial singularities associated with the Big Bang [6]. This hypothesis is known as the cosmic censorship conjecture. One way to create a naked singularity is to somehow modify either a Kerr or Reissner-Nordström black hole in a particular way. For a Kerr spacetime, having $a > M$ results in a spacetime with a naked singularity. Press and Teukolsky showed numerically that a Kerr black hole is stable to linear perturbations against being turned into a naked singularity [7]. In the Reissner-Nordström geometry, having $|Q| > M$ also results in a naked singularity.

1.3 Black hole thermodynamics

One of the great successes of semiclassical gravity is the discovery that black holes follow the laws of thermodynamics. In particular it was found by Hawking that black holes act as black bodies [8, 9]. A black hole radiates through the emission of particles and this emission is thermal. This process of emission is known as the Hawking process. The expectation value for the number of particles that are emitted from a charged rotating black hole in a particular wave mode labeled by the frequency ω , spheroidal harmonic l , axial quantum number m and polarization p is

$$\langle N_{j\omega l m p} \rangle = -Z \left[\exp \left[2\pi\kappa^{-1} (\omega - m\Omega - q\Phi) \right] \mp 1 \right]^{-1}, \quad (1.11)$$

where Z is the absorption probability for an incoming wave of that mode; κ , Ω , and Φ are the surface gravity, surface angular frequency and surface electrostatic potential respectively as measured on the event horizon of the black hole. Here the minus sign is for bosons and the plus is for fermions.

The surface gravity, for static black holes, is the limiting value of the force required to hold a unit test mass in place as measured from infinity as the mass approaches the horizon. For rotating black holes one cannot hold a mass at rest near the horizon with respect to infinity due to the object being inside the ergosphere so this description of the surface gravity doesn't hold, although this quantity is still well-defined. This quantity is used because the locally measured acceleration is infinite at the horizon. For a general rotating, charged black hole the surface gravity is given by

$$\kappa = \frac{(M^2 - a^2 - Q^2)^{1/2}}{2M \left[M + (M^2 - a^2 - Q^2)^{1/2} \right] - Q^2} \quad (1.12)$$

The temperature of a black hole is proportional to the surface gravity of the event horizon, κ :

$$T = \frac{\kappa}{2\pi} \quad (1.13)$$

For Schwarzschild black holes, the temperature is given by

$$T_{\text{Schwarz}} = \frac{1}{8\pi M} \quad (1.14)$$

Note that the temperature is inversely proportional to the mass. As a Schwarzschild

black hole evaporates, losing mass, its temperature increases. This is equivalent to saying that Schwarzschild black holes have a negative specific heat.

There is a class of black holes that have zero temperature. These black holes are described as extreme black holes. Extreme black holes have zero surface gravity, and hence zero temperature by Eq. (1.13). One example that will be discussed in Chapter 2 is a Reissner-Nordström black hole with $|Q| = M$. In that chapter we will investigate whether the presence of a quantized massive scalar field will perturb nearly extremal charged black holes to holes which have zero temperature.

The second law of thermodynamics, the total system entropy must never decrease, can be restated in terms of black holes as the total area of the event horizon must always increase. This black hole area theorem was discovered by Hawking [10]. Beckenstein was the first to propose a generalized second law for black holes correlating the entropy to the area [11]. The physical entropy, S , of the black hole is related to the area of the event horizon, A , by:

$$S = \left(\frac{1}{4}\right) A. \quad (1.15)$$

1.4 Evolution of black holes

One consequence of the fact that black holes are thermodynamic objects is that they radiate away energy, and hence their mass, as long as their temperature is greater than that of the surrounding medium, if any [2]. Our understanding of how the evolution

process occurs is clear until the black hole reaches the Planck scale. Planck, or natural, units are a system of units specified by the fundamental constants of nature G , c , \hbar , and k_B . For example the Planck length, l_P , is given by

$$l_P = \left(\frac{\hbar G}{c^3} \right)^{1/2} = 1.62 \times 10^{-33} \text{ cm} . \quad (1.16)$$

We would need a full theory of quantum gravity to understand the evolution of a black hole whose curvature has approached this length. For a small Reissner-Nordström black hole in isolation, the charge of the black hole is rapidly removed via pair creation so that the holes evolve asymptotically towards a Schwarzschild state [13, 14, 15]. Extremely large charged black holes however tend to evolve towards the extreme state initially and have lifetimes orders of magnitude longer than the lifetime of a Schwarzschild black hole with the same initial mass [12]. In the case of the Kerr black hole, the black hole will rapidly lose most of its angular momentum if it radiates in the presently known matter fields [2]. In Chapter 3 we will investigate the effects of the emission of massless scalar particles on the evolution of a Kerr black hole.

CHAPTER 2

PERTURBATIONS OF NEARLY EXTREMAL CHARGED
BLACK HOLES

The final fate of an evaporating black hole is not known. This is due to the fact that once the black hole reaches the Planck scale, the full theory of quantum gravity is needed to understand the evolution. One possibility is that a remnant of the black hole will remain. Such a remnant would have zero temperature and would remain at zero temperature if it were in isolation. If it were not in isolation matter could fall into the black hole, raising its temperature and move it away from its extreme state. The black hole would then radiate away mass, returning it to zero temperature. The exact physical characteristics of this possible remnant are not well known.

Recently Anderson and Mull have shown that size gaps exist for zero temperature static black holes in semiclassical gravity [16, 17]. This work was done using conformally invariant massless scalar fields. Conformal invariance for scalar fields requires that the field be massless and have a curvature coupling constant to gravity of $\xi = 1/6$ where ξ is the coupling constant. In general, a field ψ is conformally

invariant with the metric $g_{\mu\nu}$ if there exists a real number s such that $\tilde{\psi} = \Omega^s \psi$ is a solution for the metric $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. This transformation of the metric is known as a conformal transformation. The size gaps found for zero temperature semiclassical black holes were found by demanding that there be no scalar curvature singularities at the horizon and that the quantized field satisfied the semiclassical trace equation. The full semiclassical equations were not solved. Anderson and Mull investigated the general static spherically symmetric metric which takes the following form:

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 d\Omega^2. \quad (2.1)$$

In this chapter we will investigate the effects that a massive or massless scalar field of arbitrary curvature coupling constant will have on nearly extremal Reissner-Nordström black holes. In the case of the massive scalar field the DeWitt-Schwinger approximation will be used to calculate the expectation value for the stress-energy of the field. For the case of the massless scalar field the expectation of the stress-energy of the field will be taken from numerical work by Anderson, Hiscock, and Loran [18].

2.1 Perturbations from a massive scalar field using the DeWitt-Schwinger approximation to determine $\langle T^\mu_\nu \rangle$

The work done in this section is an extension of work done by Anderson, Hiscock, Whitesell, and York [19]. In that work they determined what effect the presence of a quantized massive scalar field would have on the temperature, or surface gravity, of

a Schwarzschild black hole. They found that for most cases the temperature of the black hole was decreased by the presence of the field [19].

The DeWitt-Schwinger expansion is a method of obtaining the renormalized stress energy for a matter field in curved space. When calculating the expectation value of stress-energy tensor one must renormalize to give sensible results. The DeWitt-Schwinger method is one way of doing this. The solution to the wave equation for the massive field can be written in terms of the Euclidean space Green's function. The Green's function will satisfy

$$[\square_x - m^2 - \xi R(x)] G_E(x, x') = -g^{-1/2}(x) \delta^4(x, x'), \quad (2.2)$$

where m is the field mass, R is the Ricci scalar, and g is the determinant of the metric. The Green's function can be expanded in terms of inverse powers of the field mass. The expression for the the stress-energy tensor is quadratic in the field operators and contains derivatives of terms like $\phi(x)\phi(x)$. Since the stress-energy is quadratic in these terms, the point x can be split into x and x' and the coincident limit taken. The expectation value of the stress-energy can then be calculated for the field in the vacuum state. The expression will contain divergent and non-divergent terms. The renormalization is accomplished by subtracting off the divergent pieces. One is then left with a finite expression for the expectation value of the stress-energy tensor.

We will be working in the Reissner-Nordström spacetime. The Reissner-Nordström metric takes the form found in Eq. (1.5). Using the DeWitt-Schwinger approximation

one obtains the following expressions for two of the components of the stress-energy tensor with a massive scalar field of arbitrary coupling constant in the Reissner-Nordström metric:

$$\begin{aligned}
\langle T_t^t \rangle = & \frac{\epsilon}{\pi^2 m^2} \left[\frac{1237M^5}{5040r^9} - \frac{25M^4}{224r^8} - \frac{1369M^5 r_+}{1008r^{10}} + \frac{41M^4 r_+}{105r^9} + \frac{3M^3 r_+}{56r^8} \right. \\
& + \frac{613M^5 r_+^2}{210r^{11}} - \frac{73M^4 r_+^2}{3360r^{10}} - \frac{41M^3 r_+^2}{210r^9} - \frac{3M^2 r_+^2}{112r^8} - \frac{2327M^5 r_+^3}{1260r^{12}} - \frac{613M^4 r_+^3}{210r^{11}} \\
& + \frac{883M^3 r_+^3}{1260r^{10}} + \frac{2327M^4 r_+^4}{840r^{12}} + \frac{613M^3 r_+^4}{840r^{11}} - \frac{883M^2 r_+^4}{5040r^{10}} - \frac{2327M^3 r_+^5}{1680r^{12}} \\
& + \frac{2327M^2 r_+^6}{10080r^{12}} + \xi \left(\frac{-11M^5}{10r^9} + \frac{M^4}{2r^8} + \frac{217M^5 r_+}{30r^{10}} - \frac{14M^4 r_+}{5r^9} - \frac{226M^5 r_+^2}{15r^{11}} \right. \\
& + \frac{77M^4 r_+^2}{180r^{10}} + \frac{7M^3 r_+^2}{5r^9} + \frac{91M^5 r_+^3}{10r^{12}} + \frac{226M^4 r_+^3}{15r^{11}} - \frac{182M^3 r_+^3}{45r^{10}} - \frac{273M^4 r_+^4}{20r^{12}} \\
& \left. - \frac{113M^3 r_+^4}{30r^{11}} + \frac{91M^2 r_+^4}{90r^{10}} + \frac{273M^3 r_+^5}{40r^{12}} - \frac{91M^2 r_+^6}{80r^{12}} \right) \Bigg], \quad (2.3)
\end{aligned}$$

$$\begin{aligned}
\langle T_r^r \rangle = & \frac{\epsilon}{\pi^2 m^2} \left[\frac{-47M^5}{720r^9} + \frac{7M^4}{160r^4} + \frac{2081M^5 r_+}{5040r^{10}} - \frac{16M^4 r_+}{63r^9} + \frac{3M^3 r_+}{280r^8} \right. \\
& - \frac{13M^5 r_+^2}{18r^{11}} + \frac{983M^4 r_+^2}{10080r^{10}} + \frac{8M^3 r_+^2}{63r^9} - \frac{3M^2 r_+^2}{560r^8} + \frac{421M^5 r_+^3}{1260r^{12}} + \frac{13M^4 r_+^3}{18r^{11}} \\
& - \frac{383M^3 r_+^3}{1260r^{10}} - \frac{421M^4 r_+^4}{840r^{12}} - \frac{13M^3 r_+^4}{72r^{11}} + \frac{383M^2 r_+^4}{5040r^{10}} + \frac{421M^3 r_+^5}{1680r^{12}} \\
& - \frac{421M^2 r_+^6}{10080r^{12}} + \xi \left(\frac{3M^5}{10r^9} - \frac{M^4}{5r^8} - \frac{49M^5 r_+}{30r^{10}} + \frac{14M^4 r_+}{15r^9} + \frac{14M^5 r_+^2}{5r^{11}} \right. \\
& - \frac{61M^4 r_+^2}{180r^{10}} - \frac{7M^3 r_+^2}{15r^9} - \frac{13M^5 r_+^3}{10r^{12}} - \frac{14M^4 r_+^3}{5r^{11}} + \frac{52M^3 r_+^3}{45r^{10}} + \frac{39M^4 r_+^4}{20r^{12}} \\
& \left. + \frac{7M^3 r_+^4}{10r^{11}} - \frac{13M^2 r_+^4}{45r^{10}} - \frac{39M^3 r_+^5}{40r^{12}} + \frac{13M^2 r_+^6}{80r^{12}} \right) \Bigg], \quad (2.4)
\end{aligned}$$

where here m is the field mass, r_+ is the event horizon as defined in Eq. (1.6), $\epsilon = 1/M^2$

is our expansion parameter, and ξ is the curvature coupling constant for the field.

Only these two components of the expectation value of the stress-energy tensor are required to find the perturbative solutions of the semiclassical Einstein field equations. The expressions given above are derived from work done by Anderson, Hiscock, and Samuel [20]. They found that these analytic expressions for the stress-energy were accurate, as compared to those calculated numerically, when $mM \gtrsim 2$ [20].

In addition to the stress-energy from the quantized field, there will also be classical background stress-energy terms due to the electromagnetic field present due to the charge of the black hole which will appear on the right side of Eq. (1.2).

It turns out that it will be easier to solve the perturbed semiclassical Einstein equations if we switch from the coordinates given in Eq. (1.5) to ingoing Eddington-Finkelstein coordinates. The Eddington-Finkelstein coordinates are coordinates which use as a basis freely falling photons. The ingoing Eddington-Finkelstein coordinates will have one coordinate which describes ingoing, radial, null geodesics. The transformation to the ingoing Eddington-Finkelstein coordinates $(v, \tilde{r}, \theta, \phi)$ from (t, r, θ, ϕ) is described by

$$\begin{aligned} \frac{\partial t}{\partial v} &= 1, \\ \frac{\partial t}{\partial \tilde{r}} &= -\left(1 - \frac{2M}{\tilde{r}} + \frac{Q^2}{\tilde{r}^2}\right)^{-1}, \\ \frac{\partial r}{\partial v} &= 0, \\ \frac{\partial r}{\partial \tilde{r}} &= 1. \end{aligned} \tag{2.5}$$

The Reissner-Nordström metric will then take the following form in ingoing Eddington-

Finklestein coordinates

$$ds^2 = - \left(1 - \frac{2M}{\tilde{r}} + \frac{Q^2}{\tilde{r}^2} \right) dv^2 + 2dv d\tilde{r} + \tilde{r}^2 d\Omega^2 . \quad (2.6)$$

The components of the expectation value of the stress-energy tensor will transform into the Eddington-Finklestein coordinates as

$$\langle T^v_v \rangle = \langle T^t_t \rangle , \quad (2.7)$$

$$\langle T^{\tilde{r}}_{\tilde{r}} \rangle = \langle T^r_r \rangle , \quad (2.8)$$

$$\langle T^v_{\tilde{r}} \rangle = \left(1 - \frac{2M}{\tilde{r}} + \frac{Q^2}{\tilde{r}^2} \right)^{-1} [\langle T^r_r \rangle - \langle T^t_t \rangle] . \quad (2.9)$$

One can rewrite the metric given in Eq. (2.6) more generally as

$$ds^2 = -e^{2\psi(r)} \left(1 - \frac{2m(r)}{r} + \frac{Q^2}{r^2} \right)^{-1} dv^2 + 2e^{\psi(r)} dv dr + r^2 d\Omega^2 , \quad (2.10)$$

where for ease \tilde{r} is written as r . The perturbations can then be introduced by doing an expansion of the exponential and $m(r)$ terms to linear order in the perturbation parameter ϵ :

$$e^{\psi(r)} = 1 + \epsilon\rho(r) , \quad (2.11)$$

$$m(r) = M[1 + \epsilon\mu(r)] . \quad (2.12)$$

One can now calculate the left-hand side of Eq. (1.2) with the above metric and

perturbation functions. One can then equate this to the right side given by Eqs. (2.7 - 2.9) and the classical stress-energy terms due to the electromagnetic field present from the charge on the black hole, then collect all terms that are of linear order in ϵ . Doing so yields two first order differential equations for $\rho(r)$ and $\mu(r)$:

$$\frac{\partial \mu}{\partial r} = -\frac{4\pi r^2}{M\epsilon} \langle T_t^t \rangle, \quad (2.13)$$

$$\frac{\partial \rho}{\partial r} = \frac{4\pi r}{\epsilon} \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} [\langle T_r^r \rangle - \langle T_t^t \rangle], \quad (2.14)$$

with $\langle T_t^t \rangle$ and $\langle T_r^r \rangle$ given by Eq. (2.3) and Eq. (2.4). The ϵ in the denominator of the leading terms in both Eq. (2.13) and Eq. (2.14) are exactly cancelled by the overall factor of epsilon in the expressions for $\langle T_t^t \rangle$ and $\langle T_r^r \rangle$. The two differential equations can then be integrated to find solutions for μ and ρ . The general solutions are

$$\begin{aligned} \mu = & C_1 + \frac{1}{\pi m^2} \left[\frac{1237M^4}{7560r^6} - \frac{5M^3}{56r^5} - \frac{4169M^4}{158760r_+^6} + \frac{461M^3}{6615r_+^5} - \frac{6607M^2}{105840r_+^4} + \frac{3007M}{158760r_+^3} \right. \\ & - \frac{1369M^4 r_+}{1764r^7} + \frac{82M^3 r_+}{315r^6} + \frac{3M^2 r_+}{70r^5} + \frac{613M^4 r_+^2}{420r^8} - \frac{73M^3 r_+^2}{5880r^7} - \frac{41M^2 r_+^2}{315r^6} \\ & - \frac{3Mr_+^2}{140r^5} - \frac{2327M^4 r_+^3}{2835r^9} - \frac{613M^3 r_+^3}{420r^8} + \frac{883M^2 r_+^3}{2205r^7} + \frac{2327M^3 r_+^4}{1809r^9} + \frac{613M^2 r_+^4}{1680r^8} \\ & - \frac{883Mr_+^4}{8820r^7} - \frac{2327M^2 r_+^5}{3780r^9} + \frac{2327Mr_+^6}{22680r^9} + \xi \left(-\frac{11M^4}{15r^6} + \frac{2M^3}{5r^5} + \frac{4M^4}{45r^6} - \frac{11M^3}{45r_+^5} \right. \\ & + \frac{41M^2}{180r_+^4} - \frac{13M}{180r_+^3} + \frac{62M^4 r_+}{15r^7} - \frac{28M^3 r_+}{15r^6} - \frac{113M^4 r_+^2}{15r^8} + \frac{11M^3 r_+^2}{45r^7} + \frac{14M^2 r_+^2}{15r^6} \\ & \left. + \frac{182M^4 r_+^3}{45r^9} + \frac{113M^3 r_+^3}{15r^8} - \frac{104M^2 r_+^3}{45r^7} - \frac{91M^3 r_+^4}{15r^9} - \frac{113M^2 r_+^4}{60r^8} + \frac{26Mr_+^4}{45r^7} \right) \end{aligned}$$

$$\left. + \frac{91M^2r_+^5}{30r_+^9} - \frac{91Mr_+^6}{180r_+^9} \right) \Bigg], \quad (2.15)$$

and

$$\begin{aligned} \rho = & C_2 + \frac{1}{\pi m^2} \left[-\frac{29M^4}{280r_+^6} + \frac{817M^4}{3528r_+^6} - \frac{3221M^3}{8820r_+^5} + \frac{253M^2}{1680r_+^4} + \frac{184M^4r_+}{441r_+^7} + \frac{M^3r_+}{35r_+^6} \right. \\ & - \frac{229M^4r_+^2}{420r_+^8} - \frac{92M^3r_+^2}{441r_+^7} - \frac{M^2r_+^2}{70r_+^6} + \frac{229M^3r_+^3}{420r_+^8} - \frac{229M^2r_+^4}{1680r_+^8} \\ & + \xi \left(\frac{7M^4}{15r_+^6} - \frac{14M^4}{15r_+^6} + \frac{23M^3}{15r_+^5} - \frac{13M^2}{20r_+^4} - \frac{32M^4r_+}{15r_+^7} + \frac{13M^4r_+^2}{5r_+^8} + \frac{16M^3r_+^2}{15r_+^7} \right. \\ & \left. \left. - \frac{13M^3r_+^3}{5r_+^8} + \frac{13M^2r_+^4}{20r_+^8} \right) \right]. \quad (2.16) \end{aligned}$$

The integration constants for both μ and ρ have been broken into two pieces, C_1 and C_2 respectively and additional terms, all of which have r_+ in the denominator. This was done so that $\mu(r_+) = C_1$ and $\rho(r_+) = C_2$. The perturbed spacetime is now defined to first order in ϵ to within the two remaining unknown constants C_1 and C_2 .

Due to the perturbation, the event horizon is no longer at r_+ . Instead, it is now located at

$$r_h = m(r) + \sqrt{m(r)^2 - Q^2} \Big|_{r \rightarrow r_h}. \quad (2.17)$$

However we do not know what value of r_h to put into the function $m(r)$. The mass of the black hole is a measurable quantity defined by the quantity of matter inside the horizon radius so we can write

$$M_{bh} = m(r_h) = M[1 + \epsilon\mu(r_+)], \quad (2.18)$$

to first order. This physical, or dressed, mass, M_{bh} , is a function of the bare, and unmeasurable mass, M plus a small perturbation:

$$M_{bh} = M + \delta M, \quad (2.19)$$

where δM is of order ϵ . The horizon radius is determined by the measured mass of the black hole

$$r_h = M_{bh} + \sqrt{M_{bh}^2 - Q^2}. \quad (2.20)$$

Substituting M_{bh} from Eq. (2.19) into Eq. (2.20) yields, to first order in ϵ ,

$$r_h = r_+ \left(1 + \frac{\delta M}{\sqrt{M^2 - Q^2}} \right). \quad (2.21)$$

Using Eq. (2.18) an expression for the perturbation δM can be found:

$$\begin{aligned} \delta M &= \epsilon M \mu(r_+) \\ &= \epsilon M C_1. \end{aligned} \quad (2.22)$$

This leaves a final expression for the horizon radius to first order in ϵ :

$$r_h = r_+ \left(1 + \frac{\epsilon M C_1}{\sqrt{M^2 - Q^2}} \right). \quad (2.23)$$

At this point the physically unmeasurable bare mass M and the integration constant C_1 still appear in the definition of the horizon. It would be preferable to express

things in terms of the dressed mass M_{bh} . This can be done and at the same time the integration constant can be absorbed into the definition of $m(r)$. This is accomplished by starting with Eq. (2.12) and noting that it can be written as

$$m(r) = M[1 + \epsilon C_1 + \epsilon \tilde{\mu}(r)] , \quad (2.24)$$

where

$$\tilde{\mu}(r) = \mu(r) |_{C_1=0} . \quad (2.25)$$

Substituting from Eq. (2.18) for M yields, to first order in ϵ ,

$$m(r) = M_{bh}[1 + \epsilon \tilde{\mu}(r)] . \quad (2.26)$$

Physically what was done was to note that both the bare mass and the integration constant C_1 cannot be measured independently so they have been absorbed into the dressed mass.

Notice also that Eq. (2.23) is ill-defined for the unperturbed extreme case when $|Q| = M$. For this reason this case must be treated independently from the non-extreme cases. To do this, set $|Q| = M$ before doing the renormalization described above. Doing this, to first order, one finds

$$r_h = M[1 + \sqrt{2\epsilon C_1} + \epsilon C_1] . \quad (2.27)$$

Now in addition to a linear order term as was found in the previous near extremal case, a term which goes as $\sqrt{\epsilon}$ has appeared. Having two contributions of different order which will both contribute up to linear order in ϵ is undesirable. To avoid this in the unperturbed extreme case we explicitly set $C_1 = 0$. This physically means that any perturbation of an initially extreme Reissner-Nordström black hole will leave that hole in an extreme state and that the temperature of the black hole will remain zero even in the presence of the quantized scalar field.

The metric can now be rewritten in terms of the dressed mass M_{bh} and $\tilde{\mu}(r)$. Because these quantities are those that can be physically measured, the bh subscript and the tilde will now be dropped, writing only M and μ respectively.

The renormalization of the black hole mass takes care of the first integration constant, C_1 , leaving only C_2 to be defined. To fix the value of C_2 microcanonical boundary conditions will be imposed. The black hole will be treated as being in the center of an ideal, massless, perfectly reflecting spherical wall located at a radius r_c , which is outside the event horizon. The scalar field will be contained inside this cavity. Outside the cavity we demand that we have precisely the classical Reissner-Nordström geometry with a mass identified with $m(r = r_c)$. It is now convenient to transform from Eddington-Finkelstein coordinates back to (t, r, θ, ϕ) coordinates. Doing so one finds the perturbed metric takes the following form:

$$ds^2 = -[1 + 2\epsilon\rho(r)] \left(1 - \frac{2m(r)}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m(r)}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 . \quad (2.28)$$

To avoid discontinuities, g_{tt} must be continuous across the cavity boundary. The form of g_{tt} must then have the following form:

$$g_{tt} = \begin{cases} \left(1 - \frac{2m(r_c)}{r} + \frac{Q^2}{r^2}\right) & r > r_c \\ \left(1 - \frac{2m(r)}{r} + \frac{Q^2}{r^2}\right) [1 + 2\epsilon\rho(r)] & r < r_c \end{cases} \quad (2.29)$$

Continuity across the cavity boundary thus requires that

$$\rho(r_c) = 0. \quad (2.30)$$

This in turn determines the value of C_2 :

$$C_2 = -\tilde{\rho}(r_c), \quad (2.31)$$

where

$$\tilde{\rho}(r) = \rho(r) |_{C_2=0}. \quad (2.32)$$

Unperturbed black holes which are nearly extremal will have nearly zero temperature. In order to investigate the effect the scalar field has on the black hole's temperature, the surface gravity must be calculated. As noted in Eq. (1.13), the temperature depends on the surface gravity of the black hole. It is easiest to calculate the surface gravity for the perturbed metric written in the form of Eq. (2.1). The metric for the perturbed black hole in this form is given in Eq. (2.28). For any static

spherically-symmetric metric the surface gravity is

$$\kappa = \left(\frac{1}{2} \right) \frac{f'}{\sqrt{fh}} \Big|_{r=r_h}, \quad (2.33)$$

where the prime represents a derivative with respect to r . For the perturbed metric in Eq. (2.28) the surface gravity to first order in ϵ is

$$\kappa = \left(\frac{r_h - M}{r_h^2} \right) + \epsilon \left[C_2 \left(\frac{r_h - M}{r_h^2} \right) - \left(\frac{M}{r_h} \right) \mu' \Big|_{r=r_h} \right]. \quad (2.34)$$

Here the first term is just the classical value of the surface gravity. The perturbation in the surface gravity is given by

$$\delta\kappa = \kappa - \left(\frac{r_h - M}{r_h^2} \right). \quad (2.35)$$

As can be seen above explicitly, for unperturbed black holes which are nearly extreme κ is approximately $\delta\kappa$.

Before discussing the results in detail, notice should be paid as to the dependencies in the expression for the surface gravity. First is the dependence on C_2 . Since C_2 is dependent on the size of the cavity radius as seen from Eq. (2.31), we need to take care in our choice of the cavity radius. The size of the cavity must be of appropriate size so that at the surface of the cavity the stress-energy is still dominated by the classical radiation terms. We also want the fractional surface gravity change to be unaffected by the choice of cavity radius. In Fig. (2.1) the fractional surface gravity

change is shown versus the cavity radius for a black hole with a charge to mass ratio of 0.10. A similar graph is shown in Fig. (2.2) for a charge to mass ratio of 0.99. In both cases, $\xi = 1/6$ and $mM = 2$. In both cases, the fractional surface gravity change is approximately constant once the cavity radius reaches a size of $5M$. For simplicity, in all further calculations the cavity radius will be assumed to be $20M$. The surface

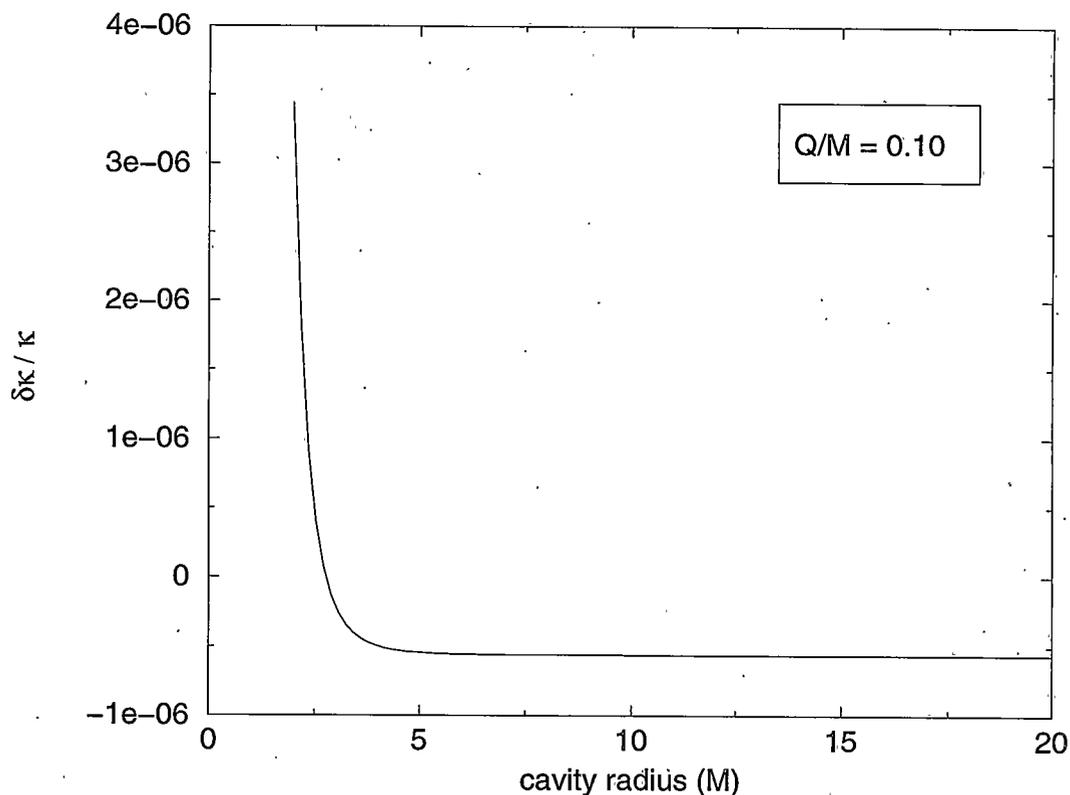


Figure 2.1: The fractional surface gravity change versus cavity radius for a Reissner-Nordström black hole with $|Q|/M = 0.10$, $\xi = 1/6$, $\epsilon = 0.01$, and $mM = 2$.

gravity also depends on the value of the curvature coupling constant, ξ , which appears in both the expression for C_2 and μ . In addition the surface gravity will also depend on the field mass m . For both μ and C_2 , m is just an overall factor. This can be seen explicitly in the expressions for μ and ρ in Eqs. (2.15, 2.16). In cases where we

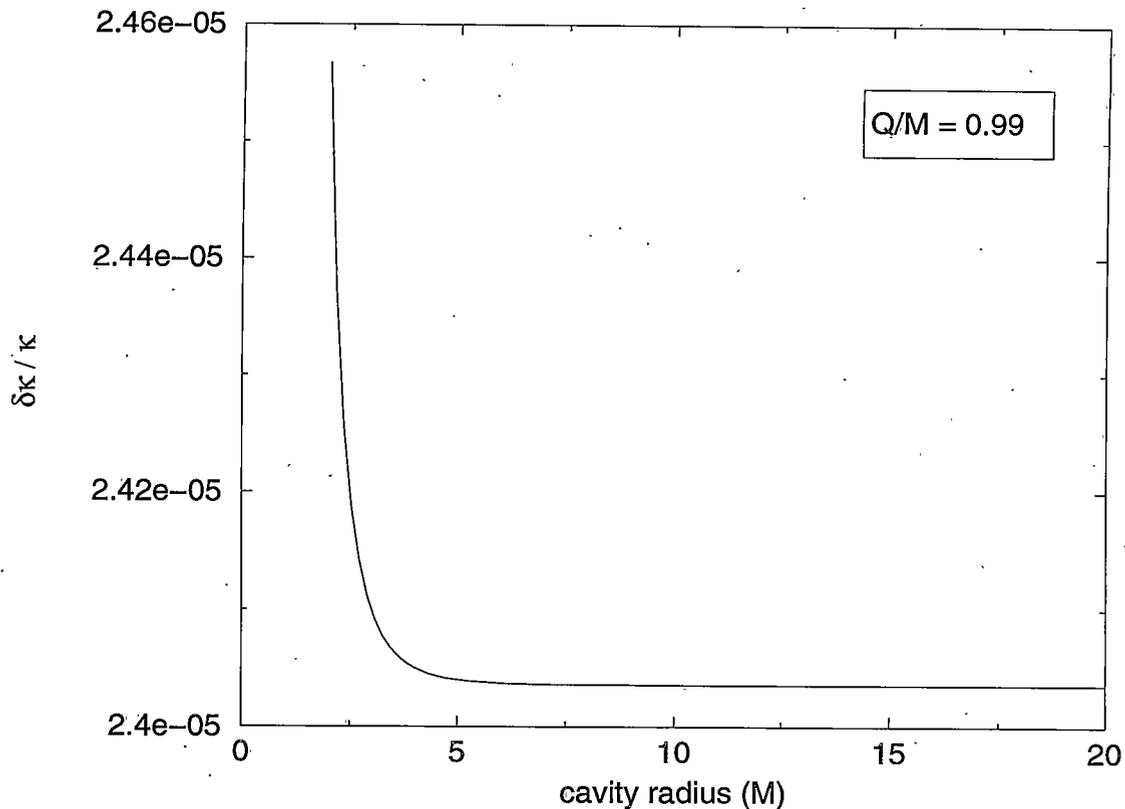


Figure 2.2: The fractional surface gravity change versus cavity radius for a Reissner-Nordström black hole with $|Q|/M = 0.99$, $\xi = 1/6$, $\epsilon = 0.01$, and $mM = 2$.

calculate values for either κ or $\delta\kappa$ explicitly we will assume that $mM = 2$. The cases when either κ or $\delta\kappa$ is zero will be looked at with interest. In the case where $\delta\kappa = 0$, the overall factor of the field mass $1/m^2$ will drop out. This will also be true for the case where $\kappa = 0$ for unperturbed black holes which are nearly extremal. Finally the expression for κ also contains the expansion parameter, ϵ . In all further calculations it will be assumed that $\epsilon = 0.01$. This corresponds to a black hole with a mass of 10 Planck masses and corresponds to the region in which the perturbation theory is believed to hold. Larger values of ϵ would make the perturbations too large to be considered under perturbation theory.

How the presence of the field affects the temperature of the black hole can now be investigated. In addition to investigating black holes which start out nearly extremal, geometries that are naked singularities in their unperturbed state can be investigated provided they have a charge only slightly greater than their mass. This can be done because the expectation value of the stress-energy calculated for the field is calculated locally. The field can only sense the geometry locally, within its Compton wavelength. In these naked singularity cases, the perturbation would act to convert the naked singularity into a zero temperature black hole.

In Fig. 2.3 the horizon radius of the perturbed black hole in terms of the mass is plotted versus the curvature coupling constant. The curve shows the set of perturbed geometries in which the black hole has zero temperature. Unperturbed near extremal black holes lie to the right of $r_h = M$ and unperturbed naked singularities would lie to the left of this value. We can see that exactly at $r_h = M$ the curvature coupling constant takes a value of $\xi = 0.357$. We can also investigate how the temperature changes in the presence of the field. In Fig. 2.4 the curve represents those black holes which have $\delta\kappa = 0$. This curve is plotted in terms of the charge and the curvature coupling constant.

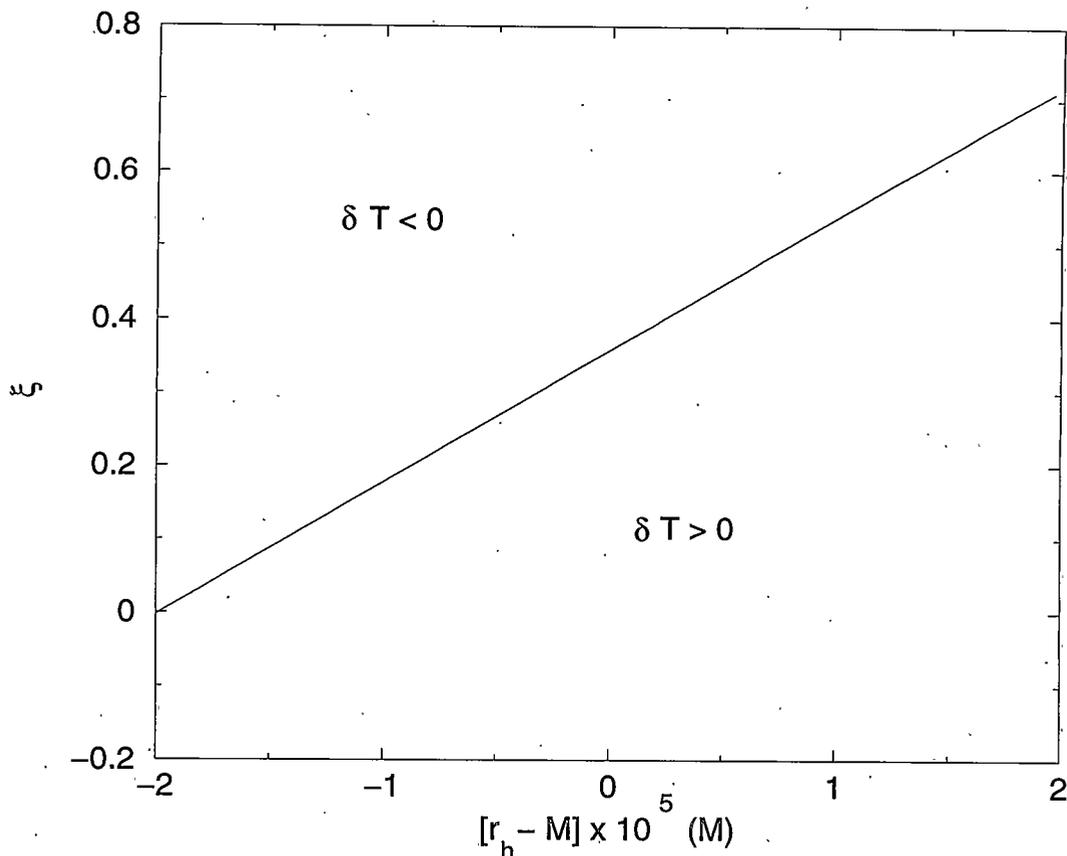


Figure 2.3: The curve shows zero temperature semiclassical black hole solutions for objects that were either initially a nearly extreme black hole or a naked singularity with a charge slightly larger than its mass in their unperturbed state. The quantity δT indicates how the presence of the field changes the temperature relative to the classical value of the temperature.

For a conformally or minimally coupled scalar field the temperature is always increased for an initial state corresponding to a near extremal black hole so, for these cases, there are not any zero temperature solutions to first order. If the unperturbed state is a naked singularity the situation is more complicated. A zero temperature solution with a conformally coupled scalar field will have a perturbed horizon radius of approximately $0.99999M$. A minimally coupled scalar field will have zero temperature with a perturbed horizon radius of approximately $0.99998M$. Note that these horizons

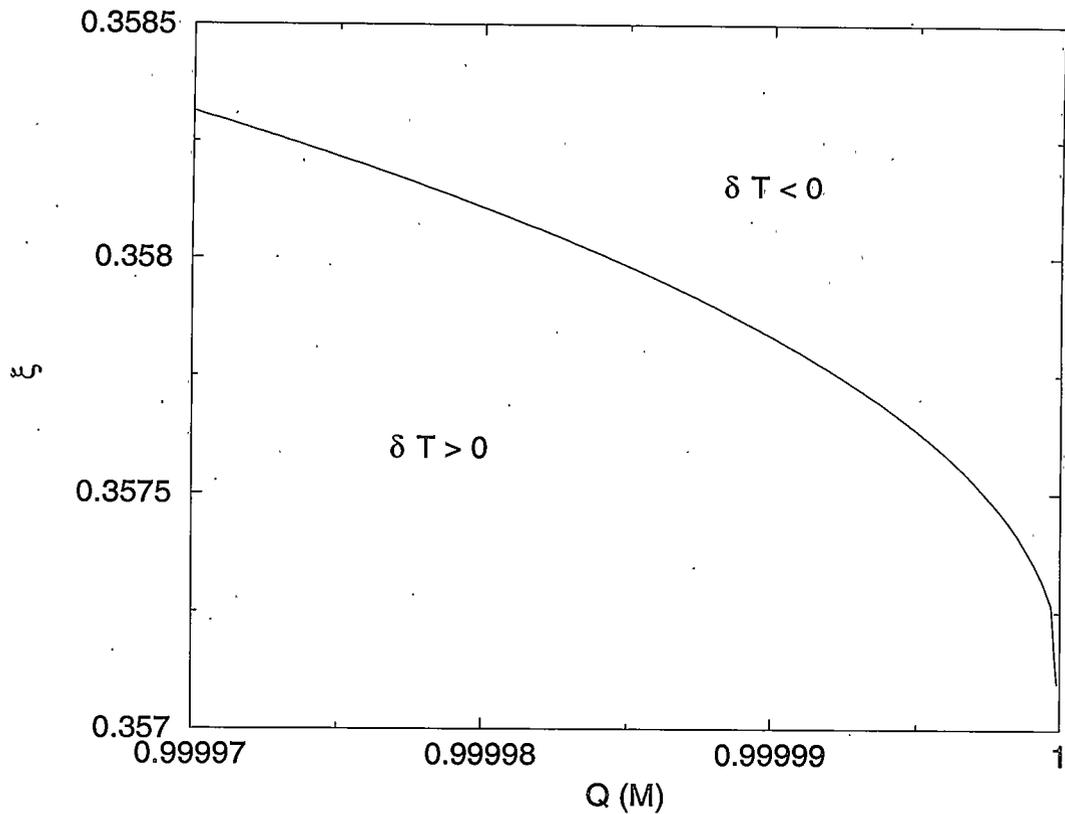


Figure 2.4: The curve represents semiclassical black hole solutions for which the change in temperature is zero for particular values of the charge and curvature coupling constant. Below the curve, the temperature is increased and above the curve the temperature is decreased by the presence of the scalar field relative to the classical temperature.

would not actually exist in the unperturbed state because these values represent naked singularities which have no horizons.

2.2 Perturbations from a massless scalar field using numerical data to determine $\langle T^\mu_\nu \rangle$

In Section 2.1 the DeWitt-Schwinger approximation was used to obtain the value for the expectation of the stress-energy tensor for a quantized massive scalar field.

In this section numerical data from work done by Anderson, Hiscock, and Lorz [18] will be used to obtain an expression for the stress-energy tensor of a quantized massless scalar field near the event horizon. The above authors were investigating the semiclassical stability of extreme Reissner-Nordström black holes. Previously Trivedi [21] had shown that in two dimensional gravity the stress-energy tensor diverged on the horizon of an extreme Reissner-Nordström black hole. Analytic approximations done in four dimensions also predicted a divergence in the stress-energy tensor of a massless scalar field at the horizon of an extreme Reissner-Nordström black hole [20, 22]. If the actual stress-energy tensor diverges, then semiclassical effects would substantially alter the classical spacetime geometry around the black hole. Such a situation could certainly not be analyzed using perturbation theory.

In their paper, Anderson, Hiscock, and Lorz numerically calculated the stress-energy tensor of an arbitrarily coupled massless scalar field near the event horizon of an extreme Reissner-Nordström black hole [18]. This result will be used here to determine how the presence of the field will affect the surface gravity, or equivalently the temperature, of the black hole. The results given here will apply only near the horizon of the black hole due to the fact that the stress-energy tensor calculation was done only for near the event horizon. In the previous section results for objects which were unperturbed naked singularities were discussed. This was possible due to the field being massive and hence sampling only its local region of spacetime. In this case the massless field prevents us from looking at naked singularities as the field has no inherent length scale over which it samples.

The metric for the perturbed extreme Reissner-Nordström black hole takes the form of the general static spherically symmetric metric given in Eq. (2.1). Its components are explicitly given by:

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2d\Omega^2, \quad (2.36)$$

where

$$f(r) = \left(1 - \frac{M}{r}\right)^2 + \epsilon\tilde{f}(r), \quad (2.37)$$

$$h(r) = \left(1 - \frac{M}{r}\right)^{-2} + \epsilon\tilde{h}(r). \quad (2.38)$$

Here $\tilde{f}(r)$ and $\tilde{h}(r)$ are the perturbation functions and ϵ is our expansion parameter, $1/M^2$.

The stress-energy tensor for the quantized massless scalar field may be written in the form

$$\langle T^\mu{}_\nu \rangle = \langle C^\mu{}_\nu \rangle + \left(\xi - \frac{1}{6}\right) \langle D^\mu{}_\nu \rangle, \quad (2.39)$$

where $C^\mu{}_\nu$ is the conformal piece of the stress energy tensor and $D^\mu{}_\nu$ is the nonconformal piece. The particular components we are concerned with have the following expressions

$$\langle C_t^t \rangle = \left(\frac{\epsilon}{368640M^2\pi^2} \right) [128.803733 - 193.782517x - 23928.9977x^2 + 266349.65x^3] , \quad (2.40)$$

$$\langle D_t^t \rangle = \left(\frac{\epsilon}{368640M^2\pi^2} \right) [5.56001333 - 1364.90448x - 215647.04x^2 + 2026455.94x^3] , \quad (2.41)$$

$$\langle C_r^r \rangle = \left(\frac{\epsilon}{368640M^2\pi^2} \right) [128.190333 - 44.7975913x - 10531.7106x^2 + 92836.0528x^3] , \quad (2.42)$$

$$\langle D_r^r \rangle = \left(\frac{\epsilon}{368640M^2\pi^2} \right) [0.70762 - 145.994095x - 63573.7436x^2 + 495390.831x^3] , \quad (2.43)$$

where x is a dimensionless quantity given by $x = (r - M)/M$. The numerical data taken from Anderson, Hiscock and Loran was fitted by a polynomial fit to third order in x to give the above expansions. As in the previous section, the angular components are not needed to completely solve the perturbed semiclassical Einstein equations.

Before the semiclassical Einstein equations given by Eq. (1.2) can be solved, the components of the Einstein tensor must be determined. The relevant components to

first order in ϵ are

$$G_t^t = -\frac{M^2}{r^4} + \epsilon \frac{(M-r)^3}{r^6} \left[3M\tilde{h} + r\tilde{h} - rM\tilde{h}' + r^2\tilde{h}'' \right] + O(\epsilon^2), \quad (2.44)$$

$$G_r^r = -\frac{M^2}{r^4} + \frac{\epsilon}{(M-r)r^6} \left[2Mr^4\tilde{f} + (M-r)^4(M+r)\tilde{h} + r^5(M-r)\tilde{f}' \right] + O(\epsilon^2), \quad (2.45)$$

where a prime represents a derivative with respect to the r coordinate. Note the expressions have a zeroth order piece. This will be exactly cancelled by the background stress-energy due to the electromagnetic field of the black hole. One can then integrate the equations given by setting Eqs. (2.44, 2.45) equal to the source terms which are made from the classical terms due to the electromagnetic field of the black hole and the semiclassical terms given by Eqs. (2.40 - 2.43) combined according to Eq. (2.39) to find expressions for the perturbative functions \tilde{f} and \tilde{h} :

$$\begin{aligned} \tilde{f} = & \frac{(M-r)^2}{r^2} \left[\frac{7.0841r^4(-0.0082994 - 0.49852\xi)}{M^4\pi} \right. \\ & + \frac{14.168M^2(0.0036411 - 0.062562\xi)}{(M-r)^2\pi} + \frac{28.336M(-0.0087825 + 0.031252\xi)}{(M-r)\pi} \\ & + \frac{28.336r(-0.015492 + 0.03218\xi)}{M\pi} + \frac{(5.6673r^5(0.016408 + 0.12073\xi))}{M^5\pi} \\ & + \frac{14.168r^2(-0.013992 + 0.12605\xi)}{M^2\pi} + \frac{9.4456r^3(-0.016133 + 0.28249\xi)}{M^3\pi} \\ & \left. + \frac{38.3364(-0.020634 + 0.00087028\xi) \log(r-M)}{\pi} \right], \quad (2.46) \end{aligned}$$

$$\tilde{h} = \frac{16.209}{M^5\pi(r-M)^4} [-0.037267M^3r^6 + 0.079725M^2r^7 - 0.060572Mr^8 + 0.014932r^9 + \xi(M^3r^6 - 2.7187M^2r^7 + 1.6856Mr^8 - 0.45219r^9)] \quad (2.47)$$

Now that we have expressions for \tilde{f} and \tilde{h} we can calculate the surface gravity. We can do this most easily by using Eq. (2.33).

The expressions we actually use for this calculation may be simplified immensely by recalling that we are doing a perturbation. In this case the perturbed state is very near an extreme black hole. For κ one need only worry about first order perturbations around the black hole. The perturbed horizon will be very near the classical extreme horizon of $r_+ = M$. This simplifies the expression for f considerably. Consider the case where

$$r_h = M + r_1. \quad (2.48)$$

Here the order of r_1 is not given explicitly, but it will be of first or lesser order in ϵ . Substituting this expansion for the event horizon given in Eq. (2.48) into the relation for \tilde{f} will yield only two terms which are up to first order in ϵ . With this in mind the expression for \tilde{f} can be simplified to

$$\tilde{f} = \frac{(M-r)^2}{r^2} \left[\frac{14.168M^2(0.0036411 - 0.062562\xi)}{(M-r)^2\pi} + \frac{28.336M(-0.0087825 + 0.031252\xi)}{(M-r)\pi} \right] \quad (2.49)$$

From Eq. (2.33), it can be shown that to zeroth order the numerator of κ is zero

leaving only a first order term. Since κ is evaluated at the event horizon we can expand the denominator about the horizon radius given in Eq. (2.48). Since only first order perturbations to κ are being investigated and since the numerator is already explicitly first order in ϵ , only the zeroth order term from the denominator will be kept. The zeroth order piece of the denominator is precisely 1.

Before explicitly calculating κ , it is convenient to define the following quantities:

$$A = \frac{14.168M^2}{\pi}(0.0036411 - 0.062562\xi), \quad (2.50)$$

$$B = \frac{28.336M}{\pi}(-0.0087825 + 0.031252\xi). \quad (2.51)$$

The function f is now given to first order in ϵ by

$$f = \left(1 - \frac{M}{r}\right)^2 + \frac{\epsilon A}{r^2} + \frac{\epsilon B(M - r)}{r^2}. \quad (2.52)$$

Using the expression given in Eq. (2.52), κ can be calculated as

$$\kappa = \frac{1}{2} \left[\frac{2M}{r^2} \left(1 - \frac{M}{r}\right) - \frac{2\epsilon A}{r^3} - \frac{2\epsilon B M}{r^3} + \frac{\epsilon B}{r^2} \right] \Big|_{r=r_h}. \quad (2.53)$$

To finally evaluate the surface gravity we need to locate the horizon of the perturbed black hole. Since this is a static spherically symmetric spacetime the horizon can be found by determining the values of r where f is zero. Doing this we find

$$r_h = M + r_1$$

$$= M \pm a\sqrt{\epsilon} + \epsilon \frac{B}{2} \quad (2.54)$$

where $a^2 = -A$, A is defined as in Eq. (2.51), and B is defined in Eq. (2.51).

The expression for κ can given in Eq. (2.53) can now be evaluated. Doing so at the horizon yields

$$\kappa = \frac{\sqrt{-A\epsilon}}{M^2} + \frac{2A\epsilon}{M^3} \quad (2.55)$$

The positive root for the $\sqrt{\epsilon}$ term in Eq. (2.54) was chosen for the horizon radius as it corresponds to the event horizon. The mass M is just an overall scaling factor. In order to plot κ , values for M and ϵ must be chosen. Since M is just a scaling factor it is simplest to set $M = 10$ which fixes the value of ϵ to be $\epsilon = 0.01$. Figure 2.5 shows the surface gravity for a black hole in the presence of a massless quantized scalar field.

Note that zero temperature solutions exist when $A = 0$. The value of ξ that is required for such a solution is $\xi = 0.05820$ as can be found from Eq. (2.51). For the semiclassical black hole solution to have a well-defined temperature, the massless scalar field must have $\xi \geq 0.05820$. The above restrictions severely limit zero temperature semiclassical black hole solutions. This result can be seen graphically in Fig. 2.5. In particular a conformally coupled scalar field will not result in a zero temperature solution and a minimally coupled scalar field will not give a well-defined temperature for the semiclassical black hole.

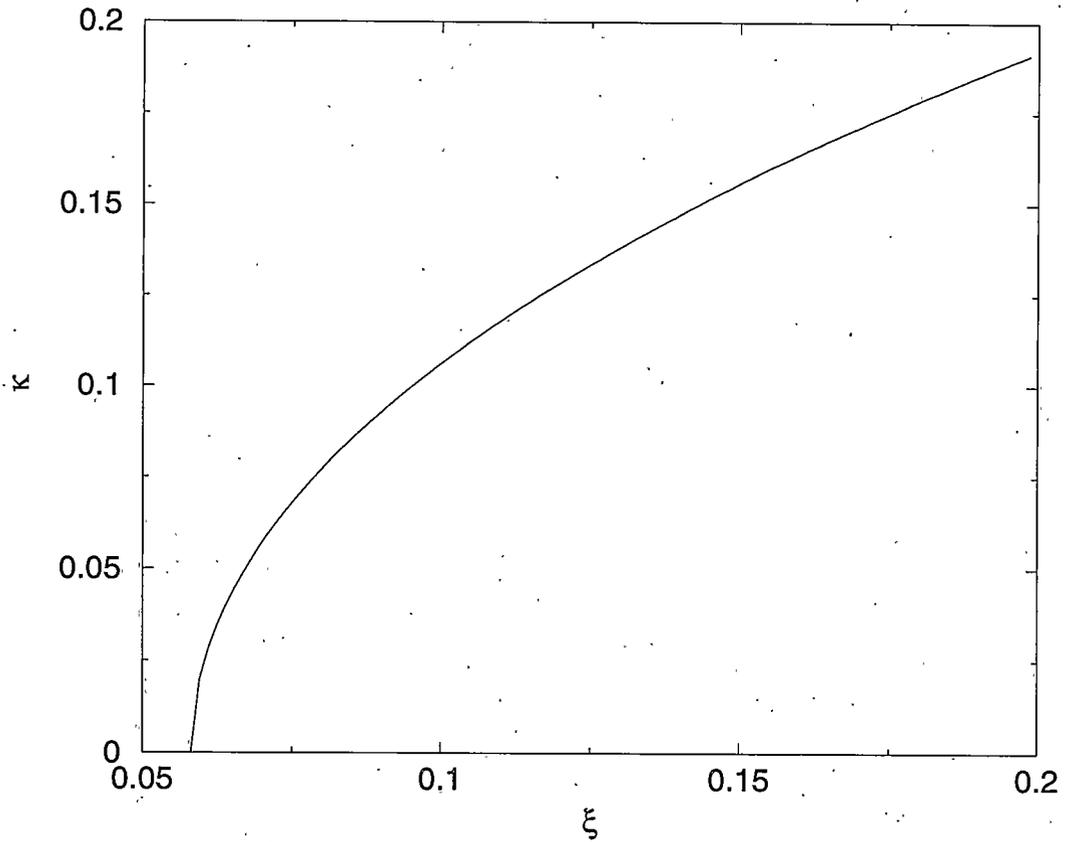


Figure 2.5: The surface gravity for a semiclassical Reissner-Nordström black hole is plotted versus the curvature coupling constant. The unperturbed black hole was extreme. The mass has been chosen such that $M = 10$.

CHAPTER 3

SPINNING DOWN A KERR BLACK HOLE DUE TO THE
EMISSION OF MASSLESS SCALAR PARTICLES

The evolution of evaporating black holes is a process that has been studied in great detail. The conventional viewpoint of black hole evolution has been that Hawking radiation will always force the black hole to asymptotically approach an uncharged, zero angular momentum final state long before all its mass has been lost, regardless of its initial state. Consequently as the black hole approaches the Planck scale, where quantum gravity will be needed to describe further evolution, the final asymptotic state of the black hole is assumed to be described by the Schwarzschild metric. Recently we showed that this conventional view of black hole evolution will not necessarily be true if the black hole is allowed to emit scalar particles as it evaporates [23, 24].

Previous calculations have shown that any charge on the black hole will be rapidly neutralized [13, 14, 15]. The number of particles that are needed to carry off charge Q goes like Q/e which is of the same order as the mass M of the black hole. Using

Hawking's result, as seen in Eq. (1.11), Carter was able to make qualitative arguments which suggested that the mass and angular momentum loss would proceed on cosmologically long time scales [15]. In the case of angular momentum loss, the number of particles needed to carry off angular momentum J goes like M^2 . The timescale on which the mass is removed goes like M^3 . These qualitative arguments could not determine whether the angular momentum or mass loss would occur at a greater rate however, so the final state of an initially rotating hole was indeterminate. Using Hawking's results on quantum emission from black holes [9], Page was able to numerically investigate this and other related questions. By performing a quantitative study of the emission of particles associated with massless fields of spin 1/2, 1, and 2 for both rotating [2] and nonrotating holes [25], Page was able to show that the dimensionless rotation parameter, $a_* = a/M = J/M^2$, tends to zero asymptotically more rapidly than the mass approaches zero. This indicates that angular momentum is lost at a greater rate than mass, which implies the final asymptotic state of an initially rotating hole is described by the Schwarzschild metric. Specifically Page found that for a canonical collection of nonzero spin fields (a single spin 1, a single spin 2, and three spin 1/2) that a hole will lose all of its angular momentum by the time it has lost half of its mass [2].

This result is important as realistic black holes which form from stellar collapse should be initially highly rotating [4]. However Page did not consider the effect that scalar fields might have on the evolution. He did however speculate that the emission of scalar particles might allow evolution to nonzero final values of a_* , in which case the

hole must still be represented by the more complicated Kerr metric, rather than the Schwarzschild metric, as it enters the late stages of its evolution. Page's speculation was based on two arguments. The first argument is related to the dominant modes of particle emission from the evaporating hole. Page found for the nonzero spin fields with spin $s = 1/2, 1, 2$ that the dominant modes were those with $l = m = s$ where l and m are the usual spheroidal harmonic indices. Only a scalar field can radiate in the $l = 0$ mode and from Page's argument one might then expect that this would be the dominant mode for scalar particle emission. This mode carries off energy but no angular momentum. If this mode were to dominate the overall emission from both scalar and nonscalar fields then the hole could lose mass faster than it loses angular momentum. This would result in the hole evolving toward an asymptotic state with nonzero a_* . Secondly, Page noted in his results a simple, and perhaps accidental, linear relationship between the spin of the field and the normalized ratio of the angular momentum loss rate to the mass loss rate as a_* tended to zero. Page noted that if this relationship were extrapolated to the zero spin scalar field then the loss rates for emission of solely scalar particles would result in a final state of nonzero angular momentum, contrary to the result he found for the nonscalar fields.

In this chapter Page's work will be extended to include the scalar field. First, the evolutionary path that a hole which is initially rotating follows if it is allowed to radiate solely into a massless scalar field will be investigated. In addition the case where the hole can radiate into not only the scalar field but a canonical set of nonscalar fields (3 neutrino, photon, and graviton) will be investigated. From this

it will be determined whether it is possible to have the black hole asymptotically evolve to a nonzero value of a_* . With these results along with those of Page from Ref. [2] one can calculate the evolution of a Kerr black hole which is evaporating into an arbitrary collection of scalar and nonscalar fields. In addition a number of other interesting quantities will be calculated for radiation into both the purely scalar and scalar plus the canonical set of nonzero spin fields. These quantities include the lifetime of a Kerr black hole for a variety of initial conditions, the minimum initial mass a primordial Kerr black hole could have possessed assuming it was observed today with a specific angular momentum of a_* , the initial mass of a primordial Kerr black hole which would have just disappeared today, and the evolution of the event horizon area, or equivalently the entropy, of Kerr black holes as they evolve.

In Section 3.1 the mathematical formulas that will be used to determine the evolution of the Kerr black hole as it evaporates are introduced. Section 3.2 will contain the numerical methods used and finally the results will be presented in Section 3.3.

3.1 Mathematical formulas

The Kerr metric describes a black hole which is uncharged and rotating. This angular momentum can be represented in a number of ways. Most commonly in the literature it is represented as the specific angular momentum given by $a = J/M$. It is more convenient in this work to use the dimensionless rotation parameter given by $a_* =$

$a/M = J/M^2$. For this the black hole will be assumed to be in isolation and that enough time has passed since its formation that any initial charge, if any, has been neutralized and that the black hole is still rotating with angular momentum described by the rotation parameter a_* . The basic procedures to be followed are the same as used by Page for the nonscalar fields [2]. It will be more convenient to use Kerr-ingoing coordinates, rather than the Boyer-Lindquist form given in Eq. (1.7). The metric has the following form in Kerr-ingoing coordinates

$$ds^2 = - \left[1 - \frac{2Mr}{\Sigma} \right] d\tilde{v}^2 + 2drd\tilde{v} + \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} [(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta] d\tilde{\phi}^2 - 2a \sin^2 \theta dr d\tilde{\phi} - \frac{4Mra \sin^2 \theta}{\Sigma} d\tilde{v} d\tilde{\phi}. \quad (3.1)$$

Here the quantities Σ and Δ are still defined as they were for Eq. (1.7).

The wave equation for the massless scalar field, $\square\phi = 0$, separates in Kerr-ingoing coordinates [3] by choosing

$$\phi = R(r)S(\theta)e^{-i\omega\nu}e^{im\phi}, \quad (3.2)$$

where $S(\theta)$ is a spheroidal harmonic [26]. The radial function $R(r)$ satisfies

$$(\partial_r \Delta \partial_r - 2iK \partial_r - 2i\omega r - \lambda) R(r) = 0, \quad (3.3)$$

where $\Delta = r^2 - 2Mr + a^2$, $K = (r^2 + a^2)\omega - am$, $\lambda = E_{lm\omega} - 2am\omega + a^2\omega^2$ and $E_{lm\omega}$ is the separation constant. A general solution to Eq. (3.3), expressed in terms

of known functions, is not known, but asymptotic solutions can be found [3],

$$R \longrightarrow \begin{cases} Z_{\text{hole}} & r \rightarrow r_+ \\ Z_{\text{in}}r^{-1} + Z_{\text{out}}r^{-1}e^{2i\omega r} & r \rightarrow \infty \end{cases} \quad (3.4)$$

The subscript “in” refers to an ingoing wave originating from past null infinity, “out” refers to an outgoing wave reflected from the hole that propagates toward future null infinity, and “hole” refers to the component of the wave that is transmitted into the black hole through the horizon at $r = r_+$. The quantities Z_{in} , Z_{out} , and Z_{hole} represent complex amplitudes of the waves. The amplification, Z (the fractional gain of energy in a scattered wave), is

$$Z = \left| \frac{Z_{\text{out}}}{Z_{\text{in}}} \right|^2 - 1. \quad (3.5)$$

This quantity can be greater than zero due to the superradiant scattering process.

Following Page [25] the rates at which the mass and angular momentum decrease are expressed by the quantities $f \equiv -M^2 dM/dt$ and $g \equiv -Ma_*^{-1} dJ/dt$ respectively, which have been scaled to remove overall dependence on the size (mass) of the black hole. The coordinate t is the usual Boyer-Lindquist time coordinate. These quantities will determine the evolution of the black hole. They are defined by

$$\begin{pmatrix} f \\ g \end{pmatrix} = - \sum_{l,m} \frac{1}{2\pi} \int_0^\infty dx \frac{Z}{e^{2\pi k/\kappa} - 1} \begin{pmatrix} x \\ ma_*^{-1} \end{pmatrix}, \quad (3.6)$$

where $k = \omega - m\Omega$, $\Omega = a_*/2r_+$ is the angular frequency of the event horizon,

$\kappa = \sqrt{(1 - a_*^2)}/2r_+$ is the surface gravity of the hole, and following Page [2] we have defined $x = M\omega$. The relative magnitude of the mass and angular momentum loss rates will determine whether or not the hole will spin down to a nonzero value of a_* .

To describe how the angular momentum and mass loss rates compare we define

$$h(a_*) \equiv \frac{d \ln a_*}{d \ln M} = \frac{g(a_*)}{f(a_*)} - 2. \quad (3.7)$$

To investigate how the mass and angular momentum evolve in time, it is convenient to define new quantities. Again following Page [2], define

$$y \equiv -\ln a_*, \quad (3.8)$$

which will be used as a new independent variable replacing t . If the initial mass of the black hole at $t = 0$ is defined to be M_1 , then a dimensionless mass variable z may be defined by

$$z \equiv -\ln(M/M_1), \quad (3.9)$$

which has the initial value $z(t = 0) = 0$. The hole's mass, now parameterized by z , with the definitions of f and g as well as Eq. (3.8) and Eq. (3.9), will then evolve according to

$$\frac{dz}{dy} = \frac{1}{h} = \frac{f}{g - 2f}, \quad (3.10)$$

Finally define a scale-invariant time parameter

$$\tau \equiv M_1^{-3}t, \quad (3.11)$$

with the initial value $\tau(t = 0) = 0$. The evolution of τ with respect to y is then determined by

$$\frac{d\tau}{dy} = \frac{e^{-3z}}{g - 2f}. \quad (3.12)$$

To see explicitly how a_* evolves with respect to time t Eq. (3.12) can be used to find

$$\frac{da_*}{dt} = -\frac{a_* h f}{M^3}. \quad (3.13)$$

At points in the evolution where $g = 2f$, or equivalently where $h = 0$, the right-hand side of Eq. (3.12) is divergent. This is due to the fact that the hole will be described by a constant value of a_* as it continues to lose mass. It can be seen from Eq. (3.13) that if the black hole is in isolation, it can only lose mass (via the Hawking process), so the function f must be positive definite. Therefore, if there is a nonzero value of $a_* = a_{*0}$, for which h is zero, then da_*/dt will be zero there and the hole will remain at that value of a_* . If dh/da_* is positive at such a point, then it represents a stable state towards which holes will asymptotically evolve. If dh/da_* is negative or zero at a point where $h = 0$, then it represents an unstable equilibrium point of a_* , and holes will evolve away from it. It will be shown that emission of scalar radiation may create stable states of a_* at points where $h = 0$ and $dh/da_* > 0$, depending on the

mix of fields present.

The evolution of the mass and angular momentum of the hole are completely determined by Eq.(3.10) and Eq.(3.12). For collections of fields for which the black hole spins down completely, evolving towards a state described by a value of $a_* = 0$, the situation resembles that analyzed by Page. First, the initial point of the numerical integration will be taken to be nearly extremal with $a_* = 0.999$. The resulting functions $z(y)$ and $\tau(y)$ may then be used to describe evolution from other initial rotations a_{*i} , with the initial values

$$y_i \equiv -\ln(a_{*i}), \quad (3.14)$$

$$z_i \equiv z(y_i) = -\ln(M_i/M_1), \quad (3.15)$$

$$\tau_i \equiv \tau(y_i) = M_i^{-3} t_i. \quad (3.16)$$

A hole which began with $a_* = 0.999$ and $M = M_1$ at $t = 0$ will then have $a_* = a_{*i}$ and $M = M_i$ at $t = t_i$.

For a collection of fields for which the hole evolves to a spinning asymptotic state described by $h(a_{*0}) = 0$ (as happens, for example, with purely scalar radiation) two distinct evolutionary cases must be investigated. One of these is to start out again with a nearly extremal hole as described above, where now the possible initial values a_{*i} range only from the initial near extremal value to the asymptotic limiting value a_{*0} for that set of fields. In the second case the initial point of the evolution will be taken to be nearly Schwarzschild with an initial value $a_* = 0.001$. In this case the value of

